«LE MATEMATICHE»
Vol. XLIV (1989) - Fasc. II, pp. 173-204

MATHEMATICAL CHARACTERIZATION OF FUNCTIONS UNDERLYING THE PRINCIPLE OF RELATIVITY

SEBASTIANO PENNISI (Catania) - MASSIMO TROVATO (Catania) (*) (**)

In a Minkowsky space V, complete representation formulae are derived for scalar-valued, vector-valued and tensor-valued functions subjected to the principle of relativity with an arbitrary number of scalars, vectors and tensors (of the second order symmetric and skewsymmetric) as variables.

1. Introduction.

The principle of relativity states that the laws of physics assume the same form in all inertial frames. This imposes conditions on the functions appearing in these laws.

Here attention is focused on those functions depending on x^α through a time-like 4-vector V^α, P other 4-vectors $V_1^\alpha, \dots, V_P^\alpha$, N second order symmetric tensors $A_1^{\alpha\beta}, \dots, A_N^{\alpha\beta}$ and M second order skew-symmetric tensors $W_1^{\alpha\beta}, \dots, W_M^{\alpha\beta}$. (For applications, see for example [1-3] where V^α is the 4-velocity of the fluid, $A^{\alpha\beta}$ is the stress-energy-momentum tensor $T^{\alpha\beta}$ and $W^{\alpha\beta}$ is the electro-magnetic

(*) Entrato in Redazione il 13 luglio 1989
(**) The present work was partially supported by the contract M.P.I.

tensor $F^{\alpha\beta}$. In the case of mixtures we have as much of these variables as the number of costituents).

These functions satisfy the principle of relativity if for every automorphism P of V we have

$$(1.1)\qquad f(P^\alpha_\lambda V^\lambda, P^\alpha_\lambda V^\lambda_i, P^\alpha_\lambda P^\beta_\mu A^{\lambda\mu}_j, P^\alpha_\lambda P^\beta_\mu W^{\lambda\mu}_k) = f(V^\alpha, V^\alpha_i, A^{\alpha\beta}_j, W^{\alpha\beta}_k)$$

$$P^\alpha_\lambda f^\lambda(P^\alpha_\lambda V^\lambda, P^\alpha_\lambda V^\lambda_i, P^\alpha_\lambda P^\beta_\mu A^{\lambda\mu}_j, P^\alpha_\lambda P^\beta_\mu W^{\lambda\mu}_k) = \\ = f^\alpha(V^\alpha, V^\alpha_i, A^{\alpha\beta}_j, W^{\alpha\beta}_k)$$

$$P^\alpha_\lambda P^\beta_\mu f^{\lambda\mu}(P^\alpha_\lambda V^\lambda, P^\alpha_\lambda V^\lambda_i, P^\alpha_\lambda P^\beta_\mu A^{\lambda\mu}_j, P^\alpha_\lambda P^\beta_\mu W^{\lambda\mu}_k) = f^{\alpha\beta}(V^\alpha, V^\alpha_i, A^{\alpha\beta}_j, W^{\alpha\beta}_k)$$

where f, f^α, $f^{\alpha\beta}$ are the components in a given basis of the function we are considering if it is scalar-valued, vector-valued or second order tensor-valued respectively; $i = 1, \dots, P$; $j = 1, \dots, N$; $k = 1, \dots, M$. If among the variables there are also some scalars, they will appear as independent variables only in the scalar-valued functions; this statement will be implicitly evident in our treatment.

Functions satisfying (1.1) will be briefly called isotropic.

The corresponding propositions in the case of a 3-dimensional euclidean vector space have been object of many papers such as [4-8].

The 4-dimensional case has been considered only by Wang [9], but only with a skew-symmetric second order tensor as variable, and by Pennisi [10] but only in the case $N = 1$, $M = 0$ and moreover $P = 0$ in the part treating the problem of irreducibility.

