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A MATHEMATICAL MODEL FOR RADIATION HYDRODYNAMICS

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We adopt here the idea of describing a radiation field by means of the radiation energy density E and the radiative flux vector $\mathbf{F}$ which must satisfy a set of evolution equations; in these equations an unknown tensorial function $\mathsf{P}(E, \mathbf{F})$ appears that is determined by the methods of extended thermodynamics.

1. Introduction.

Transport phenomena occur in many areas of physics, and in the last decades, have received an increasing attention. Typical examples of such phenomena are radiative transfer (relativistic astrophysics and cosmology are its areas of applicability), neutron transport and, for non-neutral particles, electron transport in plasmas or semiconductors.

The mathematical tool for dealing with the transport of particles of identical speed is the distribution funtion $\mathcal{F}(\mathbf{r}, \Omega, t)$, which gives the probability of finding a particle at time t in the position $\mathbf{r}$ and with propagation direction Ω, where Ω lies in the unit sphere S^2. The evolution of the distribution function is governed by the transfer

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equation which, for neutral particles, reads (see [12])

$$\frac{1}{v}\frac{\partial \mathcal{F}}{\partial t} + \boldsymbol{\Omega} \cdot \nabla \mathcal{F} = v\frac{\gamma}{4\pi}E - \sigma \mathcal{F} \tag{1}$$

where v is the speed of the particles, σ is the total interaction coefficient expressing the rate at which particles at the position r are removed from the beam with direction Ω, γ is the effective albedo here supposed to depend only on position and time, while E is the surface integral of $\mathcal{F}$ on the unit sphere

$$E = \frac{1}{v}\int_{4\pi} \mathcal{F}(\mathbf{r}, \Omega, t)d\Omega. \tag{2}$$

Notice that in eq (1) we have not taken into account the dependence of the distribution function on the modulus of the momentum p of the particle. This means that eq (1) is the transfer equation for particles with a given $|p|$.

In eq (1) it is implicit that no external forces (gravitational or other) can influence our particles. Only scattering (here supposed coherent and isotropic) and absorption by the underlying medium can change the propagation direction of our particles. Equation (1) neglects also the possibility of particle - particle collision. For photons this gives no restriction, the superposition principle being valid, but for neutrons (or any other fermions) this amounts to considering low densities with null probability of neutron-neutron scattering.

Usually in radiative transfer treatments the transport equation is written for the radiation intensity I, which differs from the proper distribution function by a factor $\frac{c^2}{2h^4\nu^3}$ (see [12]), where c is the speed of light, h is the Plank constant, and v is the frequency of the photons we are considering. Because of the linearity of (1) this makes no difference, but when written in terms of radiation intensity the quantity E, as defined by (2), is the energy density, while the vector

$$\mathbf{F}(\mathbf{r}, t) = \int_{\pi} \mathcal{F}(\mathbf{r}, \Omega, t)\, \Omega d\Omega \tag{3}$$

is the energy flux. In this paper we are mainly interested in radiative transfer; then henceforth we shall call E and $\mathbf{F}$, energy density and energy flux respectively.

Equation (1) has been the object of many theoretical enquiries. In particular existence and uniqueness questions for the solution of eq (1) with prescribed boundary and initial conditions (see for example [5], [7], [8]) have been studied. But in many instances, particularly when (1) must be coupled with the equations governing the underlying medium, solving the transfer equation can be too difficult.

In these cases an approximation procedure is needed. A common method to circumvent this difficulty is the so called moment method: taking the moments of equation (1) one obtains an infinite set of equations for the moments of distribution function

$$M^{i_1,\dots,i_k} \equiv \int_{4\pi} \mathcal{F}\Omega^{i_1} \dots \Omega^{i_k} d\Omega$$

In order to make this procedure operationally useful one needs a relation linking the $(n+1)$-th moment to the lower ones; the question of how to postulate such a relation defines the so-called closure problem. The system of the first m moments of the transfer equation, together with the closure form a closed set of equations, which is usually simpler to solve than the original transfer equation (1) , because the moments of the distribution function depend only on space and time.

In this paper we shall seek a closure at the second order, that is we impose the condition that the second moment of the distribution function, the stress tensor

$$\mathsf{P} = \frac{1}{v}\int_{4\pi} \mathcal{F}\Omega\,\Omega d\Omega, \tag{4}$$

depends only on the zeroth and first moments , i.e. (2) and (3) respectively. Under that hypothesis, the most general expression for P obeying the objectivity principle (see [13], [16], [17]) is

$$P^{ij} = \frac{1}{3}E\delta^{ij} + q(F^i F^j - \frac{1}{3}F^2\delta^{ij}) \tag{5}$$

where also the fact has been used that the trace P^i_i is equal to E. q is a scalar function depending on E and $F = |\mathbf{F}|$.

In radiative transfer a very commonly used (see [9]) closure is

$$P^{ij} = E\left(\frac{1-\chi}{2}\delta^{ij} + \frac{3\chi-1}{2}\frac{F^iF^j}{F^2}\right) \tag{6}$$

where χ, a scalar function depending on E and F, is called variable Eddington factor. The meaning of χ is apparent , if one considers the one dimensinal version of (6), viz.

$$P = \chi E. \tag{7}$$

In the collision dominated regime the radiation field is isotropic and then one has $P = \frac{1}{3}E$. In the opposite regime, the free streaming case (see [9]) all pressure is concentrated in one direction and one has $P = E$. Then the variable Eddington factor plays the role of an interpolating factor between the two opposite regimes.

Notice that the two closures (5) and (6) are equivalent, and the two unknown functions are related by $q = EF^{-2}\,(3\chi-1)/2$; moreover the problem of finding a closure at the second order is reduced to finding a variable Eddington factor. Its expression is usually introduced in a phenomenological or ad hoc manner, while here we shall find it, except for a constant h, by requiring that the system of the equations of moments (see eqs (8) and (9) below), together with the closure (5) (or (6)), admit a supplementary conservation law.

This requirement is based on the desire to have a symmetric quasilinear system of partial differential equations and it will guarantee the existence of an entropy . The exploitation of the entropy balance will allow us to find a whole class of closures (6)

In particular we shall find that the only Eddington factors consistent with the existence of an entropy principle are those satisfying an ordinary differential equation (see eq (22) below).

For a comprehensive review of the variable Eddington factors already introduced in the literature see ref. [9]. We list one, viz.

$$\chi = \frac{5}{3} - \frac{2}{3}\sqrt{4 - 3F^2E^{-2}}$$

which is appropriate to the case when there exists a preferred reference frame in which the radiation is isotropic.

This is a particular case in this paper, when the above mentioned constant h is $-5/4$, and has been found by Levermore [9] and by Anile, Pennisi and Sammartino [2] although in different contexts.

The paper is organized as follows. In Sec. II we shall state the basic equations of our treatment and some other constraints. In Sec III we shall impose an entropy principle, and find as final result an ordinary differential equation for χ depending on a real parameter, whose solution will give the only closure consistent with such entropy principle.

