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ON PROJECTIVE MANIFOLDS ADMITTING 3-GONAL OR 4-GONAL CURVE SECTIONS

SONIA BRIVIO (Milano) (*)

Let X be a projective manifold, of dimension $n \geq 3$, and L a very ample line bundle on X. In this paper we investigate the pairs (X, L) with the following property: there exists a transversal intersection of $n-1$ members of $|L|$ which is a smooth d-gonal curve ($d = 3, 4$). We prove, under suitable assumptions, that $n \leq 4$ or 5 (resp. if $d = 3$ or $d = 4$) and (X, L) is a fibration over $\mathbf{P}^1$ with genera; fibre a rational cubic (resp. quartic) scroll.

Introduction.

Let X be a projective manifold, of dimension $n \geq 2$, and let L be a very ample line bundle on it; suppose that there exists a smooth d-gonal curve C (see sec. 0), which is the transverse intersection of $(n-1)$ general elements of $|L|$.

For $d \leq 2$ such manifolds are completely classified. Actually $d = 1$ means that X has sectional genus $g(L) = 0$ and the corresponding classification is a classical result (e.g. see [4]). For $d = 2$ either $g(L) \leq 1$

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(see [4]) or C is a hyperelliptic curve (see [10]). For $d \geq 3$ the most interesting results, due to Serrano, [5] th. (3.13) (resp. (3.15)), are concerned with regular surfaces ($n = 2$) with a trigonal (resp. 4-gonal) hyperplane section. In most cases, under suitable assumptions e.g. see (0.4) and (0.5), (S, L) is rational and ruled respectively by rational cubics or quartics.

In this paper we deal with higher dimensional manifolds in cases $d = 3$ and $d = 4$. Our results are stated in th.s (1.1) for $d = 3$ and (1.3) for $d = 4$. The idea of the proof is the following. Let S be a smooth surface which is the transverse intersection of $(n-2)$ general elements of $|L|$, then S contains a trigonal (resp. 4-gonal) curve, so that to the pair (S, L) the above results apply. Now the key point is to extend ruling $\phi : S \to \mathbf{P}^1$ to the whole X. The first extension to threefolds is obtained by using a result of Serrano, [6 th. (5.2)], up to impose some numerical restrictions to the pair (X, L); the further extension is got by a result of Sommese, [7], which also works by only assuming the ampleness of L.

Assume $n = 3$ and $d = 3$. Then by a result of Sommese [9], X has a structure of a $\mathbf{P}^1$-bundle $p : X \to \tilde{S}$ over a smooth surface and S is a meromorphic (nonholomorphic) section of p i.e. S meets the general fibre of p at one point but S contains some fibres. In our case S is ruled by cubics over $\mathbf{P}^1$, hence $\tilde{S}$ is also endowed with a morphism $\pi : \tilde{S} \to \mathbf{P}^1$. Look at the composite morphism $\pi \circ p$: $X \to \mathbf{P}^1$ its general fibre F is a $\mathbf{P}^1$-bundle over the general fibre of π, which is a $\mathbf{P}^1$. Hence F is a $\mathbf{F}_n$. This provides, at least for $n = 3$, an alternative description of the fibration $X \to \mathbf{P}^1$ we obtain in th. (1.1).

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0. Background material.

(0,1) Let X be a complex projective manifold of dimension $n \geq 2$ and L a very ample line bundle on X.

We set:

$|L|$ = the complete linear system associated with L,

K_X the canonical bundle of X,

$O_X(D)$ = the invertible sheaf associate dto a divisor D on X,

$h^i(O_X(D)) = \dim H^i(O_X(D))$,

$deg(X, L)c_1(L)^n$,

L^k = the k-th tensor power of L.

If S is a surface, i.e. $n = 2$, we will write $q = h^1(O_S)$ and $p_g = h^2(O_S)$. Surfaces with $q = 0$ are said regualr. Moreover, for any line bundle L on S, we set:

$$g(L) = 1 + \frac{1}{2}(L \otimes K_S) \cdot L$$

the arithmetic genus of L.

As usual,w e do not distinguish between line bundles and the corresponding invertible sheaves.

The word curve will mean complex, projective, reduced, irreducible curve. Any smooth curve C admititng a g^1_d, but not a g^1_t for $t < d$, is said d-gonal.

Let us list some pairs (X, L) as in (0.1) frequently occurring in the sequel. A pair (X, L) as above is said to be a scroll if X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve and $L_{|f} = O_{\mathbf{P}^1}(1)$, for every fibre of X. A pair (X, L), which is not a scroll, is said to be a quadric bundle (respectively a conic bundle if $n = 2$) if there is a morphism $p : X \to C$ over a smooth curve, whose general fibre F satisfies $(F, L_F) = (Q^{n-1}, Q_{Q^{n-1}}(1))$, where $(Q^{n-1}, O_{Q^{n-1}}(1))$ is the smooth quadric $Q^{n-1} \subset \mathbf{P}^n$ polarized by its hyperplane bundle. For each $n \geq 2$, we let B_n denote the class of scrolls and C_n that of quadric bundles.