However [10] can not be considered a particular case of the present paper; for example, the sets of generators $\{\vec{f}_1, \dots, \vec{f}_r\}$ for vector-valued and second order tensor-valued isotropic functions found in [10] are irreducible in the sense that they are linearly independent, i.e.

$$(1.2)\qquad l_1\vec{f}_1 + \dots + l_r\vec{f}_r = \bar{0}$$

holds if $l_1 = \dots = l_r = 0$, while the corresponding sets of generators in this paper are expected irreducible in the sense that from (1.2) it follows that each of $l_1, \dots, l_r$ must be zero for some value of the variables; this does not permit to obtain one of the $\vec{f}_1, \dots, \vec{f}_r$ as a linear combination of the remainder because this linear combination would not be defined for all the values of the variables. This fact is similar to the equivalent prposition in the 3-dimensional case [4-8].

In section 2 we find a complete functions basis for scalar-valued isotropic functions, i.e. a set of scalars such that every other scalar-valued isotropic function can be written as a function of them. In sections 3,4 and 5 the case is considered of vector-valued, second order symmetric and second order skew-symmetric tensor-valued isotropic functions respectively.

If the reader is interested only in the results, he can find them in (2.1) and (2.2), (3.1), (4.1), (4.2), (5.1), (5.2) respectively.

2. Complete function basis for scalar-valued isotropic functions.

We prove now that every scalar-valued isotropic function f of V^α, V_p^α, $A_n^{\alpha\beta}$, $W_m^{\alpha\beta}$ ($p = 1, \dots, P; n = 1, \dots, N; m = 1, \dots, M$) can be written as a function of the following scalars (The notation $\vec{x} \cdot \vec{y} = x^\alpha g_{\alpha\beta} y^\beta$; $\vec{x} \cdot A\vec{y} = x^\alpha g_{\alpha\beta} A^{\beta\gamma} g_{\gamma\delta} y^\delta$; $\vec{x} \cdot W\vec{y} = x^\alpha g_{\alpha\beta} W^{\beta\gamma} g_{\gamma\delta} y^\delta$, $trA = A^{\alpha\beta} g_{\beta\alpha}$ with $g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ the metric tensor, will be used).

(2.1) $\vec{V} \cdot \vec{V}, \vec{V} \cdot A_{n_1} \vec{V}, \vec{V} \cdot \vec{V}_{p_1}, \vec{V}_{p_1} \cdot \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_4} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot \vec{V}_{p_2},$

$\vec{V} \cdot A_{n_4} A_{n_5} \vec{V}, \vec{V} W_{m_4} W_{m_5} \vec{V}, \vec{V} \cdot A_{n_4} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_4} \vec{V},$

$\vec{V} \cdot W_{m_4} A_{n_1} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot A_{n_1} \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_5} \vec{V}, \vec{V} \cdot W_{m_4} A_{n_1} W_{m_5} \vec{V},$

$\vec{V}_{p_1} \cdot A_{n_1} \vec{V}_{p_2}, \vec{V} \cdot A_{n_4} A_{n_1} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} A_{n_1} \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} A_{n_1} \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_4} \vec{V},$

$$\vec{V} \cdot W_{m_4} A_{n_1} A_{n_1} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot A_{n_1} A_{n_1} \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_1} A_{n_5} \vec{V},$$

$$\vec{V} \cdot W_{m_4} A_{n_1} A_{n_1} W_{m_5} \vec{V}, \vec{V}_{p_1} \cdot A_{n_1} A_{n_1} \vec{V}_{p_2}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_1} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_1} \vec{V}_{p_1},$$

$$\vec{V} \cdot W_{m_4} A_{n_1} A_{n_1} \vec{V}_{p_1}, \vec{V} \cdot A_{n_4} A_{n_1} A_{n_2} A_{n_4} \vec{V}, \vec{V} \cdot W_{m_4} A_{n_1} A_{n_2} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot A_{n_1} A_{n_2} \vec{V}_{p_1},$$

$$\vec{V} \cdot A_{n_4} (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) A_{n_5} \vec{V}, \vec{V} \cdot A_{n_4} (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) W_{m_4} \vec{V},$$

$$\vec{V} \cdot A_{n_4} (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) W_{m_5} \vec{V},$$

$$\vec{V} \cdot W_{m_4} (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) \vec{V}_{p_1}, \vec{V}_{p_1} \cdot (A_{n_1} A_{n_2} - A_{n_2} A_{n_1}) \vec{V}_{p_2}, \vec{V} \cdot A_{n_4} W_{m_1} A_{n_5} \vec{V},$$