In Sect. IV. we shall find other conditions on χ that will guarantee the hyperbolicity of our system of equations; this result will be obtained as a consequence of the convexity of entropy in some cases, or it will be proved directly in the remaining ones.

In Sect. V we shall state the existence and uniqueness for the solution χ of the aforesaid differential equation and of all the other related conditions.

In Sect VI we draw some conclusions.

2. Basic equations and constraints.

Let us now consider eq. (1) and take its zeroth moment integrating over Ω. We obtain:

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = v(\gamma - \sigma)E \tag{8}$$

Multiplying (1) by Ω and then integrating, we obtain the first order moment equation

$$\frac{1}{v}\frac{\partial \mathbf{F}}{\partial t} + v\,\nabla \cdot \mathsf{P} = -\sigma\mathbf{F} \tag{9}$$

The system (8), (9) together with the closure (6) forms a closed set of equations once χ is given as a function of E and F.

It is commonly assumed that χ depends on F and E through their ratio $f = \dfrac{F}{vE}$ (see [9]). Here we shall do so.

From the definition of F and E, we have that $0 \leq F \leq Ev$, so that $0 \leq f \leq 1$, Moreover the two limiting values, 0 and 1, are assumed by f in the two opposite regimes, the collision-dominated and the collisionless regime respectively .

Then we are interested in a function $\chi(f)$, defined in [0,1], such that $\chi(0) = \frac{1}{3}$ and $\chi(1) = 1$; that function will be found, except for a constant h, in the next section.

3. Entropy principle and determination of $\chi(f)$.

Now we impose the condition that the solution of (8) and (9) , with p^{ij} given by (5) , satisfies the relation

$$\partial_t S + \partial_i Q^i = g \tag{10}$$

where S, Q^i and g are functions of the field variables E and F^i.

From a physical point of view the existence of such a supplementary conservation law is very natural; for an exhaustive treatment on entropy principle see [1] and [15].

In [10] it is shown that the existence of an entropy principle for solutions of (8) and (9) is equivalent to the existence of four functions λ, λ_i called Lagrange multipliers, depending on E and F and such that one has identically

$$\partial_t S + \partial_i Q^i - g + \lambda[\partial_t E + \partial_i F^i - v(\gamma - \sigma E)] + \lambda_i(\partial_t F^i + v^2 \partial_j P^{ij} + v\sigma F^i) = 0 \tag{11}$$

By the representation theorems of [13], [16], [17] we have

$$\lambda_k = LF_k, \; Q_i = \alpha F_i \tag{12}$$

where L and α are scalar functions depending on E and F. We use the closure (5) and obtain

$$\begin{gathered}\frac{\partial S}{\partial E} + \lambda = 0, \; \frac{\partial S}{\partial F} + LF = 0, \; \frac{\partial \alpha}{\partial E} + Lv^2\left(\frac{1}{3} + \frac{2}{3}F^2\frac{\partial q}{\partial E}\right) = 0 \\ \frac{\partial \alpha}{\partial F} + \frac{1}{3}Lv^2Fq + \frac{2}{3}F^2L\frac{\partial q}{\partial F}v^2 = 0, \; \alpha + \lambda + Lv^2F^2q = 0, \\ g = \lambda v(\sigma E - \gamma) + v\sigma\lambda_i F^i.\end{gathered} \tag{13}$$

The integrability condition implied by equations $(13)_1$ and $(13)_2$ reads

$$\frac{\partial \lambda}{\partial F} - F\frac{\partial L}{\partial E} = 0; \tag{14}$$

if one solves eq. $(13)_5$ for α, substitutes in eqs. $(13)_3$ and $(13)_4$, one obtains the equations

$$\frac{\partial \lambda}{\partial E} = \frac{1}{3}Lv^2\left(1 - F^2\frac{\partial qq}{\partial E}\right) - qv^2F^2\frac{\partial L}{\partial E} \tag{15}$$

$$\frac{\partial \lambda}{\partial F} = -\frac{1}{3}Lv^2F^2\frac{\partial q}{\partial F} - qv^2F^2\frac{\partial L}{\partial F} - \frac{5}{3}qLv^2F.$$

Using $(15)_2$ eq (14) becomes:

$$\frac{\partial(\log L)}{\partial F}qv^2F + \frac{\partial(\log L)}{\partial E} + \frac{1}{3}v^2\left(F\frac{\partial q}{\partial F} + 5q\right) = 0, \tag{16}$$

while the integrability condition implied by $(15)_1$ and $(15)_2$ reads

$$\frac{\partial(\log L)}{\partial F}\left(1 + 2F^2\frac{\partial q}{\partial E}\right) - \frac{\partial(\log L)}{\partial E}\left(2F\frac{\partial q}{\partial F} + q\right)F + \tag{17}$$

$$+3F\frac{\partial q}{\partial E} = 0$$

Using eqs (21) and (22) and defining

$$\Delta = 1 + 2F^2\frac{\partial q}{\partial E} + qv^2F^2\left(q + 2F\frac{\partial q}{\partial F}\right)$$

one obtains

$$\frac{\partial(\log L)}{\partial F} = -\frac{1}{3}F\left[9\frac{\partial q}{\partial E} + v^2\left(5q + F\frac{\partial q}{\partial F}\right)\left(q + 2F\frac{\partial q}{\partial F}\right)\right]\Delta^{-1} \tag{18}$$

$$\frac{\partial(\log L)}{\partial E} = -\frac{1}{3}v^2\left[F^2\frac{\partial q}{\partial E}\left(q + 2F\frac{\partial q}{\partial F}\right)\left(+5q + F\frac{\partial q}{\partial F}\right)\right]\Delta^{-1}$$

Now we change our variables and use, instead of F and E, the variables, E and f; moreover we define the function ϕ as $\phi = qv^2Ef^2$.

Notice that, in order to have the equivalence of the two closures (5) and (6), we shall suppose ϕ and χ linked by the linear relation $\phi = \frac{3\chi - 1}{2}$. As we have already said (in sect. II) it is usually assumed in radiation hydrodynamics that the variable Eddington factor depends only on f and henceforth we shall do so (i.e. $\phi = \phi(f)$).

Using the new variables eqs (18) become

$$(19) \qquad \begin{aligned} \frac{\partial(\log L)}{\partial f} &= -\frac{1}{3f\Delta_1}\{2f^2(\phi')^2 + 3f\phi'(\phi - 3f^2) + 9\phi(f^2 - \phi\} \\ \frac{\partial(\log L)}{\partial E} &= -\frac{1}{3\Delta_1}\{f\phi'(8\phi + 1 - 9f^2) - 12\phi + 3\phi(1 + 3f^2)\} \cdot \frac{1}{E} \end{aligned}$$

Where $' = \frac{d}{df}$, and $\Delta_1 = f^2\Delta = 2f(\phi - f^2)\phi' - 3\phi^2 + 2f^2\phi + f^2$.

