LE MATEMATICHE
Vol. XLV (1990) - Fasc. II, pp. 407-443

NEUTRAL SURFACES IN NEUTRAL FOUR-SPACES

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Properties of the Gauss map of neutral surfaces in neutral four-spaces are studied. Special attention is given to surfaces of parallel, or zero, mean curvature. Bilagrangian structures are defined and used in ways analogous to the use of complex structures in the Riemannian case. The nonsimplicity of the structure group $SO(2,2)$ is used to factor the Gauss map and to construct analogs of the twistor space, called in this context reflector spaces.

Introduction.

Surfaces with parallel mean curvature in Euclidean space $\mathbf{R}^4$ exhibit many special properties throught the splitting of the Gauss map. This spliting is a consequence of the biholomorphic and isometric isomorphism between the Grasmanniann of oriented 2-planes in $\mathbf{R}^4$ and the product of two constant curvature 2-spheres. The twistor space of an oriented Riemannian 4-manifold provide the means to generalize most of these properties to a general 4-dimensional ambient space, or at least to identify the curvature properties needed for a satisfactory generalization (cf. [4], [6] and [7]).

The complex structure induced on an oriented surfaces by a

(*) Entrato in redazione il 13 febbraio 1990.

Riemannian metric plays a fundamental role in understanding the properties of surfaces with parallel mean curvature. The twistor spaces consist of all the oriented almost complex structures on a 4-manifold. Curvature quantization result for such surfaces depend on the maximum principle for subharmonic functions on them.

In this paper we shows that essentially all of these results carry over, in the appropriately defined form, to the case of neutral surfaces of parallel mean curvature in neutral 4-spaces. (Neutral metrics, defined in §1.1, have signature zero: equal numbers of plusses and minuses). At first glance this seems very unlikely, as a neutral surface has no relevant complex structure and its Laplace-Beltrami operator is hyperbolic rather than elliptic. However, characteristic coordinates, whose existence depends only on the Frobenius Theorem (see §1.1), take the place of isothermal coordinates and account for the existence of a bilagrangian structure on any oriented neutral surfaces. The set of all almost bilagrangian structures on an oriented neutral 4-maniflod constitute the reflector space, the neutral space analogs of the twistor spaces of Riemannian geometry.

Neutral surfaces in Minkowski space have been studied by T. Milnor (cf. [9] and references cited there). In that case the simplicity of the Lorentz group $SO(3,1)$ prevents any isometric decomposition of the relevant Grassmannian of oriented planes in Minkowski space $\mathbf{R}^{3,1}$. However, $SO(2,2)$, like $SO(4)$, is at the Lie algebra level a direct sum of ideals which in this case are isomorphic to the Lorentz group Lie algebra $o(2,1)$. As a consequence the Grassmannian $SO(2,2)/SO(1,1)\times SO(1,1)$ of oriented neutral planes in $\mathbf{R}^{2,2}$ is isomorphic, as bilagrangian neutral spaces, to the product of neutral spaces forms, $S^{1,1}\times S^{1,1}$. The Riemannian results carry over exactly to the case provided complex structures and holomorphic maps are replaced throughout by bilagrangian structures and bilagrangian maps.

If H is the mean curvature vector of a neutral surface in $\mathbf{R}^{2,2}$, then $H=0$ is the wave equation, which in this case is the uniform vibrating string equation. Surfaces with $H=0$ are called strings in the literature (cf. [2]) and we follow that terminology. In theorems 1 and 2, in §2.3, we show that bilagrangian maps are strings and we characterize strings which are bilagrangian. This is done in terms of the concept of isotropy, which is defined in terms of the vanishing of certain components of the second fundamental form which decomposes into types because of the bilagrangian structure in the surface.

Theorem 3, in §2.4, shows that a neutral surface with nonzero mean curvature vector has a parallelizable normal bundle. Theorem 4 is a surprising quantization of curvature result for such surfaces.

Theorem 5, in §3.2, summarizes the ten essential features of the Gauss map, and its factors, called the reflector maps, for a neutral surface in the flat neutral space $\mathbf{R}^{2,2}$. These properties are formulated in a way that makes transparent their analogy with the Riemannian case.

In §4 we construct the reflector boundles of an oriented neutral 4-space. Theorem 6, in §4.2, generalizes to the neutral case a theorem of Friedrich and Grunewald [5]. It shows the existence of a pair of Einstein metrics on the reflector spaces of an Einstein self-dual neutral space. Theorem 7, in §4.3, is the neutral space version of theorems of Atiyah, Hitchen and Singer [1] and Eelles and Salamon [4] concerning the integrability of the two natural bilagrangian structures on the reflector spaces. Comparable results of the latter two authors have been announced in [3], p. 446. Theorem 8, in §4.4, shows the conformal invariance of one of these bilagrangian structures.

In §4.5 we investigate the essential properties of the reflector lifts (which are the factors of the Gauss lift) for an oriented neutral surface in an arbitrary oriented neutral 4-spaces. Theorem 7 contains the generalization of eight of the results from $\mathbf{R}^{2,2}$ contained in Theorem 5. These properties hold for any neutral 4-space. The remaining two properties, concerned with the harmonicity of the reflector lifts, are contained in theorem 10, which assumes the ambient space is $\pm$self-dual and Einstein in order to obtain sharp results.

The method of moving frames is used throughout this paper. This allows all calculations to be made in terms of the structure equations of $SO(2,2)$, or more precisely, the structure equations of the Levi-Civita connection on the principal bundle of oriented null frames on the neutral space. These frames, rather than the orthonormal frames, are the natural tools for neutral spaces. The Einstein summation convention is used throughout. The index ranges, which change from place to place, are defined at the beginning of each section.

1. Neutral Spaces and Associated structures.

1.1 Basic Definitions.

Let V be a vector space over $\mathbf{R}$ of dimension $2n$. A neutral metric on V is a non-degenerate symmetric bilinear form g on V of signature zero. This means that there is a basis $e = (e_1, \dots, e_{2n})$ of V such that the matrix $(g_{ab}) = L_n$, where $g_{ab} = g(e_a, e_b)$ and

$$L_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

In the first two section we use the index ranges

$$1 \le i,\ j \le n; \quad 1 \le a, b, c, d \le 2n.$$

Any basis of V with respect to which the matrix of g is L_n will be called a null frame. Given a null frame e, any other is given by $\tilde{e} = eK$, where K belongs to the Lie group $O(n,n) = \{T \in GL(2n; \mathbf{R}) : {}^tTL_{nT} = L_n\}$. The frames e and $\tilde{e}$ have the same orientation if and only if $K \in SO(n,n) = O(n,n) \cap SL(2n; \mathbf{R})$. For future reference the Lie algebra of $O(n,n)$ is

$$o(n,n) = \left\{ \begin{pmatrix} X & Y \\ Z & -{}^tX \end{pmatrix} : X, Y, Z \in \mathbf{R}^{n \times n}, {}^tZ = -Z, {}^tY = -Y \right\}.$$

If e is a null frame of V, and if $\theta = (\theta^i, \theta^{n+i})$ denotes the basis of V^* dual to e, called the null coframe dual to e, then using the symmetric product we have $g = 2 \sum \theta^i \theta^{n+i}$.

A product structure on V is a linear operator J on V for which $J^2 = id_V$. We restrict our attention to those product structures whose eigenvalues, $\{\pm 1\}$, each occurs with multiplicity n. A bilagrangian structure J on V, g is such a product structure which satisfies $g(Jx, Jy) = -g(x, y)$ for every $x, y \in V$. The name for such a J comes from its spectral decomposition $V = V_- \oplus V_+$, because each eigenspace $V_\pm$ is a Lagrangian subspace (i.e., totally isotropic and of dimension n).

Using a null frame e and its dual coframe θ of a neutral metric g on V, we can define a bilagragian structure J_e on V by $J_e e_i = -e_i$, and $J_e e_{n+i} = e_{n+i}$. Regarding a basis e of V as an isomorphism $e : \mathbf{R}^{2n} \to V$, we see that $J_e = e \circ I_{n,n} \circ e^{-1}$, where $I_{n,n} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$.

It is easily checked that any bilagrangian structure of V, g is given by J_e for some null frame e of V. If $\tilde{e} = eK$ for some $K \in O(n,n)$, then $J_{\tilde{e}} = J_e$ if and only if $KI_{n,n}K^{-1} = I_{n,n}$. It is convenient to state this in therms of certain relevant matrix groups. Let

$$\begin{aligned} G(n,n) =& \{T \in GL(2n;\mathbf{R}) : TI_{n,n}T^{-1} = I_{n,n}\} = \\ =& \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in GL(n;\mathbf{R}) \right\} \\ \cong& GL(n;\mathbf{R}) \times GL(n;\mathbf{R}) \end{aligned}$$

and let

$$\begin{aligned} B(n) = O(n,n) \cap G(n,n) = & \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in GL(n;\mathbf{R}) \right\} \\ \cong & \ GL(n;\mathbf{R}). \end{aligned}$$

Notice that actually $B(n) \subset SL(2n;\mathbf{R})$, and that $B(n) = SO(n,n) \cap G(n,n)$. Now we have that $J_{\tilde{e}} = J_e$ if and only if $\tilde{e} = eK$ for some $K \in B(n)$.

If V, g is oriented, then the bilagrangian structures J divide into two disjoint categories, those of the form J_e where e is positively oriented, and those of the form J_e where e is negatively oriented. Call J of each category positively or negatively oriented, respectively. The set of all positively oriented bilagrangian structures on V, g is then

$$SO(n,n)/B(n).$$

The case $n = 1$ will be of special interest to us. In that case

$$O(1,1) = SO(1,1) \cup L_1 SO(1,1),$$

and

$$B(1) = SO(1,1) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : 0 \neq t \in \mathbf{R} \right\}.$$

Thus on V^2, g there are exactly two bilagrangian structures. One of these is distinguished by choosing an orientation on V. Namely, take $J = J_e$, where e is any oriented null frame, and then the other one is $-J$.

A linear transformation $T : V \to \tilde{V}$ between vector spaces with bilagrangian structures J and $\tilde{J}$, respectively, is called $\pm$bilagrangian if $T \circ J = \pm \tilde{J} \circ T$.

All of this can be extended in the usual way to even ranked vector bundless over smooth manifolds. We are concerned only with the tangent bundle of an even dimensional manifold N^{2n}. A neutral space N, g is a $2n$-dimensional manifold N with a smoothly varying neutral metric g on its tangent bundle TN. An almost product structure on N is a smooth (1,1) tensor field J on N which defines a product structure on the tangent space of each point of N. It is an almost bilagrangian structure on a neutral space N, g if it defines a bilagrangian structure on each tangent space.