$$\vec{V} \cdot A_{n_4} W_{m_1} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} W_{m_1} \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_5} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_1} \vec{V}_{p_1},$$

$$\vec{V}_{p_1} \cdot W_{m_1} \vec{V}_{p_2}, \vec{V} \cdot A_{n_4} W_{m_1} W_{m_1} A_{n_4} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_1} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot W_{m_1} W_{m_1} \vec{V}_{p_1},$$

$$\vec{V} \cdot A_{n_4} W_{m_1} W_{m_1} A_{n_5} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_1} W_{m_5} \vec{V}, \vec{V}_{p_1} \cdot W_{m_1} W_{m_1} \vec{V}_{p_2},$$

$$\vec{V} \cdot A_{n_4} W_{m_1} W_{m_1} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} W_{m_1} W_{m_1} \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_1} \vec{V}_{p_1},$$

$$\vec{V} \cdot A_{n_4} W_{m_1} W_{m_2} A_{n_4} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_2} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot W_{m_1} W_{m_2} \vec{V}_{p_1},$$

$$\vec{V} \cdot W_{m_4} W_{m_1} W_{m_1} W_{m_2} W_{m_4} \vec{V}, \vec{V} \cdot W_{m_4} W_{m_1} W_{m_2} W_{m_2} W_{m_4} \vec{V},$$

$$\vec{V} \cdot A_{n_4} (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) A_{n_5} \vec{V}, \vec{V} \cdot A_{n_4} (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) W_{m_4} \vec{V},$$

$$\vec{V} \cdot A_{n_4} (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) \vec{V}_{p_1}, \vec{V} \cdot W_{m_4} (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) W_{m_5} \vec{V},$$

$$\vec{V} \cdot W_{m_4} (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) \vec{V}_{p_1}, \vec{V}_{p_1} \cdot (W_{m_1} W_{m_2} - W_{m_2} W_{m_1}) \vec{V}_{p_2},$$

$$\vec{V} \cdot A_{n_4} A_{n_1} W_{m_1} A_{n_4} \vec{V}, \vec{V} \cdot W_{m_4} A_{n_1} W_{m_1} W_{m_4} \vec{V}, \vec{V}_{p_1} \cdot A_{n_1} W_{m_1} \vec{V}_{p_1},$$

$$\vec{V} \cdot W_{m_4} A_{n_1} A_{n_1} W_{m_1} W_{m_4} \vec{V}, \vec{V} \cdot A_{n_4} (A_{n_1} W_{m_1} - W_{m_1} A_{n_1}) A_{n_5} \vec{V},$$

$$\vec{V}\cdot A_{n_4}(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})W_{m_4}\vec{V},\ \vec{V}_{p_1}\cdot(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})A_{n_4}\vec{V},$$

$$\vec{V}_{p_1}\cdot(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})W_{m_4}\vec{V},\ \vec{V}_{p_1}\cdot(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})\vec{V}_{p_2},\ trA_{n_1},\ trA_{n_1}A_{n_1},$$

$$trA_{n_1}A_{n_2},\ trA_{n_1}A_{n_1}A_{n_1},\ trA_{n_1}A_{n_1}A_{n_2},\ trA_{n_1}A_{n_2}A_{n_2},\ trA_{n_1}A_{n_2}A_{n_3},$$

$$trW_{m_1}W_{m_1},\ trW_{m_1}W_{m_2},\ trW_{m_1}W_{m_2}W_{m_3},\ trA_{n_1}W_{m_1}W_{m_1},\ trA_{n_1}W_{m_1}W_{m_2},$$

$$trA_{n_1}A_{n_2}W_{m_1},\ \vec{V}\cdot W_{m_4}(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})W_{m_5}\vec{V},$$

$$\begin{aligned}&trA_{n_1}W_{m_1}W_{m_1}A_{n_2}W_{m_1}-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot W_{m_1}W_{m_1}A_{n_2}W_{m_1}A_{n_1}\vec{V}+\\&+\vec{V}\cdot W_{m_1}A_{n_2}W_{m_1}A_{n_1}W_{m_1}\vec{V}+\vec{V}\cdot A_{n_2}W_{m_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}+\\&+\vec{V}\cdot W_{m_1}A_{n_1}W_{m_1}W_{m_1}A_{n_2}\vec{V}+\vec{V}\cdot A_{n_1}W_{m_1}W_{m_1}A_{n_2}W_{m_1}\vec{V}),\end{aligned}$$

