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BARBIER TYPE THEOREMS FOR PLANE CURVES

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A C^1 class of simple closed and convex plane curves which contains all ovals is considered. This class is divided into subclass for which greatest lower bound of number of Barbier sets are determined.

1. Introduction and main results.

Let C_L denote the set of all the positive continuous periodic functions defined on $\mathbb{R} = (-\infty, +\infty)$ with the period $L > 0$. Then we define as in [2] the following class Q_0 of all the closed plane C^1-curves of the form

$$r_{f,k}(s) = \int_0^s k(t)f(t)\, e^{iK(t)}dt,\ s \in \mathbb{R}, \tag{1.1}$$

where $K(t) = \int_0^t k(s)\, ds$ and $f, k \in C_L$ and $K(L) = 2\pi$. Note that a curve $r_{f,k}$ is closed if and only if

$$\int_0^L k(t)f(t) \cos K(t)dt = \int_0^L k(t)f(t) \sin K(t)dt = 0. \tag{1.2}$$

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In the sequel it is denoted by [P] the set of all the plane curves, which are obtained by isometrical transformations of curves belonging to the fixed set P of plane curves. We define

$$Q = [Q_0] \tag{1.3}$$

If fk is continuously differentiable, then the curvature of $r_{f,k}$ is equal to $\frac{1}{f}$. Therefore the class of all the ovals, [4], is included in Q.

At first we consider the subclass of the Q curves of constant n-width. Each convex polygon with n sides and equal interior angles is called n-polygon. Each n-polygon imitates the belt determined by two parallel straight lines in the plane. We express this known fact as follows.

LEMMA A. *For each point belonging to the region bounded by n-polygon or belonging to the sides of the n-polygon the sum of distances of the point from the straight lines passing by the sides of the n-polygon is constant.*

Therefore the value of the sum is called the n-width of the n-polygon. Obviously 2-width means usual width of the belt bounded by two parallel straight lines.

Now we consider a curve $x(s) = r_{f,k} \in Q_0$. Then $T_s = e^{iK(s)}$ is the unit tangent vector and $N_s = iT_s$ is the unit normal one at point $x(s)$. If $s \in [0, L)$, then the unit tangent vector T_s determines all directions on the plane. It is easy to observe that the function

$$s \to \varphi_n(s) = K^{-1}\left(k(s) + \frac{2\pi}{n}\right) \tag{1.4}$$

where K^{-1} is the inverse function for K and $n = 2, 3, \ldots$, rotates the tangent vector T_s by the angle $\frac{2\pi}{n}$.

For a fixed number $s \in \mathbb{R}$, the tangents at points

$$x(s), x(\varphi_n(s)), x(\varphi_n^2(s)), \ldots, x(\varphi_n^{n-1}(s)). \tag{1.5}$$

determine the n-polygon described on $x(s)$,wy here

$$\varphi_n^v(s) = \varphi_n(\varphi_n(\ldots\varphi_n(s)\ldots)),\ v = 1, 2, \ldots$$

The n-width of the n-polygon will be called the *n-width of the curve $x(s)$ in the direction of T_s* and it will be denoted by $\omega_n(s)$. Obviously

$$\omega_n(\varphi_n(s)) = \omega_n(s), \quad \text{for} \quad s \in \mathbb{R}. \tag{1.6}$$

The curve $x(s)$ will be called *of constant n-width* if and only if $\omega_n(s)$ is a constant function. Next we say that the curve $y \in [\{x\}] \subset Q$ has constant n-widt if and only if the curve $x(s)$ has constant n-width. Now we give the first main result of the paper;

THEOREM 1. *All the curves of Q of constant n-width ω_n have the same perimeter $\frac{2\pi}{n}\omega_n$.*

If $n = 2$, we have

THEOREM A (Barbier). *All the curves of constant width b have the same circumference πb.*

The second group of the results has special character. To express it we introduce some notions. Let M denote the perimeter of curve $x \in Q_0$.