We willalso need the concept of rational normal scrolls according to [2]: (0.2) Let E be a vector bundle of rank r over $\mathbf{P}^1$,

$$E = O_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus O_{\mathbf{P}^1}(a_r)$$

with $a_r > 0$, $0 \leq a_1 \leq \ldots \leq a_r$. To avoid confusion we say that X is a genrallized rational normal scroll if it is the image of the projective bundle $\mathbf{P}(E)$ under the map associated to the linear system $|H|$, where $H = \mathcal{O}_{\mathbf{P}(E)}(1)$ is the tautological bundle. X has the following properties:

i) $dimX = r$, $deg(X, H) = \sum_{i=1}^{r} a_i$ and $X \subset \mathbf{P}^{d+r-1}$ is non degenerate.

ii) the singular locu $Sing(X)$ of X is a linear space of dimension $\leq r-2$.

iii) the fibres of the ruling $\mathbf{P}(E) \to \mathbf{P}^1$ are embedded by $|H|$ as linear spaces of dimension $(r-1)$, whose common intersection is $Sing(X)$.

iv) X is smooth if and only if either $X = \mathbf{P}^2$ or $a_i > 0 \forall i$. Note that if $a_i > 0 \forall i$, then the pair $(X, H) \in B_n$.

Let S be a $\mathbf{P}^1$-bundle over a smooth curve, f and σ will denote respectively a fibre and a fundamental section, i.e. any section of the ruling with minimal self-intersection number σ^2. If S is rational, we will use the notation:

$$\mathbf{F}_n = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n))$$

for all $n \geq 0$.

We need the following lemma:

LEMMA 0.3. *The surface $\mathbf{F}_2$ contains no irreducible conics with respect to the very ample line bundle $L = \mathcal{O}_{\mathbf{F}_2}(\sigma + 3f)$.*

Proof. Assume that $\mathbf{F}_2$ contains asmooth curve C which is embedded by $|L|$ as a conic. Then $C \equiv a\sigma + bf$, with $a, b \in Z$, and $C \cdot L = 2$. As C can be neither a section, nor a fibre, it follows from [3, p. 380] that $a > 0$ and $b \geq 2a$. Since $C \cdot L = 2$, we have:

$$2 = (\sigma + 3f) \cdot (a\sigma + bf) \geq 3a \geq 3$$

which is impossible.

Let (S, L) consist of a surface S and a very ample line bundle L on it. We say that (S, L) is ruled by cubics or quartics if the fibres of the ruling of S are embedded as rational cubics or quartics by $|L|$. We will use the follwoing result about surfaces with trigonal hyperplane sections recently proved by [1, th. (2.1)+erratum]:

PROPOSITION 0.4. *Let S be a regular surface and L a very ample line bundle on it. Suppos ethat either $(L \otimes K_S)\cdot \leq -4$ or $(L \otimes K_S) \cdot K_S \leq -3$ (if $g \leq 4$), then $|L|$ contains a trigonal curve if and only if (S, L) is one of the following pairs:*

a) is rational and ruled by cubics,

b) $(\mathbf{P}^2, O_{\mathbf{P}^2}(4))(g = 3)$,

c) S is a cubic surface of $\mathbf{P}^3$ and $L = O_S(2)$.

The following result, due to Serrano, is about surfaces with 4-gonal hyperplane sections:

PROPOSITION 0.5. [5, th. (3.15)]: *Let L be a very ample line bundle on a regular surface $S, S \neq \mathbf{P}^2$. Assume there exists a smooth curve $C \in |L|$ which is 4-gonal; If $(L \otimes K_S) \cdot K_S \leq -5$ then (S, L) is rational and ruled by rational quartics. Conversely, if (S, L) is ruled by rational quartics and $L^2 \geq 17$, then it follows that $(L \otimes K_S) \cdot K_S \leq -5$.*

1. projective manifolds admitting 3-gonal or 4-gonal curve sections.

Let (X, L) be as in (0.1) with $n = dimX \geq 3$, and let S be a smooth surface which is the transverse intersection of $(n-2)$ general elements of $|L|$. Suppose that $|L_S|$ contains a smooth d-gonal curve ($d = 3$ or $d = 4$). For $d = 3$, we have the following result:

THEOREM 1.1. *Assume that $h^1(O_X) = 0$ and there exists a trigonal* *curve of genus $g \geq 4$ in $|L_S|$.*

(1.1.1) *i)* $(L_S \otimes K_S) \cdot K_S \leq -4$,

ii) $h^0(L) \geq 12 + n - 3$,

then $n \leq 4$ *and* X *is a fibration over* $\mathbf{P}^1$*, whose general fibre is a rational cubic scroll of dimension* $(n-1)$ *in* $\mathbf{P}^{n+1}$.