Let Γ^n denote the pseudo-group of transformations of $\mathbf{R}^{2n}$ consisting of all C^∞ diffeomorphisms F of an open subset of $\mathbf{R}^{2n}$ onto an open subset of $\mathbf{R}^{2n}$ such that at every point of the domain of F its Jacobian matrix belongs to $G(n,n)$. A bilagrangian structure on a smooth manifold N^{2n} is an atlas of C^∞ charts on N compatible with Γ^n. In detail, this means that if $U, x = (x^i, x^{n+i})$ and $V, y = (y^i, y^{n+i})$ are two charts in this atlas, then at any point of $U \cap V$ we have

$$\frac{\partial y^i}{\partial x^{n+j}} = 0 = \frac{\partial y^{n+i}}{\partial x^j}.$$

A bilagrangian structure on N canonically induces an almost bilagrangian structure J as follows. If U, x is a chart in the structure, then at any point of U, $J\frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x^i}$ and $J\frac{\partial}{\partial x^{n+i}} = \frac{\partial}{\partial x^{n+i}}$. In other words, the bilagrangian splitting of the tangent space of N at any point of U is given by the equations $\{dx^i = 0\}$ and $\{dx^{n+i} = 0\}$, respectively.

Conversely, we are interstead in knowing when a given almost bilagrangian structure J on N is induced by a bilagrangian structure on N. This is an integrability condition expressible in terms of the Nijenhuis torsion tensor of J (cf. [8]). To described this we observe that an almost bilagrangian structure J on N is a geometric structure on N characterized as follows. There exists an open covering $\{U_\alpha\}$ of N such that on each U_α there exists a smooth coframe field $\theta_\alpha = \begin{pmatrix} \theta^i_\alpha \\ \theta^{n+i}_\alpha \end{pmatrix}$. If $U_\alpha \cap U_\beta \neq \emptyset$, then on their intersection $\theta_\alpha = K\theta_\beta$, where K is a smoth map into $G(n,n)$. The relationship between this description and J is that at any point $p \in U_\alpha$, $J_p = J_{e_\alpha(p)}$, where e_α is the null frame field dual to θ_α. We say that an almost bilagrangian structure is integrable if it is induced by a bilagrangian structure.

PROPOSITION 1.1.1. *An almost bilagrangian structure* $\{U_\alpha, \theta_\alpha\}$ *is integrable if and only if* $d\theta^i_\alpha \equiv 0$ *(mod* θ^i_α*) and* $d\theta^{n+i}_\alpha \equiv 0$ *(mod* θ^{n+i}_α*) for every* α.

Proof. If an almost bilagrangian structure is induced by the bilagrangian structure whose atlas of charts is $\mathcal{A} = \{U_\alpha, x_\alpha\}$, then its system of local equations is U_α, θ_α, where $\theta_\alpha = dx_\alpha$, for which the integrability condition clearly holds.

Conversely, if the integrability condition of the almost bilagrangian structure $\{U_\alpha, \theta_\alpha\}$ holds, then each of the n-plane distrubutions $\{\theta^i_\alpha = 0\}$ and $\{\theta^{n+i}_\alpha = 0\}$ is completely integrable. Let $p \in N$. Then $p \in U_\alpha$ for same α, and there exists local coordinates V, x and V, y about p, such that the integrable submanifolds of $\theta^i_\alpha = 0$ (respectively, $\theta^{n+i}_\alpha = 0$) are given in V by $x^i =$ constant (respectively, $y^{n+i} =$ constant). It follows that on V, the dx^i are a linear combination of the θ^i_α and the dy^{n+i} are a linear combination of the θ^{n+i}_α. In particular, dx^i, dy^{n+i} are linearly indipendent at p, and thus x^i, y^{n+i} are local coordinates on some neighbochood $\tilde{V}$ of p. By this construction we cover N by charts $V_a, x_a = (x^i_a, x^{n+i}_a)$ such that for each a there exists an α such that $V_a \subset U_\alpha$ and the dx^i are a linear combination of the θ^i_α and the dx^{n+i} are a linear combination of the θ^{n+i}_α. Consequently, if $V_a \cap V_b \neq \emptyset$, then at each point this intersection the Jacobian matrix of $x_a \circ x_b^{-1}$ must lie in $G(n,n)$, since the dx^i_a must be a linear combination of the dx^i_b and the dx^{n+i}_a must be a linear combination of the dx^{n+i}_b. Hence the atlas $\{U_a, x_a\}$ defines a bilagrangian structure on N which clearly induces the given almost bilagrangian structure. ■

Let $F : N \to \tilde{N}$ be a C^∞ map between manifolds with almost bilagrangian structures J and $\tilde{J}$, respectively. We say that F is bilagrangian if its differential F_* is bilagrangian at every point of N. Suppose that J and $\tilde{J}$ are induced by bilagrangian structures with atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. Then F is bilagrangian if and only if for any chart $(U, x) \in \mathcal{A}$ and $(\tilde{U}, \tilde{x}) \in \tilde{\mathcal{A}}$ for which $F(U) \subset \tilde{U}$, the functions $\tilde{x}^{\tilde{i}} \circ F$ depend only on the x^j and the functions $\tilde{x}^{\tilde{n}+\tilde{i}} \circ F$ depend only on the x^{n+j}.

1.2 Geometry of Neutral Spaces.

Let N, g be a connected neutral space of dimension $2n$. Let

$$\pi : O(N) \to N$$

denote the principal $O(n,n)$-boundle of null frames. If N is oriented we let $SO(N)$ denote the principal $SO(n,n)$-boundle of oriented null frames. Let $\theta = \begin{pmatrix} \theta^i \\ \theta^{n+i} \end{pmatrix}$ and $\omega = (\omega^a_b)$, denote the canonical form and Levi-Civita connection form, respectively, on $O(N)$. Recall that ω is characterized by being $o(n,n)$-valued and satisfying the structure equations

$$d\theta = -\omega \wedge \theta.$$

The curvature form Ω is the $o(n,n)$-valued 2-form on $O(N)$ defined by

$$\Omega = d\omega + \omega \wedge \omega.$$

The components of the curvature tensor R^a_{bcd} are defined by

$$\Omega_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d$$

where $R^a_{bcd} = -R^a_{bdc}$. If we use the matrix $(g_{ab}) = L_n$ and its inverse $(g^{ab}) = L_n$ to raise and lower indices, then the Riemann curvature tensor $R_{abcd} = g_{ae} R^e_{bcd}$ satisfy all the usual symmetries. The components of the Ricci tensor are defined by $R_{ab} = R^c_{acb} = R_{ba}$ and the scalar curvature is $s = R^a_a = g^{ab} R_{ab}$.

By definition N, g has constant curvature ε if

$$\Omega^{ab} = \varepsilon \theta^a \wedge \theta^b,$$

where $\Omega^{ab} = \Omega^a_c g^{cd}$. This condition is equivalent to

$$R_{abcd} = \varepsilon (G_{ac} g_{bd} - g_{ad} g_{bc}).$$

We call a neutral space with constant curvature a neutral space form. For neutral space forms there is no meaningful difference between positive and negative curvature, since if N, g has constant curvature ϵ, then $N, -g$ has constant curvature $-\varepsilon$.

The standard models for the space forms are given in [10]:

(1) $\mathbf{R}^{n,n}$ is the flat space, which is $\mathbf{R}^{2n}$ with the neutral metric $g = 2dx^i dx^{n+i}$. The neutral proper motion group $E(n,n) = \mathbf{R}^{2n} \times SO(n,n)$ acts transitively on $\mathbf{R}^{n,n}$ as orientation preserving isometries. Its isotropic subgroup at the origin is $SO(n,n)$.

(2) $S^{n,n} = \{v \in \mathbf{R}^{n,n+1} : (v,v) = 1\}$ where $\mathbf{R}^{n,n+1}$ is $\mathbf{R}^{2n+1}$ with the inner product $(x,y) = x^i y^{n+i} + x^{n+i} y^i + x^{n+1} y^{2n+1}$ of signature $(n, n+1)$. The induced metric g on $S^{n,n}$ is neutral with constant curvature $\varepsilon = 1$. Thus $S^{n,n}, -g$ has constant curvature $\varepsilon = -1$. The group $SO(n, n+1)$ acts transitively on $S^{n,n}$ as orientation preserving isometries. Its isotropy subgroup at ε_{2n+1} is $SO(n,n)$, so $S^{n,n} = SO(n,n+1)/SO(n,n)$. Diffeomorphically $S^{n,n} = \mathbf{R}^n \times S^n$.

1.3 Surfaces.

Let N, g be an oriented 2-dimensional neutral space. A local oriented null frame field in N is a C^∞ map $e : U \to SO(N)$, where U is an open subset of N. At each point $p \in U$, $e = (e_1, e_2)$ is an oriented null frame at p. Its dual coframe is $e^*\theta$ wich we will always write as simply $\theta = \begin{pmatrix} \theta^1 \\ \theta_2 \end{pmatrix}$. Then $g = 2\theta^1\theta^2$, and $\nu_M = \theta^1 \wedge \theta^2$ is the volume form of N, g.

As now $B(1) = SO(1,1)$, it follows that $J_{\tilde{e}} = J_e$ for any other oriented null frame $\tilde{e} = eK$, where K takes values in $SO(1,1)$. Thus g and the orientation of N induce a unique almost bilagrangian structure J. As the dimension of N is two, J is automatically integrable, so that N has an atlas $\mathcal{A}$ such that for any chart $(U,(x,y)) \in \mathcal{A}$, the oriented null coframe field in U defined by e is given by

$$\theta^1 = adx, \qquad \theta^2 = bdy,$$

for some positive C^∞ functions a and b on U. Then

$$g = 2\theta^1\theta^2 = 2abdxdy = 2Fdxdy,$$

where F is a positive C^∞ function on U. We call the charts of $\mathcal{A}$ local characteristic coordinates in N, g.

The bilagrangian structure J determined on N by g and the orientation is unchanged if g is replaced by a conformally equivalent metric λg, for any positive C^∞ function λ on N. If g is replaced by $-g$ (which is still a neutral metric), and if the orientation is unchanged, then J is replaced by $-J$.

It is easily verified that the Levi-Civita connection form $\omega = (\omega^a_b)$ with respect to e (i.e., $e^*\omega$) is given by

$$\omega^1_1 = \frac{b_x}{b}dx - \frac{a_y}{a}dy = -\omega^2_2$$
$$\omega^1_2 = 0 = \omega^2_1,$$

where $b_x = \dfrac{\partial b}{\partial x}$, etc. Then

$$\Omega^1_1 = d\omega^1_1 = -\frac{(\log F)_{xy}}{F}\theta^1 \wedge \theta^2.$$

The expression for the Laplace-Beltrami operator of g in terms of characteristic coördinates is easily verified to be

$$\Delta f = 2\frac{f_{xy}}{F}$$

for any class C^2 function f on U.

We will use the following lemma, which is proved in [2], as an aid in computing this Laplacian.

LEMMA 1.3.1. *If h and k are functions on N satisfying*

$$(dh + h\tau) \wedge \theta^1 = 0 \text{ and } (dk - k\tau) \wedge \theta^2 = 0$$

for some 1-form τ, *where* θ^1, θ^2 *is any oriented null coframe field, then*

$$\Delta \log |hk| \theta^1 \wedge \theta^2 = -2d\tau$$

wherever $hk \neq 0$.