DEFINITION 1. *For parameter $s_0 \in \mathbb{R}$, the set of points on curve x defined by (1.5) for $s = s_0$ will be called au n-Barbier set of points if and only if the equality holds*

$$M = \frac{2\pi}{n}\omega_n(s_0). \tag{1.7}$$

Obviously we extend the definition of n-Barbier set of points on each curve $y \in [\{x\}] \subset Q$.

A 2-Barbier set will be calle a *a Barbier pair of points.*

Let $A \subset Q$ be an arbitrary subset of Q. We put

$$\beta_n(A) = \min\{\beta_n(y) : y \in A\},$$

where $\beta_n(y)$ denotes the number of the n-Barbier sets of points on the curve y.

Now we consider some subsets of Q. For this purpose we recall that. The sequence

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nK(s), \; \frac{1}{\sqrt{\pi}} \sin nK(s), \; n = 1, 2, \ldots$$

is orthonormal and forms a complete system in the real Hilbert space $L^2_{[0,L]}(k)$ with weight $k(s)$, [2].

Let $f \in C_L$ and $r_{f,k} \in Q_0$. Then the Fourier series for f has the following form

$$f(s) = \frac{M}{2\pi} + \sum_{n=2}^{\infty} [A_n \cos nK(s) + B_n \sin nK(s)], \tag{1.8}$$

because $A_1 = B_1 = 0$ (see condition (1.2)).

Let $Q_0^{(n,p)} \subset Q_0$, $n = 2, 3, \ldots$, $p = 2, 3 \ldots$ denote the set of all curves $r_{f,k}$ such that

$$A_n = B_n = A_{2n} = B_{2n} = \ldots A_{n(p-1)} = B_{n(p-1)} = 0.$$

Then we define

$$Q_{n,p} = [Q_0^{(n,p)}]. \tag{1.9}$$

No we can express the following.

THEOREM 2. *The identities $\beta_n(Q) = 2$ hold for $n = 2, 3, \ldots$.*

THEOREM 3. *The identities $\beta_n(Q_{n \cdot p}) = 2p$ hold for $n = 2, 3, \ldots$ $p = 2, 3, \ldots$.*

2. Proof of Theorem 1.

To prove Th 1. we apply the following.

LEMMA 1. *Let $Ł_n$ and ω_n, $n \geq 3$, denote the perimeter and the n-width of the n-polygon, respectively. Then the equality*

$$Ł_n = 2\omega_n \tan \frac{\pi}{n}$$

holds.

Proof. Let the complex numbers $z_0, z_1, \dots, z_{n-1}$ be the vertices of the n-polygon. Then the unit vectors

$$t, \varepsilon t, \dots, \varepsilon^{n-1} t$$

where

$$t = (z_1 - z_0)|z_1 - z_0|^{-1} \text{ and } \varepsilon^n = 1$$

are parallel to the sides of the n-polygon.

Now let z be a point of the interior of the n-polygon. Then we express the sum $\omega_n(z)$ of the distances of z from the straight lines passing by the sides of the n-polygon as follows

$$\omega_n(z) = \sum_{v=0}^{n-1} [\varepsilon^v t, z - q],$$

where points $q_0, q_1, \dots, q_{n-1}$ are chosen on the sides $z_0 z_1, z_1 z_2, \dots, z_{n-1} z_0$ of the n-polygon and $[z, w]$ is the determinant of z and w. Hence

$$\omega_n(z) = \left[\sum_{v=0}^{n-1} \varepsilon^{n-v} q_v, t \right]. \tag{2.1}$$

This means that $\omega_n(z)$ *is independent* of z.

Let ξ_v and η_v denote the lengths of the sectors between $q_v z_{v+1}$, respectively, $v = 0, 1, 2, \dots, n-1$ (and $z_n = z_0$). We consider the folllowing system of equations

$$q_v + \xi_v \varepsilon^v t = q_{v+1} - \eta_v \varepsilon^{v+1} t, \; v = 0, 1, 2, \dots, n-1.$$

Obviously $Ł_n = \sum_{v=0}^{n-1} (\xi_v + \eta_n)$.

But $\xi_v \sin \frac{2\pi}{n} = -[q_v - q_{v+1}, \varepsilon^{v+1} t]$, $\quad \eta_v \sin \frac{2\pi}{n} = [q_v - q_{v+1}, \varepsilon^v t]$.