Defining

$$(20) \qquad \psi(f) = \frac{f\phi'(8\phi + 1 - 9f^2) - 12\phi^2 + 3\phi(1 + 3f^2)}{-3\Delta_1}$$

integrating the eq. $(19)_2$, and inserting the solution into $(19)_1$ one readily finds

$$(21) \qquad \psi(f) = h$$

where h is a constant. Eq. (20) becomes now

$$(22) \quad \phi' f[2(3h+4)\phi - 3(2h+3)f^2 + 1] - 3(3h+4)\phi^2 + 3[1 + (2h+3)f^2]\phi + 3hf^2 = 0.$$

this is the main result of this paper. By (22) we have obtained an ordinary differential equation for ϕ, hence for χ, depending on the parameter h; in section V it will be shown that with the boundary condition, $\chi(0) = 1/3$, $\chi(1) = 1$, eq. (20) admits a unique solution for all $h \in \mathbb{R}$.

Now using $(19)_2$ and (21) we have

$$(23) \qquad L = \beta(f)E^h$$

where $\beta(f)$ is the solution of the ordinary differential equation

$$(\log \beta)' = -\frac{1}{3f\Delta_1}[2f^2(\phi')^2 + 3f\phi'(\phi - 3f^2) + 9\phi(f^2 - \phi)];$$

this expression is equivalent to

$$(\log \beta)' = \frac{-3(h+1)\phi + (2h+3) f\phi'}{f(1-\phi)}, \qquad (24)$$

because , multiplying (22) by $(2\phi' f - 3\phi)$ we obtain

$$[2f^2(\phi')^2 + 3f\phi'(\phi - 3f^2) + 9\phi(f^2 - \phi)]f(1-\phi) + \\ +3f\Delta_1[-3(h+1)\phi + (2h+3)\phi' f] = 0$$

It will be useful to notice also that, if $h = -1$ eq. (24) gives $\beta = \bar{\beta}(1-\phi)^{-1}$; if $h = -2$ it gives $\beta = \bar{\beta}(1 + 2\phi - 3f^2)^{-1}$; in both cases $\bar{\beta}$ is a constant.

Inserting the expression (23) for L in eqs. (19), they become

$$\begin{aligned} \frac{\partial \lambda}{\partial f} &= -\frac{1}{3} E^{h+1} v^2 f^{-1} [\beta(3\phi + f\phi') + 3\phi f \beta'] \\ \frac{\partial \lambda}{\partial E} &= \frac{1}{3} v^2 E^h \beta (1 - 4\phi - 3\phi h) \end{aligned} \qquad (25)$$

These two eqs. give for λ

$$\lambda = \begin{cases} \dfrac{1}{3}[1 - (3h+4)\phi]\beta v^2 \dfrac{E^{h+1}}{h+1} + \lambda_0 & \text{if } h \neq -1 \\ \dfrac{1}{3} v^2 \bar{\beta} lg E - \bar{\beta} v^2 \displaystyle\int \frac{f}{1-\phi}\left(1 + \frac{\phi' f}{1-\phi}\right) df + \lambda_0 & \text{if } h = -1 \end{cases} \qquad (26)$$

where λ_0 is a constant. Using (26) and (23) in $(13)_{1,2}$ we obtain two equations for S which give after integration

(27)

$$S = \begin{cases} -\dfrac{1}{3}\beta v^2 \dfrac{E^{h+2}}{(h+1)(h+2)}[1 - (3h+4)\phi + 3(h+1)f^2] - \lambda_0 E + \delta \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } -1 \neq h \neq -2 \\ -\dfrac{1}{3}\bar{\beta} v^2 E(\log E - 1) + \left[\dfrac{-\bar{\beta} v^2 f^2)}{1-\phi} + \right. \\ \left. + \bar{\beta} v^2 \displaystyle\int \frac{f}{1-\phi}\left(1 + \frac{\phi' f}{1-\phi}\right) df + \lambda_0\right] E + \delta \qquad \text{if } h = -1 \\ \dfrac{1}{3}\bar{\beta} v^2 lg E - \lambda_0 E - \bar{\beta} v^2 \displaystyle\int f(1 + 2\phi - 3f^2)^{-1} df + \delta \qquad \text{if } h = 2 \end{cases}$$

where δ is a constant.

Finally $(13)_5$ gives

$$\alpha = \begin{cases} \frac{1}{3}\beta v^2 \frac{E^{h+1}}{h+1}(\phi - 1) - \lambda_0 & \text{if } h \neq -1 \\ -\frac{\bar{\beta} v^2 \phi}{1-\phi} + \bar{\beta} v^2 \int \frac{f}{1-\phi}\left(1 + \frac{\phi' f}{1-\phi}\right) df - \frac{1}{3}\bar{\beta} v^2 \log E - \lambda_0 & \text{if } h = -1 \end{cases} \tag{28}$$

In conclusion we have found that the only closures (5) ,or (6) which are consistent with an entropy principle are those given by eq. (22) . Before integrating eq (22) one must of course determine the parameter h. How to choose such an h depends on the particular transport phenomenon we are dealing with. As equation (27) shows, h determines the dependence of S on E. In [2] the case $h = -5/4$ has been examined.

In Sect V we shall prove existence and uniqueness for the solution of eq. (22) with the boundary values conditions $\phi(0) = 0$; $\phi(1) = 1$ that correspond to $\chi(0) = 1/3$; $\chi(1) = 1$. Thus $\chi(f)$ will be determined except for the constant h.

But before that, we shall now exploit the condition of hyperbolicity of the system (8) and (9). This will be considered in the next section.

4. Hyperbolicity of the field equations.

In order to obtain the hyperbolicity of our the system (8), (9) we apply a powerful and elegant theory which has been widely explained and successfully applied in [4], [11], [14], [15].

To this end we recall that a sufficient condition for the hyperbolicity, is the convexity of the entropy S, i.e. that

$$\frac{\partial^2 S}{\partial X^A \partial X^B} \text{ is positive definite} \tag{29}$$

where

$$X^A = \begin{cases} E & \text{for } A = 0 \\ F^A & \text{for } A = 1, 2, 3 \end{cases};$$

moreover let

$$Y_A = \begin{cases} -\lambda & \text{for } A = 0 \\ -\lambda_A & \text{for } A = 1, 2, 3. \end{cases}$$

Infact, from $(13)_1$ $(13)_2$, we have that

$$Y_A = \frac{\partial S}{\partial X^A} \tag{30}$$

and consequently (29) ensures the invertibility of the function $Y_A = Y_A(X^B)$ so that Y_A can be taken as independent variables.