Proof. The Hodge $*$-operator on forms on M is given by

$$*1 + \theta^1 \wedge \theta^2, \quad *\theta^1 = -\theta^1, \quad *\theta^2 = \theta^2, \quad *\theta^1 \wedge \theta^2 = -1.$$

(See §1.4 below for more details on the $*$-operator.) By hypothesis, there exist locally defined functions a and b such that $dh + h\tau = a\theta^1$ and $dk - k\tau = b\theta^2$. Multiplying the first equation by k, the second by h, and adding therm together we have $d(hk) = ka\theta^1 + hb\theta^2$. Thus

$$*d(hk) = -ka\theta^1 + hb\theta^2 = -2hk\tau + hdk - kdh.$$

Dividing through by hk and applying d we have

$$\Delta \log |hk| \theta^1 \wedge \theta^2 = d * d \log |hk| = -2d\tau.$$

The first equation is the definition of Δ, which is equivalent to $\Delta u = - * d * du$ for any functions u on M. ■

We define the Gaussian curvature K of g by $d\omega_1^1 = \Omega_1^1 = K\theta^1 \wedge \theta^2$, so that $K = R^1_{112} = R_{2112} = -R_{1212}$. It is a globally defined function on N. For $g = 2F dx dy$, we have

$$K = -\frac{(\log F)_{xy}}{F} = -\frac{1}{2} \Delta \log F.$$

Then g has constant curvature ε if and only if K is constant, equal to ε. Notice that the opposite orientation on N is given by replacing x by $-x$ in each chart, with results in replacing g by $-g$ whose Gaussian curvature is $-K$.

The 2-dimensional space forms are $\mathbf{R}^{1,1}$ and $S^{1,1}$, with Gaussian curvatures of 0 and 1 respectively. In this dimension $S^{1,1} \approx \mathbf{R} \times S^1$, which is not simply connected. A simply connected model is $\mathbf{R}^2$ with the metric $g = \operatorname{sech}^2 \left(\frac{x+y}{2} \right) dx dy$. The metric of constant negative curvature -1 is $-g = \operatorname{sech}^2 \left(\frac{y-(-x)}{2} \right) d(-x) dy$.

1.4 Four-dimensional neutral geometry.

For this section we fix the index conventions

$$1 \leq i, j, k \leq 3; \quad 1 \leq a, b, c \leq 4.$$

On $\mathbf{R}^4$ with its standard basis $\{\varepsilon_a\}$ let g denote the neutral metric defined by $(g(\varepsilon_a, \varepsilon_b)) = L_2$. With the standard orientation on $\mathbf{R}^4$, it follows that $\{\varepsilon_a\}$ is a positively oriented null frame for $\mathbf{R}^4, g$. In $\Lambda_2\mathbf{R}^4$ let $\varepsilon_{ab} = \varepsilon_a \wedge \varepsilon_b$. In the ususal way g induces a metric on $\Lambda_2\mathbf{R}^4$ which in this case has signature (4,2)= $(- - - - ++)$. By definition, g is invariant under the action of $SO(2,2)$ on $\mathbf{R}^4$. The induced metric on $\Lambda_2\mathbf{R}^4$ is invariant under the action of $SO(2,2)$ given by $u \wedge v \rightarrow Au \wedge Av$ for any $A \in SO(2,2)$.

The Hodge $*$-operator on $\Delta_2\mathbf{R}^4$ is defined by $a \wedge *b = g(a,b)\varepsilon_1 \wedge \cdots \wedge \varepsilon_4$ for any $a, b \in \Delta_2\mathbf{R}^4$. As in the Riemannian case, $*^2 = 1$. The ± 1 eigenspaces of $*$, denoted $\Delta_\pm$, have bases $E_i^\pm$, respectively, where

$$\begin{aligned} E_1^+ &= \frac{\varepsilon_{13} + \varepsilon_{24}}{\sqrt{2}}, \quad E_2^+ = \varepsilon_{12}, \quad E_3^+ = \varepsilon_{34} \\ E_1^- &= \frac{\varepsilon_{13} - \varepsilon_{24}}{\sqrt{2}}, \quad E_2^- = \varepsilon_{14}, \quad E_3^- = \varepsilon_{32}. \end{aligned} \tag{1.1}$$

With respect to the bases the metrics induced on $\Delta_\pm$ have matrix

$$q = (q_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

These subspaces are invariant and irreducible under the action of $SO(2,2)$, and thus they give rise to a 2:1 surjective homomorphism

$$\begin{aligned} \mu : SO(2,2) &\rightarrow SO(2,1) \times SO(2,1) \\ K &\mapsto (K_+, K_-), \end{aligned} \tag{1.2}$$

where for $K \in SO(2,2), K_\pm$ is the matrix of $K|\Lambda_\pm$ with respect to the above bases.

Recall that $X = (X^a_b)$ belongs to $o(2,2)$ if and only if ${}^tXL_2 + L_2X = 0$. In terms of components, if we set $X^{ab} = (XL_2^{-1})^{ab} = X^a_c g^{cb}$, then

$X \in o(2,2)$ if and only if $X^{ab} = -X^{ba}$. There is an isomorphism

$$o(2,2) \cong \Lambda_2 \mathbf{R}^4$$

$$X \leftrightarrow \hat{X} = \frac{1}{2} X^{ab} \varepsilon_{ab}$$

under which the adjoint action of $SO(2,2)$ on $o(2,2)$ corresponds to the action of $SO(2,2)$ on $\Lambda_2 \mathbf{R}^4$ by

$$KXK^{-1} \leftrightarrow K\hat{X}.$$

Let N, g be an oriented connected four-dimensional neutral space. As in §1.2, Letθ and Ω denote the canonical form and the curvature form of the Levi-Civita connection on $O(N)$. Define $\Lambda_{\pm}$-valued 2-forms $\alpha_{\pm}^i \otimes E_i^{\pm}$ on $O(N)$ by

$$(1.3) \qquad \begin{aligned} \alpha_+^1 &= \frac{\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4}{\sqrt{2}}, \quad \alpha_+^2 = \theta^1 \wedge \theta^2, \quad \alpha_+^3 = \theta^3 \wedge \theta^4 \\ \alpha_-^1 &= \frac{\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4}{\sqrt{2}}, \quad \alpha_-^2 = \theta^1 \wedge \theta^4, \quad \alpha_-^3 = \theta^3 \wedge \theta^2 \end{aligned}$$

For any $K \in SO(2,2)$, the transformation rule $R_{K^{-1}}^* \theta = K\theta$ translates into

$$R_{K^{-1}}^* \alpha_{\pm} = K_{\pm} \alpha_{\pm} = K_{\pm j}^i \alpha_{\pm}^i E_i^{\pm}.$$

The curvature form transforms by $R_K^* \Omega = K^{-1} \Omega K$, which translates into

$$R_K^* \hat{\Omega} = K^{-1} \hat{\Omega}.$$

Now $\hat{\Omega} = \dfrac{1}{2} \Omega^{ab} \varepsilon_{ab}$, where $\Omega^{ab} = \dfrac{1}{2} R_{cd}^{ab} \theta^c \wedge \theta^d$, and $R_{cd}^{ab} = g^{be} R_{ecd}^a$. Using (1.3) and (1.1), we have

$$(1.4) \qquad \hat{\Omega} = E_i^+ \otimes A_j^i \alpha_+^j + E_i^- \otimes B_j^i \alpha_+^j + E_i^+ \otimes D_j^i \alpha_-^j + E_i^- \otimes C_j^i \alpha_-^j,$$

where $A = (A_j^i)$, $B = (B_j^i)$, $C = (C_j^i)$ and $D = (D_j^i)$ are 3×3 matrix valued functions on $SO(N)$. If $K \in SO(2,2)$, then

$$R_{K^{-1}}^* A = K_+ A K_+^{-1}, \quad R_{K^{-1}}^* B = K_- B K_+^{-1},$$

$$R_{K^{-1}}^* D = K_+ D K_-^{-1}, \quad R_{K^{-1}}^* C = K_- C K_-^{-1}.$$

For any point $p \in N$ the curvature operator

$$R(p) : \Lambda_2 T_p N \to \Lambda_2 T_p N$$

is defined by $\hat{\Omega}$ as follows. Let $e = (e_1, \ldots, e_4)$ be an oriented null frame field defined on a neighborood of p in N. Its dual coframe field is $e^*\theta$. As usual we interpret a basis $e(p)$ of $T_p N$ as a linear isomorphism $e(p) : \mathbf{R}^4 \to T_p N$ given by $e(p)x = x^a e_a(p)$. Thus $e(p)E^{\pm}$ are bases of $\Lambda_{\pm} T_p N$, the ± 1 eigenspaces of the Hodge $*$ operator on $T_p N$. Then

$$R(p) = \frac{1}{4} R^{ab}_{cd}(e(p)) e_a \wedge e_b \otimes e^*_p(\theta^c \wedge \theta^d),$$

which in terms of the basis $e(p)E^{\pm}$ is given by

$$R(p) = eE^+ \otimes A(e) e^* \alpha_+ + eE^- \otimes B(e) e^* \alpha_+ + eE^+ \otimes D(e) e^* \alpha_- + eE^- \otimes C(e) e^* \alpha_- .$$

Since the curvature tensors R_{abcd} satisfy the usual symmetries, it follows that the curvature operator is symmetric with respect to the inner product induced on $\Lambda_2 T_p N$ from that on $T_p N$. In terms of the bases we are using, this translates into the conditions

$$A_{ij} = A_{ji}, \quad B_{ji} = D_{ij}, \quad C_{ij} = C_{ji}$$

where the indices were lowerd with q, (i.e., $A_{ij} = q_{ik} A^k_j$, etc.), Furthermore,

$$\text{Trace } A = \text{Trace } C = \frac{s}{4}$$

where the trace is defined by A^i_i, etc; and N, g is Einstein if and only if $B = 0$.

The Weyl curvature form is the $o(2,2)$-valued 2-form $\Psi = (\Psi^a_b)$ defined by

$$\Psi^{ab} = \Omega^{ab} - \frac{1}{2}(R^a_c \theta^c \wedge \theta^b + R^b_c \theta^a \wedge \theta^c) + \frac{s}{6} \theta^a \wedge \theta^b$$

so that

$$\hat{\Psi} = \Psi^{ab} \otimes \varepsilon_a \wedge \varepsilon_b = \hat{\Omega} + \left(\frac{s}{6} \theta^a \wedge \theta^b - R^a_c \theta^c \wedge \theta^b \right) \otimes \varepsilon_a \wedge \varepsilon_b .$$

In terms of our special frames we have

$$\hat{\Psi} = E^+ \otimes \left(A - \frac{s}{12} I \right) \alpha_+ + E^- \otimes \left(C - \frac{s}{12} I \right) \alpha_- ,$$

where I is the 3×3 identity matrix.