Proof. Consider the pair (S, L_S): by the Lefschetz theorem S is a regular surface, moreover $|L_S|$ contains a trigonal curve. By the assumption (1.1.1)i), we can apply (0.4) and conclude that the pair (S, L_S) is rational and ruled by cubics.

Our goal now is to extend the ruling $\phi : S \to \mathbf{P}^1$ to the whole X and this will be done in two steps: first we will get the extension for manifolds of dimension 3, the for $n \geq 4$ we will obtain the result by recurrence.

Step 1. Assume that $n = 3$.

By the assumption (1.1.1) we can apply a result of Serrano [6, th. (5.2)] so that

(1.1.2) $$\phi \text{ is the restriction to } S \text{ of a morphism } \mathbf{P} : X \to \mathbf{P}^1.$$

Note that th. (5.2) requires two conditions: the first one is true in view of (1.1.1) ii), the second one is about the degree of the pair (X, L), but in our case it is unnecessary because the assertion is equally achieved with (1.1.1) i).

Let $\mathcal{F} = \phi^{-1}(t)$, be the general fibre of Φ, $\mathcal{F}$ is a smooth surface and satisfies

(1.1.3) $$\mathcal{F} \cdot S = \phi^{-1}(t) = D \simeq \mathbf{P}^1.$$

Consider on $F\mathcal{F}$ the very ample line bundle $L_{\mathcal{F}}$, by the Lefschetz theorem it follows that $0 = g(L_{\mathcal{F}}) \geq q(\mathcal{F})$, we see that $\mathcal{F}$ is regular and satisfies $g(L_{\mathcal{F}}) = h^1(O_{\mathcal{F}})$, so by [8], the pair $(\mathcal{F}, L_{\mathcal{F}})$ is either

(1.1.4) $$(\mathbf{P}^2, O_{\mathbf{P}^2}(i))i = 1, 2, \text{ or a rational scroll.}$$

By (1.1.3) we have $deg(\mathcal{F}, L_{\mathcal{F}}) = 3$, and this implies that $(\mathcal{F}, L_{\mathcal{F}})$ is the rational cubic scroll of $\mathbf{P}^4$, i.e. $(\mathcal{F}, L_{\mathcal{F}}) = \mathbf{F}_1, O_{\mathbf{F}_1}(\sigma + 2f))$.

Step 1 concludes the proof if $n = 3$.

Step 2. Assume that $n \geq 4$.

(1.1.5) Suppose that the assertion is true for manifolds of dimension $(n-1)$, let's prove it for a pair (X, L) with $dimX = n \geq 4$.

Choose $Y \in |L|$ a smooth hyperplane containing S, by recurrence there is a morphism $\varphi : Y \to \mathbf{P}^1$ whose general fibre is a rational cubic scroll of dimension $(n-2)$ in $\mathbf{P}^n$.

Since $dimY - dim\mathbf{P}^1 = n - 2 \geq 2$, by [7], there exists a morphism $\Psi : X \to \mathbf{P}^1$ whose restriction to Y is φ.

Let $\mathcal{F} = \Psi^{-1}(t)$ be the general fibre of Ψ, $\mathcal{F}$ is smooth. We claim that $(\mathcal{F}, L_{\mathcal{F}})$ is a rational cubic scroll of dimension $(n-1)$ in $\mathbf{P}^{n+1}$. To see this note that $Y \cdot \mathcal{F} = \varphi^{-1}(t)$, then by recurrence $(\varphi^{-1}(t),\ L_{\varphi^{-1}(t)})$ is a rational cubic scroll of dimension $(n-2)$ in $\mathbf{P}^n$. Thus the claim followsw by [4].

Since $\mathcal{F}$ is smooth, by (0.2) this can happern only when

$$3 = deg(\mathcal{F}, L_{\mathcal{F}}) = a_1 + \ldots + a_{n-1}$$

with $a_i > 0$ for $i = 1, \ldots, n-1$, and this implies $n \leq 4$ and concludes the proof.