In the same way, defining $Q^{i0} = F^i$, $Q^{ij} = v^2 p^{ij}$, $\bar{Q}^i = -Q^i + Y_B Q^{iB}$, $\bar{S} = -S + Y_A X^A$ we have that $(13)_3$, $(13)_4$ can be written as

$$\frac{\partial Q^i}{\partial X_c} - Y_B \frac{\partial Q^{iB}}{\partial X_c} = 0 \tag{31}$$

consequently we have

$$\frac{\partial \bar{S}}{\partial Y_A} = -\frac{\partial S}{\partial Y_A} + X^A + Y_B \frac{\partial X^B}{\partial Y_A} = -\frac{\partial S}{\partial Y_A} + X^A + \tag{32}$$

$$+ \frac{\partial S}{\partial X^B}\frac{\partial X^B}{\partial Y_A} = X^A$$

$$\frac{\partial \bar{Q}^i}{\partial Y_A} = -\frac{\partial Q^i}{\partial Y_A} + Q^{iA} + Y_B \frac{\partial Q^{iB}}{\partial Y_A} = Q^{iA} -$$

$$-\left(\frac{\partial Q^i}{\partial X_c} - Y_B \frac{\partial Q^{iB}}{\partial X_c}\right)\frac{\partial X^c}{\partial Y_A} = Q^{iA},$$

where (30) and (31) have been taken into account. By using these results, the field equations (8), (9), i.e. $\dfrac{\partial X^A}{\partial t} + \partial_i Q^{iA} + P^A = 0$, may be written as

$$\frac{\partial^2 \bar{S}}{\partial Y_A \partial Y_B}\frac{\partial Y_B}{\partial t} + \frac{\partial^2 \bar{Q}^i}{\partial Y_A \partial Y_B}\partial_i Y_B + P^A = 0 \tag{33}$$

Moreover, by using (32) we have that

$$\frac{\partial^2 \bar{S}}{\partial Y_A \partial Y_B}\delta Y_A \delta Y_B = \delta\left(\frac{\partial \bar{S}}{\partial Y_A}\right)\delta Y_A = \delta X^A \delta Y_A = \delta X^A \delta\left(\frac{\partial S}{\partial X^A}\right) =$$

$$= \frac{\partial^2 S}{\partial X^A \partial X^B}\partial X^B \delta X^A$$

and then $\dfrac{\partial^2 \bar{S}}{\partial Y_A \partial Y_B}$ is positive definite if and only if (29) is verified.

Thus, by imposing the convexity condition (29) we obtain that our system can be put in the symmetric hyperbolic form (33).

Moreover the well - posedness of the Cauchy problem for smooth initial data is guaranteed (see. [6]).

Let us then impose the condition (29); we have

$$Q = \frac{\partial^2 S}{\partial X_A \partial X_A} \delta X_A \delta X_B = \delta \left(\frac{\partial S}{\partial X_A} \right) \delta X_A = \delta \left(\frac{\partial^2 S}{\partial E} \right) \delta E + \\ + \delta \left(\frac{\partial^2 S}{\partial F_k} \right) \delta F_k;$$

but $S(E, F^k) = \tilde{S}(E, f(E, F^k))$, where $f(E, F^k) = v^{-1} E^{-1} (F^k F_k)^{1/2}$ and $F^k = vfEn^k$; then we have (by using $n_k \delta n^k = 0$)

$$Q = \delta \left(\frac{\partial \tilde{S}}{\partial E} - \frac{\partial \tilde{S}}{\partial f} E^{-1} f \right) \delta E + \delta \left(\frac{\partial \tilde{S}}{\partial f} v^{-1} E^{-1} n^k \right) \delta (vfEn_k) = \\ = \delta \left(\frac{\partial \tilde{S}}{\partial E} - \frac{\partial \tilde{S}}{\partial f} E^{-1} f \right) \delta E + \delta \left(\frac{\partial \tilde{S}}{\partial f} E^{-1} \right) \delta (fE) + \\ + \frac{\partial \tilde{S}}{\partial f} f \delta n^k \delta n_k = \frac{\partial^2 \tilde{S}}{\partial E^2} (\delta E)^2 + 2 \left(\frac{\partial^2 \tilde{S}}{\partial E \partial f} - \\ \frac{\partial \tilde{S}}{\partial f} E^{-1} \right) \delta E \delta f + \frac{\partial^2 \tilde{S}}{\partial f^2} (\delta f)^2 + f \frac{\partial \tilde{S}}{\partial f} \delta n^k \delta n_k .$$

By using (27), (24) and (22) we obtain

$$\frac{\partial \tilde{S}}{\partial f} = -\beta f E^{h+2} v^2 ; \frac{\partial^2 \tilde{S}}{\partial E^2} = -\frac{v^2}{3} \beta E^h [1 - (3h+4)\phi + 3(h+1) f^2];$$

$$\frac{\partial^2 \tilde{S}}{\partial E \partial f} = -(h+2) \beta f E^{h+1} v^2$$

which are also true for $h = -1$ and $h = -2$.

Consequently

$$\frac{\partial^2 \tilde{S}}{\partial f^2} = v^2 E^{h+2} \frac{\beta}{1-\phi} [-(2h+3)\phi' f + (3h+4)\phi - 1];$$

$$M = \begin{vmatrix} \dfrac{\partial^2 \tilde{S}}{\partial E^2} & \dfrac{\partial^2 \tilde{S}}{\partial E \partial f} - \dfrac{\partial \tilde{S}}{\partial f} E^{-1} \\ \dfrac{\partial^2 \tilde{S}}{\partial E \partial f} - \dfrac{\partial \tilde{S}}{\partial f} E^{-1} & \dfrac{\partial^2 \tilde{S}}{\partial f^2} \end{vmatrix} = \tag{34}$$

$$= -\frac{1}{3} \frac{\beta^2 v^4 E^{2h+2}}{1-\phi} \{[(3h+4)\phi - 3(h+1)f^2 - 1] \cdot$$

$$\cdot [(2h+3)\phi' f - (3h+4)\phi + 1] + 3(h+1)^2(1-\phi)f^2\}$$

For $2(3h+4)\phi + 1 - 3(2h+3)f^2 \neq 0$ this relation becomes equal to

$$M = \frac{1}{3}\beta^2 E^{2h+2} v^4 [2(3h+4)\phi - 3(2h+3)f^2 + 1]^{-1} \{[(3h+4)\phi - 1]^2 - \\ - 9(h+1)^2 f^2\}. \tag{35}$$

Now Q is positive definite if and only if $\dfrac{\partial^2 \tilde{S}}{\partial E^2} > 0$, M is positive and $\frac{\partial \tilde{S}}{\partial f} > 0$ and then the following conditions must be imposed

(36) a) $1 - (3h+4)\phi + 3(h+1)f^2 > 0$
b) (34) is positive

[$\beta < 0$ should also hold , but from (24) it can be seen that this amounts to choosing a negative initial value for β and does not effect the other considerations.]

Clearly (36) is verified in $(f, \phi) = (0, 0)$; therefore it must be exploited only for $f > 0$.

Let us now distinguish four cases

case i) $-2 \leq h \leq -1$;

case ii) $h > -1$ and $\phi'\left(\dfrac{1}{2h+3}\right) \neq -3\dfrac{h+1}{3h+4}$;

case iii) $h > -1$ and $\phi'\left(\dfrac{1}{2h+3}\right) = -3\dfrac{h+1}{3h+4}$;

case iiii) $h < -2$.