Defining the Weyl curvature operator $W(p)$ at $p \in N$ in the same way as we defined the curvature operator, it is evident that $W(p)$ preserves the eigenspaces of the Hodge $*$ operator. If $W^{\pm}(p)$ denotes the restriction of $W(p)$ to $\Lambda_{\pm} T_p N$, then

$$W^+(p) = eE^+ \otimes \left(A(e) - \frac{s(p)}{12} I \right) e^* \alpha_+$$

$$W^-(p) = eE^- \otimes \left(C(e) - \frac{s(p)}{12} I \right) e^* \alpha_- .$$

The oriented neutral space N, g is $\pm$-dual (read self-dual and anti-self-dual, respectively) if at every point of N we have $W^- = 0$, respectively $W^+ = 0$.

PROPOSITION 1.4.1. *Let N, g be an oriented neutral space. Then*

(1) N, g is $+$ dual if and only if $C = \frac{s}{12} I$ on $SO(N)$

(2) N, g is $-$ dual if and only if $A = \frac{s}{12} I$ on $SO(N)$.

2. Immersed Neutral surfaces.

2.1. Basic Formulas.

In this section we follow the index conventions

$$i, j, k, m = 1, 3; \ \alpha, \beta, \gamma, \delta = 2, 4; \ 1 \leq a, b, c \leq 4.$$

Let N, g be a four dimensional neutral space and let M be a connected surface. Let

$$f : M \to N$$

be an immersion whose induced metric $ds^2 = f^* g$ has signature $(-+)$. A Darboux frame field along f is a map $e : U \subseteq M \to O(N)$ such that

$$e^* \theta^\alpha = 0. \tag{2.1}$$

We will usually omit the e^* in such expressions since the context will make its presence understood.

For a Darboux frame e

$$ds^2 = 2\theta^1\theta^3,$$

so that θ^1, θ^3 is a null coframe field for ds^2 in M. If M and N are oriented, then e is called oriented if it takes values in $SO(N)$ and $\theta^1 \wedge \theta^3$ is positive on M.

From the structure equations of §1.2, with respect to a Darboux frame e we have

$$d\theta^i = -\omega^i_j \wedge \theta^j \qquad \textit{and } (\omega^i_j) \in o(1,1), \tag{2.2}$$

i.e., $\omega^1_1 + \omega^3_3 = 0$ and $\omega^1_3 = \omega^3_1 = 0$. Thus the ω^i_j are the Levi-Civita conenction forms of ds^2 with respect to θ^1, θ^3.

From the exterior derivative of (2.1) we have

$$\omega^\alpha_i = h^\alpha_{ij}\theta^j, \; h^\alpha_{ij} = h^\alpha_{ji} \tag{2.3}$$

where the functions h^α_{ij} are the components of the second fundamental tensor

$$II = h^\alpha_{ij}\theta^i\theta^j \otimes e_\alpha.$$

The mean curvature vector is

$$H = \frac{1}{2}g^{ij}h^\alpha_{ij}e_\alpha = h^\alpha_{13}e_\alpha,$$

where, as in §1.1, $(g_{ab}) = L_2$ so that $(g_{ij}) = L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The immersed timelike surface f is called a string if $H = 0$.

The Levi-Civita connection of N, g induces a metric connection in the normal bundle $TM^\perp$ of f given by

$$\nabla e_\alpha = \omega^\beta_\alpha \otimes e_\beta.$$

The covariant differential of II is given by

$$\nabla II = h^\alpha_{ijk}\theta^i \otimes \theta^j \otimes \theta^k \otimes e_\alpha,$$

where the functions h^α_{ijk} are defined by

$$dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta = h^\alpha_{ijk}\theta^k. \tag{2.4}$$

From $h^\alpha_{ij} = h^\alpha_{ji}$ we have

$$h^\alpha_{ijk} = h^\alpha_{jik},$$

while from differentiating (2.3) we have

$$h^\alpha_{ijk} = h^\alpha_{ikj} - R^\alpha_{ijk}. \tag{2.5}$$

Since the metric is parallel, we have

$$\nabla H = \frac{1}{2} g^{ij} h^\alpha_{ijk} \theta^k \otimes e_\alpha = h^\alpha_{13k} \theta^k e_\alpha. \tag{2.6}$$

We say that f has parallel mean curvature vector if $\nabla H = 0$.

Applying the structure equations of N to $d\omega^1_1 = K\theta^1 \wedge \theta^3$ and to $d\omega^2_2 = K^\perp \theta^1 \wedge \theta^3$ we obtain the Gauss equations

$$K = -R_{1313} - h^2_{11} h^4_{33} - h^2_{33} h^4_{11} + 2h^2_{13} h^4_{13}, \tag{2.7}$$

where K is the Gaussian curvature of ds^2 on M; and the Ricci equations

$$K^\perp = -R_{1324} - h^2_{33} h^4_{11} + h^2_{11} h^4_{33}, \tag{2.8}$$

where $K^\perp$ is the curvature in the normal bundle. Taking the exterior derivative of (2.3) we obtain the Codazzi-Mainardi equations

$$\begin{aligned} &[dh^2_{11} + h^2_{11}(\omega^2_2 - 2\omega^1_1)] \wedge \theta^1 + h^2_{131} \theta^1 \wedge \theta^3 = \Omega^2_1 \\ &[dh^4_{33} - h^4_{33}(\omega^2_2 - 2\omega^1_1)] \wedge \theta^3 - h^4_{313} \theta^1 \wedge \theta^3 = \Omega^4_3 \end{aligned} \tag{2.9}$$

$$\begin{aligned} &[dh^4_{11} + h^4_{11}(2\omega^3_3 + \omega^4_4)] \wedge \theta^1 + h^4_{131} \theta^1 \wedge \theta^3 = \Omega^4_1 \\ &[dh^2_{33} - h^2_{33}(2\omega^3_3 + \omega^4_4)] \wedge \theta^3 - h^2_{313} \theta^1 \wedge \theta^3 = \Omega^2_3 \end{aligned} \tag{2.10}$$

The first term in each of these equations is also a covariant derivative. For example, $dh^2_{11} + h^2_{11}\ (2\omega^3_3 - \omega^4_4) = h^2_{11i}\theta^i$. From these equations one derives (2.6). We derive some additional consequences in the next section.

2.2. Change of Darboux frame.

The second fundamental tensor of f can be rewritten as

$$II = L\theta^1\theta^1 + H ds^2 + \bar{L}\theta^3\theta^3,$$

where the local normal vector fields L and $\bar{L}$ are defined by

$$L = h^{\alpha}_{11} e_{\alpha}, \qquad \bar{L} = h^{\alpha}_{33} e_{\alpha}.$$

If $\tilde{e}$ is any oriented Darboux frame along f on U, then $\tilde{e} = eK$ where K takes values in $SO(1,1) \times SO(1,1)$, so has the form $K = \operatorname{diag}(r^{-1}, t^{-1}, r, t)$, for some real functions r and t on U. Then $\tilde{\theta}^1 = r\theta^1$, and $\tilde{\theta}3 = r^{-1}\theta^3$, from which it follows that

$$\tilde{L} = r^{-2} L, \qquad \tilde{\bar{L}} = r^2 \bar{L}.$$

In general the components of the second fundamental tensor transform by

$$\tilde{h}^2_{11} = r^2 t^{-1} h^2_{11}, \quad \tilde{h}^2_{13} = t^{-1} h^2_{13}, \quad \tilde{h}^2_{33} = r^{-2} t^{-1} h^2_{33}$$
$$\tilde{h}^4_{11} = r^2 t h^4_{11}, \quad \tilde{h}^4_{13} = t h^4_{13}, \quad \tilde{h}^4_{33} = r^{-2} t h^4_{33}.$$

Thus

$$g(L, \bar{L}) \quad \text{and} \quad L \wedge \bar{L}$$

are globally defined, and the zeros of $g(L, L)$ and $g(\bar{L}, \bar{L})$ are well-defined (independent of choice of oriented Darboux frame).

It is convenient to set

$$s_+ = g(L, \bar{L}) - \frac{L \wedge \bar{L}}{e_2 \wedge e_4}$$
$$s_- = g(L, \bar{L}) + \frac{L \wedge \bar{L}}{e_2 \wedge e_4},$$

globally defined functions on M whose local expressions with respect to an oriented Darboux frame field are

$$s_- = 2h^2_{11} h^4_{33} \quad \text{and} \quad s_+ = 2h^2_{33} h^4_{11}. \tag{2.11}$$

In terms of these invariants the Gauss and Ricci equations become

$$\frac{1}{2}(s_+ + s_-) = -R_{1313} - K + g(H,H) \tag{2.12}$$
$$\frac{1}{2}(s_+ - s_-) = -R_{1324} - K^{\perp}.$$

2.3. Isotropy.

Let e be any oriented Darboux frame field along f. We say that f is isotropic at a point $p \in M$ if eitehr

$$h^2_{11}(p) = 0 = h^4_{33}(p) \qquad (-\text{spin}) \tag{2.13}$$

or

$$h^2_{33}(p) = 0 = h^4_{11}(p) \qquad (+\text{spin}) \tag{2.14}$$

This is well defined since the vanishing of any component h^α_{ij} is independent of choice of oriented Darboux frame. We say that f is isotropic if it is isotropic at every point of M.

We refer to conditions (2.13) and (2.14) as isotropy with $\pm$-spin, respectively. A reversal of the orientation in the normal bundle of f reverses the spin.

Isotropy at $p \in M$ implies that $L(p)$ and $\bar{L}(p)$ are null vectors. The converse is true if we assume the nondegeneracy condition that $L(p)$ and $\bar{L}(p)$ span the normal space of f at p. Notice that p is an umbilical point of f if and only if $L(p) = 0 = \bar{L}(p)$.

Isotropy has a geometric interpretation in terms of the hyperbola of curvature. At a point $p \in M$ consider the curve $X(t) = te_1 + t^{-1}e_3$, where e is an oriented Darboux frame at p. This curve is a constant radius hyperbola in T_pM in the sense that $ds^2(X(t), X(t)) = 2$ for all $t \in \mathbf{R}$. The hyperbola of curvature of f at p is the curve

$$II(X(t)) = t^2L + 2H + t^{-2}\bar{L}$$

in the normal space $T_pM^{\perp}$. It is a non-degenerate (i.e., does not lie in a line) constant radius hyperbola if and only if f is isotropic at p, which case its center is at $2H$ and its radius squared is $2g(L, \bar{L})$.

Although isotropy is defined without the presence of any bilagrangian structures, it is a property intimately related to them.

THEOREM 1. *Let* N, g, J *be a four-dimensional oriented neutral space with a parallel almost bilagrangian structure. Let* M, ds^2 *be an oriented neutral surface, and let* J_M *denote its unique positively oriented bilagrangian structure. If* $f : M \to N$ *is isometric and* $\pm$ *bilagrangian, then it is an isotropic string.*

Proof. Let e be an oriented Darboux frame field along f defined on a connected open set $U \subset M$, with dual coframe θ. Then θ^1, θ^3 is an oriented null coframe field in U, and $df = \theta^i e_i$. We have $\theta^1 \circ J_M = -\theta^1$ and $\theta^3 \circ J_M = \theta^3$. If f is bilagrangian (the anti-bilagrangian case is similar), then $J \circ df = df \circ J_M = -\theta^1 e_1 + \theta^3 e_3$ so that $Je_1 = -e_1$ and $Je_3 = e_3$. Thus J must satisfy either

$$Je_2 = -e_2 \quad \text{and} \quad Je_4 = e_4$$

or

$$Je_2 = e_2 \quad \text{and } Je_4 = -e_4.$$

We assume that the former case holds on U. A similar argument applies for the latter case.