Finally $Ł_n = 2\omega_n \tan \frac{\pi}{n}$. This completes the proof.

Proof. of Th 1. Lm 1. means that each curve $y \in Q$ has constant n-width $\omega_n(s)$, if and only if all n-polygons described on y have the same perimeter.

let

$$M = \int_0^L k(t)f(t)dt$$

be the length of the curve $y \in Q$. Similarly as in Th1. [1] we show that

$$\int_0^L k(s)Ł_n(s)ds = 2nM \tan\frac{\pi}{n}. \tag{2.2}$$

But $Ł_n(s) \equiv Ł_n$, (i.e. $Ł_n$ is a constant function), thus $2\pi Ł_n = 2Mn \tan\frac{\pi}{n}$.

So $M = \frac{\pi}{n} Ł_n c\tan\frac{\pi}{n} \overset{Lm1.}{=} \frac{2\pi}{n}\omega_n$.

This completes the proof.

3. Proof of Theorems 2 and 3.

By equality (2.1) we observe that the n-width $\omega_n(s)$, corresponding to direction T_s is expressed as follows

$$\omega_n(s) = [p_n(s), T_s], \tag{3.1}$$

where $p_n(s) = \sum_{v=0}^{n-1} \varepsilon^{n-v} x(\varphi_n^v(s))$ and $x(s) = r_{f,k}(s)$.

LEMMA 2. *If $y \in [\{x\}]cQ$, then the Fourier series for $\omega_n(s)$ has the form*

$$\omega_n(s) = \frac{nM}{2\pi} - n\sum_{\mu=1}^{\infty}\left[\frac{A_{n\mu}}{n^2\mu^2-1}\cos n\mu K(s) + \frac{B_{n\mu}}{n^2\mu^2-1}\sin n\mu K(s)\right],$$

$n = 2, 3, \ldots$.

Proof. From (3.1) we obtain

$$\omega_n(s) = \sum_{v=0}^{n-1}\int_0^{\varphi_n^v(s)} k(t)f(t)\sin\left(K(s) + \frac{2\pi}{n}v - K(t)\right)dt.$$

Inserting series (1.8) and formulas (1.4) into the last formula, we obtain the Fourier series for $\omega_n(s)$ in the form:

$$\omega_n(s) = \frac{1}{2}a_0 + \sum_{j=2}^{\infty}(a_j \cos jK(s) + b_j \sin jK(s))$$

with coefficients $a_0 = \frac{nM}{\pi}$ and

$$a_j = \frac{-A_j}{j^2-1}\left[\sum_{v=0}^{n-1} \cos\frac{2\pi}{n}jv + \sum_{v=0}^{n-1} \sin\frac{2\pi}{n}jv\right] \tag{3.2}$$

and

$$b_j = \frac{B_j}{j^2-1}\left[\sum_{v=0}^{n-1} \sin\frac{2\pi}{n}jv - \sum_{v=0}^{n-1} \cos\frac{2\pi}{n}jv\right] \tag{3.3}$$

Hence $a_j = \dfrac{nA_j}{j^2-1}$ and $b_j = \dfrac{-nB_j}{j^2-1}$ whenever n divides j in the remaining cases we get $a_j = b_j = 0$. This completes the proof.

Let g_v $(v =, 2, \ldots)$ be a 2π-periodic continuous real values function such that its Fourier series has form

$$g_v(t) = \sum_{\mu=v}^{\infty}[a_\mu \cos\mu t + b_\mu \sin\mu t]. \tag{3.4}$$

A zero $t_0 \in [0, 2\pi)$ of g_v is called *simple* if we have either $t_0 = 0$ or $t_0 \in (0, 2\pi)$ and $g_v(t_0 - \varepsilon)g_v(t + \varepsilon)' < 0$ for all sufficiently small $\varepsilon > 0$; In [3] the following lemma is proved:

LEMMA B. *The function g_v has at least $2v$ simple zeros in the interval* $[0, 2\pi)$.