In the next section we shall prove existence and uniqueness of the solution of eq. (22) with $\phi(0) = 0$; $\phi(1) = 1$. We shall also prove that in the cases i) and ii) this solution satisfies the conditions (36) for $0 \leq f < 1$, while in the case iii) this condition is not verified if and only if $f = \dfrac{1}{2h+3}$ and in the case iiii) it is not verified if and only if

$\frac{1}{2h+3} \leq f < 1$. Therefore in these last two cases the function (30) is not invertible for these values of f and consequently we cannot take the Lagrange multipliers as variables . Neverthless we shall prove in Appendix A_1 that also in these cases the system (8),(9) is hyperbolic .

5. Existence and uniqueness of the solution of eq. (22) with $\phi(0) = 0; \phi(1) = 1$.

Let $\phi(f) = y[x(f)]$ with $x(f) = f^2$; then eq. (22) becomes

$$y' = \frac{3}{2} \frac{(3h+4)y^2 - [1+(2h+3)x]y - hx}{x[2(3h+4)y - 3(2h+3)x+1]} = \frac{G(x,y)}{H(x,y)} \tag{37}$$

where

$$y' = \frac{dy}{dx}$$

$$H(x,y) = 2x[2(3h+4)y - 3(2h+3)x + 1]$$

and $\phi(0) = 0$, $\phi(1) = 1$ become

$$y(0) = 0; y(1) = 1. \tag{38}$$

The simplest case is $h = -4/3$ because for such value of h eq. (5.1) is a linear differential equation whose solution is

$$y = \frac{3}{4}(x-1)^2 x^{-3/2} \left[\lg \left(\frac{x^{1/2}+1}{x^{1/2}-1} \right) + \frac{4}{3}c \right] - \frac{3}{2}x^{-1} + \frac{5}{2}$$

Where c is a constant of integration; moreover we have

$$\lim_{x \to 0} y(x) = 0 \Leftrightarrow c = 0$$

and

$$\lim_{x \to 1} y(x) = 1,$$

so that in this case the existence and uniqueness of the boundary value problem (8), (9) is proved ; furthermore the condition (36) is valid except for $x = 1$; in that case (34) is equal to (35) because $2(3h+4)y + 1 - 3(2h+3)x \neq 0$. Thus the convexity of entropy is assured. In the remaining cases we now have $h \neq -4/3$.

The solutions of $G(x,y) = H(x,y) = 0$ are the critical points $(0,0)$;

$$(1.1);\ \left(0, \frac{1}{3h+4}\right)\ \left(\frac{1}{(2h+3)^2}, \frac{-h}{(2h+3)(3h+4)}\right) \equiv Q$$

Where obviously Q must not be considered if $h = -3/2$. There is one and only one solution of (37) or of its inverse $\frac{dx}{dy} = \frac{H(x,y)}{G(x,y)}$ that reach every given point P distinct from the critical ones.

We can also see that the functions

$$(39) \qquad y = (3h+4)^{-1}[1 + 3(h+1)x^{1/2}];\ y = (3h+4)^{-1}[1 - 3(h+1)x^{1/2}]$$

are two solutions of (37); they do not satisfy the condition (38) but are useful, because they cannot be crossed by other solutions of (37) outside the critical points.

Every solution of (38) outside these critical points may be obtained eliminating the parameter μ between two functions $y = y(\mu)$; $x = x(\mu)$ satisfying

$$(40) \qquad \begin{cases} \dfrac{dy}{d\mu} = G(x,y) \\ \dfrac{dx}{d\mu} = H(x,y) \end{cases}$$

Now if we consider the linear terms of G, H in x, y, we can see that the critical point $(0,0)$ is a saddle point for the system (40) and therefore (see ref [3]) there are four and only four solutions of (40) coming from or arising in $(0,0)$; two of them are

$$x = 0; \mu - \mu_0 = \frac{1}{3} \lg\{[1 - (3h+4)y]y^{-1}\} \Rightarrow \lim_{y \to 0+} \mu = +\infty$$

and

$$x = 0; \mu - \mu_0 = \frac{1}{3} \lg\{-[1 - (3h+4)y]y^{-1}\} \Rightarrow \lim_{y \to 0-} \mu = +\infty$$

Consequently our system (40) has two and only two other solutions reaching $(0,0)$ of which one lies in the half plane $x < 0$ (and therefore must not be considered) and the other in the half plane $x > 0$. We have then to restrict ourselves to this solution ; if it reaches also $(1,1)$ then the existence of solution of (37), (38) is proved; uniqueness will follow as consequence of what we have just said.

It will be also useful henceforth to know the values that may be assumed by $y'(x)$ in $(0,0)$, $(1,1)$ and Q; to this end we can apply the l'Hopital theorem to the second member of (37) in these critical points obtaining in this way a second degree algebraic equation for the unknown $y'(x)$ whose solutions are

$$y'\left(\frac{1}{(2h+3)^2}\right) = -\frac{3}{2}h\frac{2h+3}{3h+4}$$

or

$$y'\left(\frac{1}{(2h+3)^2}\right) = -\frac{3}{2}(h+1)\frac{2h+3}{3h+4}$$

$$y'(1) = \frac{3}{2} \text{ or } y'(1) = \frac{3}{2}\frac{h+1}{3h+4}$$

$$y'(0) \to \infty \text{ or } y'(0) = -\frac{3}{5}h$$

where this last one is obviously the value assumed on the solution S while $y'(0) \to \infty$ refers to the solutions with $x = 0$.

By using all these properties it can be proved existence and uniqueness of the solution of eq. (22) with $\phi(0) = 0$, $\phi(1) = 1$; moreover it can be proved that the condition (36) holds when $-2 \leq h \leq -1$; or $h > -1$ and $\phi'\left(\frac{1}{2h+3}\right) \neq -3\frac{h+1}{3h+4}$ or $h > -1$, $\phi'\left(\frac{1}{2h+3}\right) = -3\frac{h+1}{3h+4}$ and $f \neq \frac{1}{2h+3}$; or $h < -2$ and $0 \leq f < -\frac{1}{2h+3}$;

(This fact and the results of Appendix 1, assure the hyperbolicity of the equations (8) and (9)).

The effective proof will be exposed in Appendix 2, where the following four cases will be distinguished

case 1: $h < -2$; case 2: $-2 \leq h < -4/3$; case 3: $-4/3 < h \leq -1$; case 4: $-1 < h$.

(The case $h = -4/3$ has already been treated in this section).

6. Conclusions.

We are satisfied to see how a powerful methodology such that explained in ref. [4], [11], [14], [15] always find a greater field of applicability. In this paper it has furnished a closure for the system (8), (9) as given by (5) with $q = \phi(f)v^{-2}f^{-2}E^{-1}$ and $\phi(f)$ determined except for a constant h by the differential equation (22) and the boundary values $\phi(0) = 0$; $\phi(1) = 1$.