Using the notation of §1.1 we have now along f that $J = J_e$ and

$$\nabla J = 2\omega_3^2(\theta^3 e_2 - \theta^4 e_1) + 2\omega_1^4(\theta^2 e_3 - \theta^1 e_4).$$

As J is parallel it follows then that $\omega_3^2 = 0 = \omega_1^4$, which written out in terms of components becomes

$$0 = h_{31}^2 = h_{33}^2 = h_{11}^4 = h_{13}^4.$$

These are precisely the conditions that f be an isotropic string. ■

THEOREM 2. *Let* $\mathbf{R}^{2,2}$ *denote* $\mathbf{R}^4$ *with the standard neutral metric* $g = 2(dx^1dx^3 + dx^2dx^4)$ *and standard orientation. Let* M, ds^2 *be an oriented neutral surface. An isometric immersion* $f : M \to \mathbf{R}^{2,2}$ *is bilagrangian with respect to some constant bilagrangian structure* J *on* $\mathbf{R}^{2,2}$ *if and only if* f *is an isotropic string.*

Proof. By Theorem 1 it suffices to prove the implication beginning with the assumption that f is an isotropic string. Let e be an oriented Darboux frame field along f on the connected open set $U \subset M$. Then f is a string means that $h^{\alpha}_{13} = 0$, and f is isotropic means that either $h^2_{33} = 0 = h^4_{11}$ or $h^2_{11} = 0 = h^4_{33}$. We assume the former case as the argument for the latter case is similar.

A constant product structure J is given on $\mathbf{R}^{2,2}$ by specifying its spectral decomposition. It will be bilagrangian if and only if these eigenspaces are null spaces. To this end consider the two-dimensional subspaces of $\mathbf{R}^4$ spanned by $e_1(p)$, $e_2(p)$ and $e_3(p)$, $e_4(p)$, for each $p \in U$. These subspaces are constant, independent of p, because

$$d(e_1 \wedge e_2) = (\omega^1_1 + \omega^2_2)e_1 \wedge e_2 + \omega^4_1(e_4 \wedge e_2 - e_1 \wedge e_3)$$

$$d(e_3 \wedge e_4) = (\omega^3_3 + \omega^4_4)e_3 \wedge e_4 + \omega^2_3(e_2 \wedge e_4 - e_3 \wedge e_1)$$

Our assumptions above amount to $\omega^4_1 = 0 = \omega^2_3$ on U. Hence, as mappings from U into the projective space $P(\Lambda_2\mathbf{R}^4)$, both $[e_1 \wedge e_2]$ and $[e_3 \wedge e_4]$ are constant. Define J by making the former subspace its -1 eigenspace, and the latter subspace its $+1$ eigenspace. From the proof of Theorem 1 we see that f is bilagrangian with respect to J and J_M. ∎

2.4. Parallel Mean curvature Vector.

THEOREM 3. *If the isometric immersion $f : M, ds^2 \to N, g$ has a nonzero parallel mean curvature vector, then the normal bundle $TM^{\perp}$ is parallelizable, thus trivial and flat.*

Proof. Let e be a local oriented Darboux frame field along f. Suppose that the mean curvature vector $H = h^2_{13}e_2 + h^4_{13}e_4$ is non-null, which implies that $g(H,H)$ is a nonzero constant. Then the normal vector field

$$\hat{H} = -h^2_{13}e_2 + h^4_{13}e_4$$

is globally defined, parallel, and $H, \hat{H}$ are linearly independent at every point of M, since $H \wedge \hat{H} = g(H,H)e_2 \wedge e_4 \neq 0$ at every point of M. Hence $TM^{\perp}$ is parallelized by $H, \hat{H}$.

Suppose now that $g(H, H) = 0$ on M. As $H(p) \neq 0$ at some point $p \in M$, it follows that H is nonzero at every point of M, because parallel translation along any curve of M is an isometry, thus nonsingular. At any point of M one of h^2_{13}, h^4_{13} must be zero, the other non-zero; Assuming that the domain of definition U of e is connected, it follows by a simple continuity argument that the same component must remain zero throughout U. As M is assumed connected, it follows from the change of frame formulas that the same component vanishes identically for any oriented Darboux frame field e.

To be definite, let us suppose that $H = h^2_{13} e_2$ for any such e. Ad h^2_{13} is never zero on U, we may replace e by another oriented Darboux frame field defined on U such that $H = e_2$ on U. M can be covered by the domains of such frame fields. As $H = e_2 = \tilde{e}_2$ for any pair of such frame fields, it follows that $e_4 = \tilde{e}_4$ as well. Thus $\hat{H} = e_4$ is a globally defined null vector, never zero, linearly independent of H. it is parallel because $0 = \nabla H = \nabla e_2 = \omega^2_2 e_2$ implies that $\omega^4_4 = -\omega^2_2 = 0$, and thus $\nabla e_4 = \omega^4_4 e_4 = 0$. Hence, $TM^\perp$ is parallelized. ∎

Let $N(\epsilon)$ denote a four-dimensional neutral space form with constant curvature ϵ.

PROPOSITION 3.4.1. *Let $f : M \to N(\epsilon)$ be an isometric immersion with paralell mean curvature vector H. Then on the open set where the argument of log is nonzero in each case, we have*

$$\Delta \log |s_-| = 2(2K - K^\perp)$$

$$\Delta \log |s_+| = 2(2K + K^\perp)$$

Proof. These follow from an application of Lemma 1.2.1 to (2.9) and (2.10), respectively. ∎

THEOREM 4. *Let $f : M, ds^2 \to N(\epsilon)$ be an isometric immersion with nonzero parallel mean curvature vector H. If the Gaussian curvature K of M is constant, then either $K = 0$ or $K = \epsilon + g(H, H)$.*

Proof. We have $K^\perp = 0$ by Theorem 3, and thus by (2.12) and that $R_{1324} = 0$ we have

$$s_+ = s_- = -K + \epsilon + g(H, H),$$

which is constant. If $K \neq \epsilon + g(H, H)$, then Proposition 3.4.1 implies that $0 = \Delta \log |s_+| = 4K$. ■

3. Neutral surfaces in $\mathbf{R}^{2,2}$.

3.1. The Grassmannian.

Let $G_{1,1}(2,2)$ denote the set of all oriented neutral planes in $\mathbf{R}^{2,2}$. The group $SO(2,2)$ acts transitively on $G_{1,1}(2,2)$, and its isotropy subgroup at $o = [\epsilon_1, \epsilon_3]$ is

$$G_0 = SO(1,1) \times SO(1,1) = \{\mathrm{diag}\,(s, t, s^{-1}, t^{-1}) : 0 \neq s, t \in \mathbf{R}\}.$$

Thus $G_{1,1}(2,2) = SO(2,2)/G_0$, which is a pseudo-Riemannian symmetric space with $SO(2,2)$-invariant metric g defined as follows.

If $\omega = (\omega^a_b)$ denotes the Maurer-Cartan form of $SO(2,2)$, then by §1.1,

$$\omega^1_3 = \omega^2_4 = \omega^3_1 = \omega^4_2 = 0$$

$$\omega^3_4 = -\omega^2_1, \quad \omega^4_3 = -\omega^1_2,$$

$$\omega^2_3 = -\omega^1_4, \quad \omega^4_1 = -\omega^3_2.$$

The symmetric left invariant bilinear form

$$Q = 2(\omega^4_3\omega^2_1 + \omega^2_3\omega^4_1)$$

on $SO(2,2)$ descends to an $SO(2,2)$-invariant neutral metric g on $G_{1,1}(2,2)$ such that $\pi^* g = Q$, where

$$\pi : SO(2,2) \to G_{1,1}(2,2) \tag{3.1}$$

is the projection. If u is any local section of (3.1), then

$$u^*\{\omega^4_3, \omega^2_3, \omega^2_1, \omega^4_1\}$$

is a local oriented null coframe field for g in $G_{1,1}(2,2)$, where the orientation is defined by

$$\nu = \omega_3^4 \wedge \omega_1^2 \wedge \omega_3^2 \wedge \omega_1^4.$$

There is an $SO(2,2)$-invariant bilagrangian structure J defined on $G_{1,1}(2,2)$ by

$$\omega_3^4 \circ J = -\omega_3^4, \quad \omega_3^2 \circ J = -\omega_3^2, \quad \omega_1^2 \circ J = \omega_1^2, \quad \omega_1^4 \circ J = \omega_1^4.$$

It is integrable by Proposition 1.1.1, since by the structure equations of $SO(2,2)$,

$$\begin{aligned} d\omega_3^4 &= -(\omega_4^4 - \omega_3^3) \wedge \omega_3^4, \quad d\omega_3^2 = -(\omega_2^2 - \omega_3^3) \wedge \omega_3^2 \\ d\omega_1^2 &= (\omega_4^4 - \omega_3^3) \wedge \omega_1^2, \quad d\omega_1^4 = (\omega_2^2 - \omega_3^3) \wedge \omega_1^4, \end{aligned} \tag{3.2}$$

This local product structure is global. Let $B_+ = B(2)$, as defined in §1.1, and let $B_- = \{A \in SO(2,2) : AI_- = I_-A\}$, where $I_- = \mathrm{diag}(-1,1,1,-1)$. It is evident that B_- is conjugate in $O(2,2)$ to B_+. Let $S_\pm = SO(2,2)/B_\pm$, and let

$$\pi_\pm : SO(2,2) \to SO(2,2)/B_\pm = S_\pm$$

denote the projections. The symmetric left invariant bilinear forms

$$Q_- = 2\omega_3^4\omega_1^2, \quad Q_+ = 2\omega_3^2\omega_1^4$$

descend to $SO(2,2)$-invariant neutral metrics g_- and g_+ on S_- and S_+, respectively. Each has constant Gaussian curvature equal to 2, as one sees from (3.2) and the equations

$$d(\omega_4^4 - \omega_3^3) = 2\omega_3^4 \wedge \omega_1^2, \quad d(\omega_2^2 - \omega_3^3) = 2\omega_2^2 \wedge \omega_1^4.$$

We orient these spaces by

$$\nu_- = \omega_3^4 \wedge \omega_1^2 \quad \text{and} \quad \nu_+ = \omega_3^2 \wedge \omega_1^4,$$

respectively, which uniquely determines bilagrangian structures on each.