Proof. of Th 2. Now let $r_{f,k}$ be an arbitrary curve in $Q_0^{n,p}$. We consider the following function (by LM 2).

$$g_n(t) = \sum_{\mu=1}^{\infty}\left[\frac{A_{n\mu}}{n^2\mu^2-1}\cos\mu nt + \frac{B_{n\mu}}{n^2\mu^2-1}\sin\mu nt\right]$$

where $t = k(s) \in [0, 2\pi)$, $n = 2, 3, \ldots$. hence by Lm B the function $g_n(t)$ has at least $2n$ simple zeros in $[0, 2\pi)$, $n = 2, 3, \ldots$. This means that teh

n-width $\omega_n(s)$ has the value $\frac{nM}{2\pi}$ at least in $2n$ points (see LM 2.). But the n-Barbier set consists of n points in $[0, L)$. If $\omega_n(s) = \frac{nM}{2\pi}$, then $\omega_n(\varphi_n(s)) = \frac{nM}{2\pi}$, (by (1.6)). Thus there exist at least two different n-Barbier sets of points on $r_{f,k}$.

This means that $\beta_n(Q) \geq 2$. Let the curve $r_{f,k} \in Q$ be defined by the formula (1.1) with $k(s) \equiv 1$, $L = 2\pi$, $K(s) = s$ and $f(s) = 2 - \cos ns$.

Then

$$g_n(s) = \frac{n}{n^2 - 1} \cos ns.$$

The nonnegative roots of the equation $g_n(s) = 0$ are equal to

$$s_v = \frac{2v + 1}{n} \cdot \frac{\pi}{2}, \quad v = 0, 1, 2, \ldots.$$

it is easy to observe that $s_v \in [02\pi)$ if and only if $v = 0, 1, 2, \ldots 2n-1$; Next we observe that $\varphi_n^l(s_v) = s_{v+2l}$. Denoting the n-Barbier set by $B(s_v) = \{x(s_v), x(\varphi_n(s_v)), \ldots, x(\varphi_n^{n-1}(s_v)0\}$ we have

$$B(s_0) = B(s_2) = \ldots = B(s_{2j}); \; j = 0, 1, 2, \ldots$$

and

$$B(s_1) = B(s_3) = \ldots B(s_{2j+1}), \; j = 0, 1, 2, \ldots.$$

Hence there are exactly two n-Barbier set of points on the curve. Thus $\beta_n(Q) = 2$. This completes the proof.

Proof. of Th 3. Similarly considering the functions

$$g_{n,p}(t) = n \sum_{\mu=p}^{\infty} \cdot \left[\frac{A_{n\mu}}{n^2\mu^2 - 1} \cos \mu nt + \frac{B_{n\mu}}{n^2\mu^2 - 1} \sin \mu nt \right]$$

we deduce that $\beta_n(Q - n, p) \geq 2p$.

Next in order that to prove $\beta_n(Q_{n,p}) = 2p$ we examine zeros of the function $f(s) = 2 - \cos nps$, where $k(s) \equiv 1$, $L = 2\pi$. It is not hard to see that $s_v = \frac{2v + 1}{np} \cdot \frac{\pi}{2}$ $v = 0, 1, 2, \ldots, 2np - 1$ are simple zeros of $g_{n,p}$ belonging to $[0, 2\pi)$. Moreover $\varphi_n^1(s_v) = s_v + 21p$. Hence we obtain the following equalities:

$$B(s_0) = B(s_{2p}) = \dots B(s_{2jp}),\ j = 0, 1, 2, \dots$$

$$B(s_1) = B(s_{2p+1}) = \dots B(s_{2pj+1}),\ j = 0, 1, 2, \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$B(s_{2p-1}) = B(s_{4p-1}) = \dots B(s_{2pj+2p-1}),\ j = 0, 1, 2, \dots.$$

Finally $\beta_n(Q_{n,p}) = 2p$. This completes the proof.

Remark In the extreme case for $f \in C_L$, if $A_{nj} = B_{nj} = 0$, $j = 1, 2, \dots$, then for each $s \in [0, L)$ the formulas (1.5) and (1.6) determine are n-Barbier set of points on the curve $r_{f,k}$, (compare Th 1. of the paper).

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