If $h \in [-2, -1]$ this closure allows the system (8), (9) to be put in the symmetric hyperbolic form (33) with $\bar{S}$ a convex function of its variables; hyperbolicity and well - posedness of the Cauchy problem for smooth initial data is then guranteed ([6]).

The same thing can be said if $h \in]-1, +\infty[$ but

$$\phi'\left(\frac{1}{2h+3}\right) \neq -3\frac{h+1}{3h+4}.$$

In the other case, i.e., $h \in]-\infty, -2[$ or $h \in]-1+\infty[$ but with

$$\phi'\left(\frac{1}{2h+3}\right) = -3\frac{h+1}{3h+4},$$

we have that the hyperbolicity of our system (8), (9) still holds and moreover the characteristic velocities do not exceed the speed of the particles v that is not greater than that of light.

Therefore we consider these results as greatly improved over those already known in the literature regarding the variable Eddington factors.

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Appendix 1 - Hyperbolicity of equations (8),(9).

We have already proved in sect. IV that in the cases i) and ii) the system (8),(9) is hyperbolic ; the same thing we have proved in the case iii) except for $f = \frac{1}{2h+3}$ and in the case iiii) except for $-\frac{1}{2h+3} \leq f \leq 1$. We prove now that also in these cases the hyperbolicity requirement holds.

Infact, if we call $F^i = vEf^i$ and substitute $\frac{\partial E}{\partial t}$ from (8) in (9) this system assumes the form

$$A^{0AB}\partial_t U_B + A^{iAB}\partial_i U_B + G^A = 0$$

where

$$U_B = (E, f_k)^T,\ G^A = (vE(-\gamma+\sigma),\ v\gamma f^j)^T,$$

$$(A.1) \qquad A^{0AB} = \begin{pmatrix} 1 & 0^k \\ 0^j & \delta^{jk} \end{pmatrix};$$

$$A^{iAB} = v \begin{pmatrix} fn^i & E\delta^{ik} \\ q_1\delta^{ij} + q_2 n^i n^j & \begin{matrix} p_1 n^i n^j n^k + p_2 \delta^{ij} n^k + \\ + p_3 \delta^{ik} n^j + p_4 \delta^{jk} n^i \end{matrix} \end{pmatrix}$$

Let us consider firstly the case iii); here hyperbolicity remains to be proved only for $f = \frac{1}{2h+3}$, $\phi'\left(\frac{1}{2h+3}\right) = -3\frac{h+1}{3h+4}$ and consequently $\phi = \frac{-h}{((2h+3)(3h+4)}$,

$$q_1 = 2\frac{(h+1)(h+2)}{(2h+3)(3h+4)}E^{-1};$$

$$q_2 = -2\frac{(h+1)((h+2)}{(2h+3)^2(3h+4)}E^{-1};$$

$$p_1 = -\frac{h+3}{3h+4};$$

$$p_2 = \frac{h+1}{3h+4};$$

$$p_3 = -2\frac{(h+1)(h+2)}{3h+4} \cdot \frac{1}{2h+3};$$

$$P_4 = -\frac{h}{3h+4}.$$

Consequently the hyperbolicity requirement holds , if the eigenvalue problem

$$(\varepsilon_i A^{iAB} - \lambda A^{0AB})V_B = 0$$

admits real eigenvalues and 4 linearly independent eigenvectors for every unit vector ε_i.

In the reference Σ where $n^i \equiv (1,0,0)$; $\varepsilon^i \equiv (\varepsilon^1, \varepsilon^2, 0)$ the characteristic equation is

$$\left(\lambda + \frac{h}{3h+4}\varepsilon^1 v\right)\left\{\lambda^3 + \lambda^2 \frac{3h+2}{3h+4}\varepsilon^1 v^2 - \right.$$

$$-\lambda\left[(\varepsilon^1)^2 \frac{h^2+8h+8}{(3h+4)^2} + 2(\varepsilon^2)^2 \frac{(h+1)(h+2)}{(3h+4)^2}\right] v^3 +$$

$$\left.(\varepsilon^1)^3 \cdot v^3 \frac{h(h+2)}{(3h+4)^2} + 2(\varepsilon^2)^2 v^1 \cdot v^3 \frac{(h+1)(h+2)}{(3h+4)^2}\right\} = 0,$$

whose solutions are real and given by

$$\lambda_1 = -\frac{h}{3h+4}\varepsilon^1 v;$$

$$\lambda_2 = -\varepsilon^1 v$$

$$\lambda_{3,4} = v\frac{1}{3h+4}[(\varepsilon^1)^2 \pm \sqrt{(\varepsilon^1)^2(h+1)^2 + 2(\varepsilon^2)^2(h+1)(h+2)}];$$

if $\varepsilon^1\varepsilon^2 \neq 0$ these eigenvalues are all distinct and then the hyperbolicity requirement holds;

if $\varepsilon^1 = 0$ only the first two of them are coincident but there are two independent corresponding eigenvectors

$$((2h+3), -2(h+2), 0, 0); (0, 0, 0, 1);$$

if $\varepsilon^2 = 0$ only one of the above eigenvalues has multiplicity greater than 1, i.e. $\lambda = -\frac{h}{3h+4}$ with multiplicity 2; its corresponding independent eigenvectors are $(0,0,1,0)$ and $(0,0,0,1)$.

Consequently the hyperbolicity requirement in $f = \frac{1}{2h+3}$ holds in every case; it can be also appreciated that all the above eigenvalues are such that $|\lambda_i| \leq v$ and then the characteristic velocities do not exceed that of the particles nor the speed of light. Let us consider now the case iiii) for $-\frac{1}{2h+3} \leq f \leq 1$; in the section V we have proved that in this case the solution is $\phi = (3h+4)^{-1}[1+3(h+1)f]$. Consequently in the matrix (A.1) we have

$$q_1 = (3h+4)^{-1}(h+1)(1-f)E^{-1}$$

$$q_2 = (3h+4)^{-1}[1+3(h+1)f-(3h+4)f^2]E^{-1}$$

$$p_1 = -(3h+4)^{-1}[3(h+1)+2f^{-1}];$$

$$p_2 = -(3h+4)^{-1}(h+1);$$

$$p_3 = (3h+4)^{-1}[3(h+1)+f^{-1}]-f;$$

$$p_4 = (3h+4)^{-1}[3(h+1)+f^{-1}]$$

and then we find the real eigenvalues

$$\lambda_1 = \varepsilon^1 v;$$

$$\lambda_2 = (3h+4)^{-1}[3(h+1)+f^{-1}]\varepsilon^1 v$$

$$\lambda_{3,4} = vf^{-1}(6h+8)^{-1}\{[(1+(2h+1)f]\varepsilon^1 \pm$$

$$\pm\sqrt{[1+(4h+5)f]^2(\varepsilon^1)^2+4(h+1)f(f-1)(\varepsilon^2)^2}\};$$

When $\varepsilon^1\varepsilon^2 \neq 0$ they are all distinct;

When $\varepsilon_1 = 0$ we have $\lambda_1 = \lambda_2$ and $\lambda_1, \lambda_3, \lambda_4$ are all distinct; moreover there are two independent eigenvectors corresponding to λ_1 such as $(1, 1-f, 0, 0)$; $(0,0,0,1)$.