LEMMA 3.1.1. *There is an $SO(2,2)$-equivariant isometry*

$$\psi = (\psi_-, \psi_+) : G_{1,1}(2,2) \to S_- \times S_+.$$

Geometrically, given an oriented neutral plane $P \subset \mathbf{R}^{2,2}$, then $\psi(P)$ is the pair of bilagrangian structures (J_-, J_+) on $\mathbf{R}^{2,2}$ whose restriction to P is the unique bilagrangian structure determined by its orientation, and whose restriction to the orthogonal complement $P^\perp$ is, respectively, minus or plus the bilagrangian structure determined by its orientation (inherited from P and $\mathbf{R}^{2,2}$).

Proof. Let $e = (e_1, \dots, e_4) \in SO(2,2)$ be an oriented null frame such that e_1, e_3 is an oriented null frame for P. Then e_2, e_4 is an oriented null frame for $P^\perp$, and $J_+ = J_e$ (as defined in §1.1) while $J_- = J_{e'}$, where $e' = (e_1, e_4, e_3, e_2)$. The map ψ is an isometry since $Q = Q_- + Q_+$. ∎

3.2. The Gauss Map.

Let $f : M \to \mathbf{R}^{2,2}$ be an isometric immersion of an oriented neutral surface. define its Gauss map to be

$$\gamma_f : M \to G_{1,1}(2,2),$$

where $\gamma_f(p)$ is the tangent plane $f_* T_p M$ parallel translated to the origin of $\mathbf{R}^{2,2}$. From the splititng of the Grassmannian the Gauss map factors into $\gamma_f = (\varphi_-, \varphi_+)$, where $\varphi_\pm = \psi_\pm \circ \gamma_f$. We call $\varphi_\pm$ the reflector maps of f. These are the analogues of the twistor maps for surfaces in $\mathbf{R}^4$. The name change seems justified by the fact that a bilagrangian structure on an oriented neutral plane is a reflection, whereas an orthogonal complex structure on an oriented Euclidean plane is a quarter turn, that is, a twist.

THEOREM 5. *Let H, γ_f, $\varphi_\pm$ denote the mean curvature vector, Gauss map and reflector maps, respectively, of f. Then:*

(1) f is isotropic with $\pm$spin if and only if $\varphi_\pm$ is bilagrangian.

(2) f is totally umbilical if and only if γ_f is bilagrangian.

(3) f is a string if and only if φ_- is anti-bilagrangian if and only if φ_+ is antibilagrangian if and only if γ_f is anti-bilagrangian.

(4) If f is a string then $\varphi_\pm^* g_\pm = \frac{1}{2} s_\pm ds^2$.

(5) If f is isotropic with ±spin then $\varphi_\pm^* g_\pm = \frac{1}{2} g(H,H) ds^2$.

(6) f is an isotropic string with ±spin if and only if $\varphi_\pm$ is constant.

(7) $\varphi_\pm$ is harmonic if and only if $\nabla H : TM \to TM^\perp$ is ±bilagrangian.

(8) f has paralell mean curvature vector if and only if γ_f is harmonic.

(9) $\varphi_+^* \nu_+ + \varphi_-^* \nu_- = K \nu_M$.

(10) $\varphi_+^* \nu_+ - \varphi_-^* \nu_- = K^\perp \nu_M$.

Proof. Let u be a local section of (3.1). Then $u \circ \gamma_f = e$ is a local oriented Darboux frame field along f. Consider the local oriented null coframe field

$$u^* \{\omega_3^4, \omega_3^2, \omega_1^2, \omega_1^4\} \tag{3.3}$$

in $G_{1,1}(2)$ and let $\{F_1, \dots, F_4\}$ denote the oriented null frame field which is its dual. We may regard $u^*\{\omega_3^4, \omega_1^2\}$ as a local oriented null coframe field in S_-, and $u^*\{\omega_3^2, \omega_1^4\}$ as one in S_+, with dual frame fields $\{F_1, F_3\}$ and $\{F_2, F_4\}$, respectively. (To be more precise, oriented null coframes exist in $S_\pm$ which pull back by $\psi_\pm^*$ to these forms in $G_{1,1}(2,2)$.) The metric on $G_{1,1}(2,2)$ is $g = g_- + g_+$, where $g_- = 2\omega_3^4 \omega_1^2$ and $g_+ = 2\omega_3^2 \omega_1^4$. The positive volume element in $G_{1,1}(2,2)$ is $\nu = \nu_- + \nu_+$, where (omitting u^* and $\psi_\pm^*$) $\nu_- = \omega_3^4 \wedge \omega_1^2$ and $\nu_+ = \omega_3^2 \wedge \omega_1^4$ are the positive volume, elements in $S_\pm$, respectively.

Since e is an oriented Darboux frame field along f, we have $\gamma_f = \{e_1, e_3\}$, $\varphi_+ = J_e$ and $\varphi_- = J_-$, where J_- is given by $J_- e_1 = -e_1$, $J_- e_2 = e_2$, $J_- e_3 = e_3$, $J_- e_4 = -e_4$.

Then

$$d\varphi_- = \omega_3^4 F_1 + \omega_1^2 F_3 = (h_{31}^4 \theta^1 + h_{33}^4 \theta^3) F_1 + (h_{11}^2 \theta^1 + h_{13}^2 \theta^3) F_3, \tag{3.4}$$

$$d\varphi_+ = \omega_3^2 F_2 + \omega_1^4 F_4 = (h_{31}^2 \theta^1 + h_{33}^2 \theta^3) F_2 + (h_{11}^4 \theta^1 + h_{13}^4 \theta^3) F_4, \tag{3.5}$$

and $d\gamma_f = d\varphi_- + d\varphi_+$. Furthermore,

$$\varphi_-^* g_- = 2\omega_3^4 \omega_1^2 = (h_{31}^4 h_{13}^2 + h_{33}^4 h_{11}^2) ds^2 + 2h_{31}^4 h_{11}^2 \theta^1 \theta^1 + 2h_{33}^4 h_{13}^2 \theta^3 \theta^3, \tag{3.6}$$

and

$$(3.7)\quad \varphi_+^* g_+ = 2\omega_3^2\omega_1^4 = (h_{31}^2 h_{13}^4 + h_{33}^2 h_{11}^4)ds^2 + 2h_{31}^2 h_{11}^4 \theta^1\theta^1 + 2h_{33}^2 h_{13}^4 \theta^3\theta^3,$$

where θ^1, θ^3 is the oriented null coframe field induced in M by e.

If J denotes the bilagrangian structure on one of $G_{1,1}(2,2)$ or $S_\pm$, then $JF_i = -F_i$ and $JF_\alpha = F_\alpha$. The structure on M satisfies $\theta^1 \circ J_M = -\theta^1$ and $\theta^3 \circ J_M = \theta^3$. Hence φ_- is bilagrangian $\Leftrightarrow J \circ d\varphi_- = d\varphi_- \circ J_M \Leftrightarrow h_{33}^4 = 0 = h_{11}^2 \Leftrightarrow f$ is isotropic with $-$spin by (2.13). The proof for φ_+ goes the same way, and φ_f is bilagrangian $\Leftrightarrow \varphi_-$ and γ_+ are bilagrangian $\Leftrightarrow f$ is isotropic with $+$ and $-$ spin $\Leftrightarrow f$ is totally umbilical.

Similarly, φ_- is anti-bilagrangian $\Leftrightarrow J \circ d\varphi_- = -d\varphi_- \circ J_M \Leftrightarrow h_{13}^4 = 0 = h_{13}^2 \Leftrightarrow H = 0 \Leftrightarrow f$ is a string. The proofs for φ_+ and γ_f are similar.

If f is a string, then (2.11), (3.6) and (3.7) prove (4). If f is isotropic with $\pm$spin, then (5) follows from (3.6) and (3.7); Clearly (6) follows from (3.4) and (3.5), respectively.

Using (2.4), (2.6), (3.2) and (3.4)-(3.7), we compute the tension fields of $\varphi_\pm$ to be

$$\tau(\varphi_-) = H_3^4 F^1 + H_1^2 F_3, \qquad \tau(\varphi_+) = H_3^2 F_2 + H_1^4 F_4.$$

The covariant differential of H is a bundle map $\nabla H : TM \to TM^\perp$, given in terms of e by $\nabla H = H_i^\alpha \theta^i \otimes e_\alpha$. Then ∇H is bilagrangian $\Leftrightarrow J \circ \nabla H = \nabla H \circ J \Leftrightarrow H_1^2 = 0 = H_1^4 \Leftrightarrow \varphi_+$ is harmonic. Similary, ∇H is antibilagrangian $\Leftrightarrow H_1^2 = 0 = H_3^4 \Leftrightarrow \varphi_-$ is harmonic.

Since $\tau(\varphi_-)$ and $\tau(\varphi_+)$ are linearly independent, γ_f is harmonic $\Leftrightarrow 0 = \tau(\gamma_f) = \tau(\varphi_-) + \tau(\varphi_+) \Leftrightarrow \varphi_-$ and φ_+ are harmonic $\Leftrightarrow H$ is parallel.

Since $\varphi_\pm = \psi_\pm \circ \gamma_f$, we have

$$\varphi_+^* \nu_+ = \gamma_f^* u^* (\omega_3^2 \wedge \omega_1^4) = (h_{31}^2 h_{13}^4 - h_{33}^2 h_{11}^4)\nu_M$$

and

$$\varphi_-^* \nu_- = \gamma_f^* u^* (\omega_3^4 \wedge \omega_1^2) = (h_{31}^4 h_{13}^2 - h_{33}^4 h_{11}^2)\nu_M.$$

Now 9) and 10) follow from these equations together with (2.7) and (2.8). ∎

4. The presence of holonomy.

4.1. The Grassmann and Reflector Bundles.

If a four-dimensional neutral space N, g has holonomy, then there is no unambiguous way to parallel transport tangent 2-planes to a fixed origin of N. In order to generalize the Gauss map for an oriented neutral surface

$$f : M, ds^2 \to N, g \tag{4.1}$$

we consider sections of the Grassmann bundle

$$G_{1,1}(N) = \{(p, P) : P \text{ is an oriented neutral plane in } T_pN\}.$$

Using the standard action of $SO(2,2)$ on $G_{1,1}(2,2)$, we have

$$G_{1,1}(N) = SO(N) \times_{SO(2,2)} G_{1,1}(2,2) = SO(N)/SO(1,1) \times SO(1,1).$$

The Gauss lift of f is given by

$$\gamma_f : M \to G_{1,1}(N)$$

where $\gamma_f(p) = (f(p), f_*T_pM)$.

The splitting of $G_{1,1}(2,2)$ given in Lemma 3.1.1 leads us to consider the reflector bundles

$$r_\pm : Z_\pm \to N \tag{4.2}$$

defined by

$$\begin{aligned} Z_\pm = \{(p, J) : J &\text{ is a bilagrangian structure on} \\ T_pN, g_p &\text{ of } \pm \text{ orientation}\} \\ = SO(N) &\times_{SO(2,2)} S_\pm = SO(N)/B_\pm, \end{aligned} \tag{4.3}$$

where $r_\pm(p, J) = p$. We let

$$\sigma_\pm : SO(N) \to Z_\pm = SO(N)/B_\pm \tag{4.4}$$

denote the projection. There are natural maps

$$\psi_\pm : G_{1,1}(N) \to Z_\pm, \tag{4.5}$$

where $\psi_\pm(p, P) = (p, J_\pm)$. To describe $J_\pm$, let e be an oriented null frame of T_pN such that $P = \{e_1, e_3\}$. Then as in Lemma 3.1.1, $J_+ = J_e$ while $J_- = J_{e'}$, where $e' = (e_1, e_4, e_3, e_2)$.