Remark 1.2. The assumption i) th. (1.1) can be replaced by the following one: $(L \otimes K_S) \cdot K_{\S} \leq -4$ or $(L \otimes K_S) \cdot K_S \leq -3$ (if $g = 3$ or $g = 4$), and the assertio still holds. In fact, (S, L_S) is one of the pairs listed in (0.4), nevertheless the only case occurring is the first one. By a result of [7], case b) cannot be lifted to higher dimension. Case c) has to be excluded because of numerical invarints. Let $n = dimX = 3$, suppose that (S, L_S) is a cubic surface of $\mathbf{P}^3$ and $L_S = K_S^{-2}$: then $((K_X \otimes L)^2 \otimes L)_S = O_S$. By the Lefschetz theorem this implies $K_X^2 \otimes L^3 = O_X$ so that

$$c_1(K_X)^3 = \left(\frac{3}{2}\right)^3 \cdot c_1(L)^3 = \frac{27}{8} \cdot 12 \notin Z.$$

For $d = 4$, we have a similar result:

THEOREM 1.3. *Assume that* $h^1(O_X) = 0$ *and that there exists a smooth 4-gonal curve in* $|L_S|$. *If:*

(1.3.1) *i)* $(L_S \otimes K_S) \cdot K_S \leq -5$,

ii) $h^0(L) \geq 15 + n - 3$,

then $n \leq 5$ *and* X *is a fibration over* $\mathbf{P}^1$, *whose general fibre is either a rational quartic scroll of dimension* $(n-1)$ *in* $\mathbf{P}^{n+2}$, *or a Veronese surface of* $\mathbf{P}^5$.

Proof. As in th. (1.1) let's consider the pair (S, L_S): by the Leftschetz theorem S is a regular surface and it is not isomorphic to $\mathbf{P}^2$ by [7], moreover $|L_S|$ contains a 4-gonal curve. Due to the assumption (1.3.1)i), we can apply (0.5) and conclude that (S, L_S) is rational and ruled by quartics. As in th. (1.1) the extension of the ruling of S to the whole X will be done by two steps.

Step 1. Assume that $n = 3$.

The proof runs as that of th. (1.1). By using [6] we see that (1.1.2) holds. Let $\mathcal{F} = \Phi^{-1}(t)$ be the general of Φ and consider teh very ample line bundle $L_{\mathcal{F}}$ on it; then $(\mathcal{F}, L_{\mathcal{F}})$ is one of pairs in (1.1.4). Since $deg(\mathcal{F}, L_{\mathcal{F}}) = 4$, this implies the following possibilities:

(1.3.2) i) $(\mathcal{F}, L_{\mathcal{F}}) = \mathbf{P}^2, O_{\mathbf{P}^2}(2))$,

ii) $(\mathcal{F}, L_{\mathcal{F}})$ is a rational quartic scroll, namely either $(\mathbf{F}_0, O_{\mathbf{F}_0}(\sigma + 2f))$ or $(\mathbf{F}_2, O_{\mathbf{F}_2}(\sigma + 3f))$.

Note that if a fibre is as i) or respectively ii), then the general fibre is so. As $(K_X \otimes [\mathcal{F}])_{\mathcal{F}} = K_{\mathcal{F}}$ we have that $c_1^2(\mathcal{F})$ is indipendnet of the fibres. On the other hand $c_1^2(\mathcal{F}) = 8$ or 9 according to cases i) and ii). In fact, if a fibre $\mathcal{F} = \mathbf{F}_0$, then every smooth fibre is $\mathbf{F}_0$. Consider on X the line bundle $H = (K_X \otimes [\mathcal{F}] \otimes L)^{-1}$ and compute $H_{\mathcal{F}} = K_{\mathcal{F}}^{-1} \otimes L_{\mathcal{F}}^{-1}$. If $\mathcal{F} = \mathbf{F}_0$, then $H_{\mathcal{F}} = O_{\mathbf{F}_0}(\sigma)$ and $\sigma \cdot L_{\mathcal{F}} = 2$, hence σ is embedded by $|L_{\mathcal{F}}|$ as a, irreducible conic for every fibre $\mathcal{F}$ of X, but by (0.3) this cannot happen if $\mathcal{F} = \mathbf{F}_2$.

Therefore (X, L) is one pof the following:

(1.3.3) i) X is a fibration over $\mathbf{P}^1$ with general fibre a Veronese surface,

ii) X is a fibration over $\mathbf{P}^1$ with general fibre a rational quartic scrollo $\mathbf{F}_0$,

iii) X is a fibration over $\mathbf{P}^1$ with general fibre a rational quartic scroll $\mathbf{F}_2$. If $n = 3$ step 1 concludes the proof.

Step 2. Assume that $n \geq 4$.

By assuming the assertion true for manifolds of dimension $(n-1)$, in the same way of th. (1.1) one can prove that X is a fibration over $\mathbf{P}^1$ whose general fibre $\mathcal{F}$ is a rational quartic scroll of dimension $(n-1)$ in $\mathbf{P}^{n+2}$. SInce $\mathcal{F}$ is smooth, this can happern, by (0.2), only when

$$4 = deg(\mathcal{F}, L_{\mathcal{F}}) = a_1 + \ldots + a_{n-1}$$

with $a_i > 0$, for $i = 1, \ldots, n-1$. This implies $n \leq 5$ and concludes the proof.

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Dipartimento di Matematica
"F. Enriques"
Via C. Saldini, 50, - 20133 Milano