When $\varepsilon_2 = 0$ the distinct eigenvalues are λ_2 (with multiplicity 2), λ_1 and $-(3h+4)^{-1}\ (h+2)\varepsilon^1$; there are two independent eigenvectors corresponding to λ_2 such us $(0,0,1,0)$ and $(0,0,0,1)$.

Then also in this case the hyperbolicity requirement holds and it can be seen that the characteristic velocities do not exceed the speed of the particles nor that of light.

Appendix 2. Study of the solution S from $f = 0$ to $f = 1$.

Let us now study the solution S (discovered in sect. V) for all values of the constant h. We shall prove that it reaches the point $(x, y) = (1, 1)$ or it can be extended in an unique way to a solution reaching this point.

Four cases will be distinguished .

Case 1: $h < -2$. Let us consider the domains

$$D_1 = \{(x,y) : y_1(x) \leq y \leq y_2(x), 0 \leq x \leq \frac{1}{(2h+3)^2}\}$$

$$D_2 = \{(x,y) : y_2(x) \leq y \leq y_3(x), 0 \leq x \leq \frac{1}{(2h+3)^2}\},$$

where

$$y_1(x) = \frac{1 + 3(h+1)x^{1/2}}{3h+4}$$

i.e. the solution $(39)_1$,

$$y_2(x) = \frac{1 + (2h+3)x - \sqrt{1 + (2h+3)^2x^2 + 2(6h^2+10h+3)x}}{2(3h+4)},$$

i.e. a part of $G(x,y) = 0$

$$y_3(x) = (6h+8)^{-1}[3(2h+3)x - 1],$$

i.e. a part of $H(x,y) = 0$.

We have that $G(x,y) < 0$, $H(x,y) > 0$ and then $y'(x) < 0$ in the internal points of D_2, while $G(x,y) > 0$, $H(x,y) > 0$ and then $y'(x) > 0$ in the internal points of D_1.

Consequently the solution S is initially contained in D_1. It cannot reach a point P of the curve $(x, y_2(x))$ outside the critical points because P is a point of minimum for the solution of (40) passing through P. The solution S cannot also reach a point of the curve $(x, y_1(x))$ outside the critical points because such curve is another solution of (37).

Then if we call $\bar{y}(x)$ the solution S; we have $y_1(x) < \bar{y}(x) < y_2(x)$ for $o < x < \dfrac{1}{(2h+3)^2}$; taking the limit for $x \to \dfrac{1}{(2h+3)^2}$ we obtain

$$\bar{y}\left(\frac{1}{(2h+3)^2}\right) = \frac{-h}{(2h+3)(3h+4)},$$

i.e. the solution S reachs the point Q. The expression of $\bar{y}(x)$ for $\dfrac{1}{(2h+3)^2} \leq x \leq 1$ is $(39)_1$; in fact $y = y_1(x)$ is a solution of (37) coming from Q and reaching $(1,1)$; there is no other solution satisfying this property because, for $h < -2$, Q is a knot while $(1,1)$ is a saddle (if $\gamma_1(x) < \gamma_2(x)$ were two solution, coming from a knot and going to a saddle, through every point $(\bar{x}, \bar{y})$ such that $\gamma_1(\bar{x}) < \bar{y} < \gamma_2(\bar{x})$ would pass another solution satisfying the same properties; then there will be infinite solutions reaching the saddle and this is not possible).

For $0 < x < \dfrac{1}{(2h+3)^2}$ the condition (36) is verified; in fact (36.a) becomes

$$(3h+4)(y_1(x) - \bar{y}) > 0 \tag{A.2}$$

that is true because $\bar{y}(x) > y_1(x)$.

Moreover (34) is equal to (35), because

$$2(3h+4)\phi + 1 - 3(2h+3)f^2 = 2(3h+4)(\bar{y}(x) - y_3(x)) \neq 0 \tag{A.3}$$

and then (36.b) becomes

$$\frac{(3h+4)^2(\bar{y}(x) - y_1(x))(\bar{y}(x) - y_4(x))}{2(3h+4)(\bar{y}(x) - y_3(x))} > 0 \tag{A.4}$$

Where $y_4(x) = \dfrac{1 - 3(h+1)x^{1/2}}{3h+4}$ i.e. the solution $(39)_2$; the above inequality is verified because

$$\bar{y}(x) > y_1(x) > y_4(x)$$

and moreover

$$\bar{y}(x) < y_2(x) < y_3(x).$$

Then hyperbolicity holds for $x < \dfrac{1}{(2h+3)^2}$; it holds also for $\dfrac{1}{(2h+3)^2} \leq x < 1$ because this has been proved in Appendix 1, case iiii).

Let us now consider the

Case 2: $-2 \leq h < -4/3$. As before we may consider the sets

$$D_3 = \{(x, y) : y_1(x) \leq y \leq y_2(x), 0 \leq x \leq 1\}$$

$$D_4 = \{(x, y) : y_2(x) \leq y \leq y_3(x), 0 \leq x \leq 1\}$$

We have $y'(x) < 0$ in the internal points of D_4 and $y'(x) > 0$ in the internal points of D_3: then S is initially contained in D_3.

The curve $(x, y_2(x))$, outside the critical points, is constituted by points of minimum for the solutions of (37) passing through them; then it cannot be crossed by S outside the critical points.

The curve $(x, y_1(x))$ is also a solution of (37) and then it cannot be crossed by S outside the critical points; the critical points contained in D_3 are reached only for $x = 0$ and $x = 1$ (and then $Q \notin D_3$, differently from the previous case except for $h = -2$ when $Q \equiv (1, 1)$), so that $y_1(x) < \bar{y}(x) < y_2(x)$ for $0 < x < 1$ and then taking the limit for $x \to 1$ we obtain $\bar{y}(1) = 1$ as we desidered to prove.

Moreover $\bar{y}(x) > y_1(x) > y_4(x)$ and $\bar{y}(x) < y_2(x) < y_3(x)$ assure that the conditions (A.2), (A.3) and (A.4) are verified and then hyperbolicity holds because S is convex.

The case $h = -\dfrac{4}{3}$ has already been succesfully considered.

The next case to be considered is

Case 3: $-\dfrac{4}{3} < h \leq -1$.

We consider the sets

$$D_5 = \{(x, y) : y_2(x) \leq y \leq y_1(x), 0 \leq x \leq 1\}$$

$$D_6 = \{(x, y) : y_3(x) \leq y \leq y_2(x), 0 \leq x \leq 1\}$$

$$D_7 = \{(x, y) : y \leq y_3(x), 0 \leq x \leq 1\}$$

and notice that

$y(x)$ is decreasing in the internal points of D_5,

$y(x)$ is increasing in the internal points of D_6,

$y(x)$ is decreasing in the internal points of D_7.