The reflector lifts of f are $\varphi_\pm = \psi_\pm \circ \gamma_f$, so that

$$\varphi_\pm : M \to Z_\pm \tag{4.6}$$

are sections of (4.2) in the sense that $r_\pm \circ \varphi_\pm = f$ (i.e., they are sections of $f^{-1}Z_\pm \to M$). In terms of a local oriented Darboux frame field e along f they are given by $\varphi_\pm = (f, J_\pm)$.

4.2. *The Metrics on* $Z_\pm$.

To construct our 1-parameter family of neutral metrics on $Z_\pm$, recall the notation of §1.2: $\theta = (\theta^a)$ is the canonical form and $\omega = (\omega^a_b)$ is the Levi-Civita connection form on $SO(N)$. Consider the symmetric bilinear forms

$$Q_+ = 2\omega^2_3\omega^4_1 \quad \text{and} \quad Q_- = 2\omega^4_3\omega^2_1$$

on $SO(N)$. For any nonzero real number t, the form

$$2(\theta^1\theta^3 + \theta^2\theta^4) + tQ_\pm \tag{4.7}$$

descends to a neutral metric g_t on $Z_\pm = SO(N)/B_\pm$.

It u is any local section of (4.4), then u^* of

$$\theta^1, \theta^2, r\omega^2_3, \theta^3, \theta^4, \epsilon r\omega^4_1 \tag{4.8}$$

and

$$\theta^1, \theta^4, r\omega^4_3, \theta^2, \theta^3, \epsilon r\omega^2_1 \tag{4.9}$$

are oriented null coframe fields for g_t on Z_+ and Z_-, respectively. Here $r > 0$, $\varepsilon = \pm 1$ and $\epsilon r^2 = t$.

LEMMA 4.2.1. *Let u be any local section of (4.4). The Levi-Civita connection forms of g_t on Z_+ with respect to the null coframe field*

$u^*(4.8) = (\varphi^1, \ldots, \varphi^6)$ *are* u^* *of the forms on* $SO(N)$ *(here* $i, j = 1, 2$*):*

$$\varphi^i_j = \omega^i_j - \frac{r}{2}(R^4_{12+ij}\epsilon\varphi^3 + R^2_{32+ij}\varphi^6) = -\varphi^{3+j}_{3+i}$$

$$\varphi^i_3 = -\frac{r\varepsilon}{2}(R^4_{12+ij}\varphi^j + R^4_{12+i2+j}\varphi^{3+j}) = -\varphi^6_{3+i}$$

$$\varphi^i_{3+j} = \omega^i_{2+j} - \frac{r}{2}(R^4_{12+i2+j}\epsilon\varphi^3 + R^2_{32+i2+j}\varphi^6)$$

$$\varphi^i_6 = -\frac{r}{2}(R^2_{32+ij}\varphi^j + R^2_{22+i2+j}\varphi^{3+j}) = -\varphi^3_{3+i}$$

$$\varphi^3_i = -\frac{r}{2}(R^2_{3ji}\varphi^j + R^2_{32+ji}\varphi^{3+j}) = -\varphi^{3+i}_6$$

$$\varphi^3_3 = \omega^2_2 - \omega^3_3 = -\varphi^6_6$$

$$\varphi^3_6 = 0 = \varphi^6_3$$

$$\varphi^{3+i}_j = \omega^{2+i}_j - \frac{r}{2}(R^4_{1ij}\epsilon\varphi^3 + R^2_{3ij}\varphi^6)$$

$$\varphi^{3+i}_3 = -\frac{r\epsilon}{2}(R^4_{1ij}\varphi^j + R^4_{1i2+j}\varphi^{3+j}) = -\varphi^6_i.$$

The Levi-Civita connection forms of g_t *on* Z_- *with respect to the null coframe field* u^* *(4.9) are given by similar formulas on* $SO(N)$.

Proof. Using the structure equations of $SO(N)$, one checks directly that $d\varphi^p = -\varphi^p_q \wedge \varphi^q$ and that $(\varphi^p_q) \in o(3,3)$, for $1 \leq p, q \leq 6$. ■

Computing the Ricci tensor of g_t as in [JR 1], and adapting the proof of Theorem 1 of [5] to the neutral case, one can prove

THEOREM 4. *Let* N, g *be an oriented neutral space which is Einstein and self-dual (respectively, anti-self-dual). Then* g_t *on* Z_- *(respectively, on* Z_+*) is Einstein if and only if* $t = \dfrac{12}{s}$ *or* $t = \frac{6}{s}$*, where* s *is the scalar curvature of* g *on* N.

4.3. Bilagrangian Structures on $Z_\pm$.

On the reflector spaces $Z_\pm, g_t$ there exist a pair of bilagrangian structures J_1 and J_2. Described geometrically, at a point $(p, J) \in Z_\pm$ the tangent space decomposes as $T_{(p,J)}Z_\pm = T_pN \oplus T_JS_\pm$ so that

$J_i|T_pN = J$, for $i = 1, 2$, and $J_1|T_J S_\pm = J_\pm$ while $J_2|T_J S_\pm = -J_\pm$, where $J_\pm$ is the natural bilagrangian structure on $S_\pm$ (cf. §3.2). An analytical description of J_1 and J_2 is given in the proof below.

THEOREM 5. *The bilagrangian structure J_1 on Z_+ (respectively, Z_-) is integrable if and only if N, g is anti-self-dual (respectively, self-dual). The bilagrangian structure J_2 on $Z_\pm$ is never integrable.*

Proof. Let u be a local section of (4.4); so that $(\varphi^1, \ldots, \varphi^6) = u^*(4.8)$ is a local oriented null coframe field in Z_+. Let $E = (E_1, \ldots, E_6)$ be the dual frame field. Then $J_1 = J_E$ while $J_2 = J_{E'}$, where $E' = (E_1, E_2, E_6, E_4, E_5, E_3)$.

The $+1$ and -1 eigenspaces of J_2 are the subspaces annihilated by $(\varphi^1, \varphi^2, \varphi^6)$ and $(\varphi^4, \varphi^5, \varphi^3)$, respectively. By the structure equations of $SO(N)$,

$$d\varphi^1 \equiv -\varphi^3 \wedge \varphi^5 \pmod{\varphi^1, \varphi^2}$$

which is never zero modulo $(\varphi^1, \varphi^2, \varphi^6)$. Hence J_2 is never integrable.

The ± 1 eigenspaces of J_1 are the subspaces annihilated by $(\varphi^1, \varphi^2, \varphi^3)$ and $(\varphi^4, \varphi^5, \varphi^6)$, respectively. Now

$$d\varphi^i \equiv 0 \pmod{\varphi^1, \varphi^2, \varphi^3},\ i = 1, 2$$

$$d\varphi^3 \equiv \Omega^2_3 \equiv R^2_{334}\varphi^4 \wedge \varphi^5 \pmod{\varphi^1, \varphi^2, \varphi^3}$$

$$d\varphi^{3+i} \equiv 0 \pmod{\varphi^4, \varphi^5, \varphi^6},\ i = 1, 2$$

$$d\varphi^6 \equiv \Omega^4_1 \equiv R^4_{112}\varphi^1 \wedge \varphi^2 \pmod{\varphi^4, \varphi^5, \varphi^6}$$

Hence, J_1 is integrable if and only if $R^2_{334} = 0 = R^4_{112}$ on $SO(N)$, because this must hold for any section u. From §1.4, we see that $R^2_{334} = R^{21}_{34} = -A^2_3$ and $R^4_{112} = R^{43}_{12} = -A^3_2$. Thus, J_1 is integrable if and only if

$$A^3_2 = 0 = A^2_3 \tag{4.10}$$

on $SO(N)$.

If $u \in SO(N)$ and $K \in SO(2, 2)$, then from §1.4,

$$A(uK) = K_+^{-1} A(u) K_+ \tag{4.11}$$

where K_+ is the projection of $SO(2,2)$ onto $SO(2,1)$ defined in §1.4.

By an elementary calculation, using (4.11) as K_+ ranges over $SO(2,1)$, one shows that for fixed $u \in SO(N)$, (4.10) holds for all $K \in SO(2,2)$ if and only if

$$A(u) = \lambda(u) I_3,$$

where λ is a function on $SO(N)$ constant on each fiber, thus a function on N. But then Trace $A = 3\lambda = \frac{s}{4}$, so that $\lambda = \frac{s}{12}$. Hence J_1 is integrable if and only if N, g is anti-self-dual by Proposition 1.4.1.

The proof for Z_- goes the same way. The spectral decomposition of J_1 has equations $u^*(\theta^1, \theta^4, \omega_3^4)$ and $u*(\theta^2, \theta^3, \omega_1^2)$. Using the structure equations of $SO(N)$, we see as above that J_1 on Z_- is integrable if and only if $R^4_{323} = 0 = R^2_{114}$ on $SO(N)$. But $R^4_{323} = R^{41}_{23} = C^2_3$ and $R^2_{114} = R^{23}_{14} = -C^3_2$. As above then, J_1 on Z_- is integrable if and only if $C = \frac{s}{12} I_3$ on $SO(N)$ if and only if N, g is self-dual. ■

4.4. Conformal Invariance.

Given an oriented neutral space N, g consider the conformally related metric $\tilde{g} = \lambda^2 g$, where λ is any smooth positive function on N. Since the null spaces of $\tilde{g}$ and g are the same, it follows that the reflector bundless $Z_\pm$ for $\tilde{g}$ are the same as those for g.

Using the bundle isomorphism

$$F : SO(N, \tilde{g}) \to SO(N, g)$$

defined by $F(\tilde{e}_1, \ldots, \tilde{e}_4) = \frac{1}{\lambda}(\tilde{e}_1, \ldots, \tilde{e}_4)$, the argument of §5 of [6] carries over to prove the following theorem.

THEOREM. *The bilagragian structure J_1 on $Z_\pm$ does not change with a conformal change of metric on N. The structure J_2 is invariant only under homothetic change of metric on N.*

Remark. Both J_1 and J_2 are unchanged when g is replaced by $-g$.

4.5. The Reflector Lifts.

Let f be an isometric immersion (4.1). We consider now the generalizations of Theorem 1 in §3.2. All properties but (7) and (8) carry over to a general N, g.

Observe that the 2-forms $\omega_2^2 \wedge \omega_1^4$ and $\omega_3^4 \wedge \omega_1^2$ on $SO(N)$ descend to 2-forms $\nu_\pm$ on $Z_\pm$, respectively, so that for any section u of (4.4) we have

$$u^*(\omega_3^2 \wedge \omega_1^4) = \nu_+ \quad \text{and} \quad u^*(\omega_3^4 \wedge \omega_1^2) = \nu_- .$$

THEOREM 7. *Let $\varphi_\pm : M \to Z_\pm$ be the reflector lifts (4.6) of f; Then*

(1) f is isotropic with $\pm$spin if and only if $\varphi_\pm$ is J_1-bilagrangian.

(2) f is totally umbilical if and only if both φ_+ and φ_- are J_1-bilagrangian.

(3) f is a string if and only if φ_- is J_2-bilagrangian if and only if φ_+ is J_2-bilagragian.

(4) If f is a string, then $\varphi_\pm^* g_t = \left(1 + \frac{t}{2} s_\pm\right) ds^2$.

(5) If f is isotropic with $\pm$spin, then $\varphi_\pm^* g_t = \left(1 + \frac{t}{2} g(H, H)\right) ds^2$.

(6) f is an isotropic string with $\pm$spin if and only if $\varphi_\pm$ is horizontal.

(7) $\varphi_+^* \nu_+ + \varphi_-^* \nu_- = K \nu_M$.

(8) $\varphi_+^* \nu_+ - \varphi_-^* \nu_- = K^\perp \nu_M$.

Proof. Let u be a local section of (4.4); Then $e = u \circ \varphi_\pm$ is a local Darboux frame field along f. Using the coframes (4.8) and (4.9) we have (omitting u^* and e^* as usual)

$$\begin{aligned} \varphi_+^* \theta^i = \theta^i, \ \varphi_+^* \theta^{1+i} = 0, \ i = 1, 3 \\ \varphi_+^* \omega_3^2 = h_{3i}^2 \theta^i, \ \varphi_+^* \omega_1^4 = h_{1i}^4 \theta^i \end{aligned} \tag{4.12}$$

Thus, by (4.7),

$$\varphi_+^* g_t = (1 + t(h_{31}^2 h_{13}^4 + h_{33}^2 h_{11}^4)) ds^2 + 2t(h_{31}^2 h_{11}^4 \theta^1 \theta^1 + h_{33}^2 h_{13}^4 \theta^3 \theta^3). \tag{4.13}$$

Similarly,

$$\begin{aligned} \varphi_-^*\theta^i &= \theta^i,\ \varphi_+^*\theta^{1+i},\ i=1,3 \\ \varphi_-^*\omega_3^4 &= h_{3i}^4\theta^i,\ \varphi_-^*\omega_1^2 = h_{1i}^2\theta^i \end{aligned} \tag{4.14}$$

and thus

$$\varphi_-^* g_t = (1 + t(h_{31}^4 h_{13}^2 + h_{33}^4 h_{11}^2))ds^2 + 2t(h_{31}^4 h_{11}^2 \theta^1\theta^1 + h_{33}^4 h_{13}^2 \theta^3\theta^3). \tag{4.15}$$

Let $E = (E_1, \ldots, E_6)$ be the frame field in Z_+ dual to $u^*(4.8)$. Then

$$d\varphi_+ = (E_1 + rh_{31}^2 E_3 + \epsilon r h_{11}^4 E_6)\theta^1 + (E_4 + rh_{33}^2 E_3 + \epsilon r h_{13}^4 E_6)\theta^3. \tag{4.16}$$

On Z_+, we have $J_1 = J_E$, so that

$$J_1 \circ d\varphi_+ = (-E_1 - rh_{31}^2 E_3 + \epsilon r h_{11}^4 E_6)\theta^1 + (E_4 - rh_{33}^2 E_3 + \epsilon r h_{13}^4 E_6)\theta^3. \tag{4.17}$$

Since $\theta^1 \circ J_M = -\theta^1$, and $\theta^3 \circ J_M = \theta^3$, we have

$$d\varphi_+ \circ J_M = (-E_1 r h_{31}^2 E_3 - \epsilon r h_{11}^4 E_6)\theta^1 + (E_4 + rh_{33}^2 E_3 + \epsilon r h_{13}^4 E_6)\theta^3. \tag{4.18}$$

Therefore, φ_+ is J_1-bilagrangian $\Leftrightarrow h_{11}^4 = 0 = h_{33}^2 \Leftrightarrow f$ is isotropic with +spin by (2.14). This proves (1) for φ_+. The proof φ_- is similar.

For the proof of (2), recall from §2.3 that f is totally umbilical $\Leftrightarrow h_{11}^\alpha = 0 = h_{33}^\alpha$ for $\alpha = 2,4 \Leftrightarrow$ both φ_+ and φ_- are J_1-bilagrangian.

On Z_+, $J_2 = J_{E'}$, where $E' = (E_1, E_2, E_6, E_4, E_5, E_3)$, so that

$$J_2 \circ d\varphi_+ = (-E_1 + rh_{31}^2 E_3 - \epsilon r h_{11}^4 E_6)\theta^1 + (E_4 + rh_{33}^2 E_3 - \epsilon r h_{13}^4 E_6)\theta^3. \tag{4.19}$$

Hence, comparing (4.18) and (4.19) we see that φ_+ is J_2-bilagrangian $\Leftrightarrow h_{31}^2 = 0 = h_{13}^4 \Leftrightarrow H = 0 \Leftrightarrow f$ is a string. This proves (3) for φ_+. The proof of (3) for φ_- is similar.

Assertions (4) and (5) follows from (4.13) and (4.15), respectively.

To prove assertion (6), recall that by definition φ_+ is horizontal means that

$$d\varphi_+ \subset \text{ span } \{E_1, E_2, E_4, E_5\},$$

which by (4.16) is equivalent to f being an isotropic string with ±spin. We observe further that if $\varphi_\pm$ is horizontal, then it is J_1-bilagrangian. The proof for φ_- is completely similar.

By (4.12) and (4.14),

$$\varphi_+^*\nu_+ = (h_{31}^2 h_{13}^4 - h_{33}^2 h_{11}^4)\nu_M$$
$$\varphi_-^*\nu_- = (h_{31}^4 h_{13}^2 - h_{33}^4 h_{11}^2)\nu_M.$$

Thus (7) and (8) follow from these equations combined with (2.7) and (2.8). ■

To generalize the remaining statements of Theorem 1 requires additional assumptions about N;

THEOREM 9. *Let $f : M, ds^2 \to N, g$ be an isometric immersion into a neutral Einstein space N, g. let $\varphi_\pm : M, ds^2 \to Z_\pm, g_t$ be the reflector lifts. Suppose that N, g is anti-self-dual (respectively, self-dual). If $st = 24$, then φ_+ (respectively, φ_-) is harmonic if and only if ∇H is bilagrangian. If $st \neq 24$, then φ_+ (respectively, φ_-) is harmonic if and only if f is a string.*

Proof. We summarize the calculations only for φ_+, as those for φ_- are very similar. To calculate the tension field $\tau(\varphi_+)$, let u be a section of (4.4) and use the local null coframe field for g_t

$$(\varphi^1, \dots, \varphi^6) = u^*(\theta^1, \theta^2, r\omega_3^2, \theta^3, \theta^4, \epsilon r\omega_1^4),$$

where, as in (4.8), $r > 0$, $\epsilon = \pm 1$ and $\epsilon r^2 = t$. Let $(E_1, \dots, E_6)$ denote the dual frame field in Z_+. The corresponding Levi-Civita connection forms for g_t are given by Lemma 4.2.1.

If we write

$$d\varphi_+ = \sum_{p=1}^{6} (b_1^p\theta^1 - b_3^p\theta^3)E_p$$

and compare this with (4.16) we see that

$$\begin{aligned} &b_1^1 = 1,\ b_3^1 = 0,\ b_1^2 = 0,\ b_1^3 = rh_{31}^2,\ b_3^3 = rh_{33}^2 \\ &b_1^4 = 1,\ b_3^4 = 1,\ b_1^5 = 0 = b_3^5,\ b_1^6 = \epsilon rh_{11}^4,\ b_3^6 = \epsilon rh_{13}^4 \end{aligned} \tag{4.20}$$

Then

$$\tau(\varphi_+) = \sum_{p=1}^{6} b_{13}^p E_p,$$

where

$$db_1^p - b_1^p\omega_1^1 + b_3^p\omega_1^3 + \sum_{q=1}^{6} b_1^q(\varphi_+^*\varphi_q^p) = b_{11}^p\theta^1 + b_{13}^p\theta^3.$$

Carrying out these calculations and using (1.4) in §1.4, we find that

$$b_{13}^1 = -\frac{t}{2}\left(\frac{A_1^3 + B_3^1}{\sqrt{2}}h_{33}^2 + \frac{A_1^2 + B_2^1}{\sqrt{2}}h_{13}^4\right)$$

$$b_{13}^2 = h_{13}^2\left(1 - \frac{t}{2}A_3^3\right) - \frac{t}{2}(h_{33}^2 B_3^2 + h_{11}^4 A_3^2 + h_{13}^4 B_2^2)$$

$$b_{13}^3 = r\left(h_{313}^2 - \frac{1}{2}\frac{A_1^2 + B_2^1}{\sqrt{2}}\right)$$

$$b_{13}^4 = \frac{t}{2}\left(h_{11}^4\frac{A_1^2 + B_2^1}{\sqrt{2}} + h_{31}^2\frac{A_1^3 + B_3^1}{\sqrt{2}}\right)$$

$$b_{13}^5 = h_{13}^4\left(1 - \frac{t}{2}A_2^2\right) - \frac{t}{2}(h_{33}^2 A_2^3 + h_{31}^2 B_3^3 + h_{11}^4 B_2^3)$$

$$b_{13}^6 = \epsilon r\left(h_{113}^4 - \frac{1}{2}\frac{A_1^3 + B_3^1}{\sqrt{2}}\right).$$

By (2.5) and (1.4) we have that

$$h_{113}^4 = h_{131}^4 + \frac{A_1^3 B_3^1}{\sqrt{2}}.$$

In terms of the oriented Darboux frame field $e = u \circ \varphi_+$ along f, we have

$$\nabla H : TM \to TM^\perp$$

given by (reintroducing the index ranges $i = 1,3$ and $\alpha = 2,4$)

$$\nabla H = H_i^\alpha \theta^i e_\alpha = h_{13i}^\alpha \theta^i e_\alpha,$$

where the last equality comes from (2.6). The orientations (from M and N) and neutral metrics in TM and $TM^\perp$ determine bilagrangian structures J_M and $J_\perp$. Then ∇H is bilagrangian means $\nabla H \circ J_M = J_\perp \circ \nabla H$, which occurs if and only if $H_1^4 = 0 = H_3^2$.

Suppose now that N, g is anti-self-dual and Einstein. Then by Proposition 1.4.1, we have $A = \frac{s}{12}$ and $B = 0$. Thus

$$\tau(\varphi_+) = h_{13}^2\left(1 - \frac{ts}{24}\right)E_2 + rH_3^2 E_3 + h_{13}^4\left(1 - \frac{ts}{24}\right)E_5 + \epsilon r H_1^4 E_6.$$

The theorem now follows from this formula, since by definition, φ_+ is harmonic if and only if $\tau(\varphi_+) = 0$. ∎

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