Consequently S is initially contained in D_6; moreover we have:

a) The curve $(x, y_2(x))$ outside the critical points is constituted by points of minimum of the solutions $y(x)$ of (38) passing through them,

b) the curve $(x, y_3(x))$, outside the critical points is constituted by points of minimum of the solutions $x(y)$ of (37) passing through them.

Then both curves cannot be crossed by S outside the critical points of D_3, i.e. those with $x = 0$ and $x = 1$, (we have that $Q \in D_3 \Leftrightarrow h = -1$ when $Q \equiv (1, 1)$) so that $y_3(x) < \bar{y}(x) < y_2(x)$ for $0 < x < 1$ from which $\bar{y}(1) = 1$ thus assuring existence and uniqueness.

Moreover (A.2), (A.3) and (A.4) hold, from which the convexity of entropy and the hyperbolicity requirement; in fact we have

$$\bar{y}(x) < y_2(x) < y_1(x) \leq y_4(x)$$

and

$$y_3(x) < \bar{y}(x).$$

Let us consider now

case 4: $-1 < h$, and the sets

$$D_8 = \left\{(x, y) : y_2(x) \leq y \leq y_4(x), 0 \leq x \leq \frac{1}{(2h+3)^2}\right\}$$

$$D_9 = \left\{(x,y) : y_3(x) \leq y \leq y_2(x), 0 \leq x \leq \frac{1}{(2h+3)^2}\right\}$$

$$D_{10} = \left\{(x,y) : y \leq y_3(x), 0 \leq x \leq \frac{1}{(2h+3)^2}\right\}$$

$$D_{11} = \left\{(x,y) : y_4(x) \leq y \leq y_3(x), \frac{1}{(2h+3)^2} \leq x \leq 1\right\}$$

$$D_{12} = \left\{(x,y) : y_2(x) \leq y \leq y_3(x), \frac{1}{(2h+3)^2} \leq x \leq 1\right\}$$

$$D_{13} = \left\{(x,y) : y_3(x) \leq y \leq y_5(x), \frac{1}{(2h+3)^2} \leq x \leq 1\right\}$$

Where

$$y_5(x) = \frac{1+(2h+3)x+\sqrt{1+(2h+3)^2x^2+2(6h^2+10h+3)x}}{2(3h+4)}$$

In this case, as in that with $h < -2$, we have $\frac{1}{(2h+3)^2} \leq 1$ and it will be seen that Q is reached by S before this solution arrives in (1,1). Infact

if $h < 0$ we have $y'(x) < 0$ in the internal points of D_8 and D_{10}, while $y'(x) > 0$ in the internal points of D_9 so that initially S lies in D_9; it cannot reach the curves $(x, y_2(x))$ or $(x, y_3(x))$ that are consituted for $0 < x < \frac{1}{(2h+3)^2}$ by points of minimum of the solutions $y(x)$ or $x(y)$, respectively, of (37) passing through them. Consequently $y_3(x) < \bar{y}(x) < y_2(x)$ that for $x \to \frac{1}{(2h+3)^2}$ proves that Q is reached.

If $h = 0$ the solution S for $0 \leq x \leq \frac{1}{(2h+3)^2}$ is given by $\bar{y}(x) = y_2(x) = 0$ as it can be directly verified from (37);

If $h > 0$ we have that $y'(x) < 0$ in the internal points of D_8 and $y'(x) > 0$ in the internal points of D_9 so that initially S lies in D_8; for $0 < x < \frac{1}{(2h+3)^2}$ it cannot reach the curves $(x, y_4(x))$ or $(x, y_2(x))$ because the first one is also a solution of (37) while the second one is constituted by points of maximum of the solutions of (37) passing through them. Then $y_2(x) < \bar{y}(x) < y_4(x)$, that for $x \to (2h+3)^{-2}$ proves that Q is reached by $(x, \bar{y}(x))$.

It remains now to be proved that there is a solution of (37) leaving Q and reaching $(1,1)$. To this end let us notice that $(1,1)$ is a saddle and then there are 4 and only 4 solutions of (37) leaving or reaching it. Two of them are constituted by $(39)_1$ with $x \leq 1$ and $x \geq 1$ respectively and then they are such that $y'(1) = \frac{3}{2}\frac{h+1}{3h+4}$; of the other two solutions one is such that $y \geq 1$ and for the other $y \leq 1$; then we are interested in this last solution that we call $\bar{y}(x)$ and is such that $\bar{y}'(1) = \frac{3}{2} > y_3'(1) > 0$.

Consequently, in a neighbourbood of (1.1), $(x, \bar{y}(x))$ lies in D_{11}. Moreover we have that $y'(x) > 0$ in the internal points of D_{12}, while $y'(x) < 0$ in the internal points of D_{13}, so that the curve $(x, y_3(x))$ for $\frac{1}{(2h+3)^2} < x < 1$ is constituted by points of maximum for the solution $x(y)$ of (37) passing through them and then $\bar{y}(x)$ cannot come from a point of this curve.

Furthermore $\bar{y}(x)$ cannot come from a point of the curve $(x, y_4(x))$ for $x > \frac{1}{(2h+3)^2}$ because it is also a solution of (37) and does not contain critical points; then $y_4(x) < \bar{y}(x) < y_3(x)$ for $\frac{1}{(2h+3)^2} < x < 1$, that, for $x \to \frac{1}{(2h+3)^2}$ proves that $\bar{y}(x)$ comes from Q.

Let us prove at last that the hyperbolicity requirement holds. To this end we can see that for $0 < x < (2h+3)^{-2}$ we have

$$y_1(x) > y_4(x) > y_2(x) > y_3(x)$$

so that

$$y_1(x) - \bar{y}(x) > 0; \bar{y}(x) - y_3(x) > 0; \bar{y}(x) - y_4(x) < 0$$

and then (A.2), (A.3), (A.44) are verified.

For $(2h+3)^{-2} < x < 1$ we have $y_1(x) > y_3(x)$ and then $y_1(x) - \bar{y}(x) > 0$ $\bar{y}(x) - y_3(x) < 0$; $\bar{y}(x) - y_4(x) > 0$ and then (A.2), (A.3) and (A.4) hold.

For $x = (2h+3)^{-2}$ and $y'[(2h+3)^{-2}] = -\frac{3}{2}h\frac{2h+3}{3h+4}$ we have that the first member of (36a) is $6(h+1)(h+2)(2h+3)^{-2}$ and then (36a) is verified; moreover (34) is equal to $v^4\beta^2E^{2h+2}\,(2h+3)^{-1}$ and then (36b) is verified.

In all these cases the convexity of entropy holds and consequently our system is hyperbolic. The remaining case is when $x = (2h+3)^{-2}$ and $y'[(2h+3)^{-2}] = -\frac{3}{2}(h+1)\frac{2h+3}{3h+4}$; in this case (36b) does not hold, but the hyperbolicity requirement is valid as it has been proved in Appendix 1, case iii).

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