$$\begin{aligned}&\vec{V}\cdot A_{n_4}W_{m_1}W_{m_1}W_{m_2}A_{n_4}\vec{V},\ \vec{V}_{p_1}\cdot W_{m_1}W_{m_1}W_{m_2}\vec{V}_{p_1},\\&\vec{V}\cdot A_{n_4}W_{m_1}W_{m_2}W_{m_2}A_{n_4}\vec{V},\end{aligned}\tag{2.2}$$

$$\vec{V}\cdot A_{n_4}W_{m_1}A_{n_1}W_{m_1}W_{m_1}A_{n_4}\vec{V}-$$

$$(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_4}\vec{V})(\vec{V}\cdot A_{n_4}W_{m_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}+$$

$$+\vec{V}\cdot W_{m_1}A_{n_1}W_{m_1}W_{m_1}A_{n_4}\vec{V}),\ \vec{V}_{p_1}\cdot W_{m_1}W_{m_2}W_{m_2}\vec{V}_{p_1},$$

$$\vec{V}\cdot W_{m_4}W_{m_1}A_{n_1}W_{m_1}W_{m_1}W_{m_4}\vec{V},\ \vec{V}_{p_1}\cdot W_{m_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(\vec{V}_{p_1}\cdot W_{m_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}+\vec{V}\cdot W_{m_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}_{p_1}),$$

$$\vec{V}\cdot A_{n_4}A_{n_1}A_{n_1}W_{m_1}A_{n_4}\vec{V},\ \vec{V}_{p_1}\cdot A_{n_1}A_{n_1}W_{m_1}\vec{V}_{p_1},\ trA_{n_1}A_{n_1}A_{n_2}A_{n_2},$$

$$trA_{n_1}A_{n_1}W_{m_1}W_{m_1},\ trA_{n_1}W_{m_1}W_{m_2}W_{m_2},\ trA_{n_1}W_{m_1}W_{m_1}W_{m_2},$$

$$trA_{n_1}A_{n_2}A_{n_2}W_{m_1},\ trA_{n_1}A_{n_1}A_{n_2}W_{m_1},$$

$$trA_{n_1}A_{n_1}W_{m_1}W_{m_1}A_{n_1}W_{m_1} -$$

$$(2.2)\qquad \begin{aligned} &- (\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot W_{m_1}W_{m_1}A_{n_1}W_{m_1}A_{n_1}A_{n_1}\vec{V} + \\ &+ \vec{V}\cdot W_{m_1}A_{n_1}W_{m_1}A_{n_1}A_{n_1}W_{m_1}\vec{V} + \\ &+ \vec{V}\cdot A_{n_1}W_{m_1}A_{n_1}A_{n_1}W_{m_1}W_{m_1}\vec{V} + \\ &+ \vec{V}\cdot W_{m_1}A_{n_1}A_{n_1}W_{m_1}W_{m_1}A_{n_1}\vec{V} + \\ &+ \vec{V}\cdot A_{n_1}A_{n_1}W_{m_1}W_{m_1}A_{n_1}W_{m_1}\vec{V}). \end{aligned}$$

with $p_1, p_2 = 1, \ldots, P$ and $p_1 < p_2$,

$$n_1, n_2, n_3, n_4, n_5 = 1, \ldots, N \quad \text{and} \quad n_1 < n_2 < n_3, n_4 < n_5,$$

$$m_1, m_2, m_3, m_4, m_5 = 1, \ldots, M \quad \text{and} \quad m_1 < m_2 < m_3, m_4 < m_5,$$

To prove this representation let us take $\vec{V}\cdot(-\vec{V}\cdot\vec{V})^{-\frac{1}{2}}$ as 0-axis (this is possible because $\vec{V}$ is timelike) and complete it to an orthonormal base of the vector space.

Let us consider the restriction of condition $(1.1)_1$ to the automorphisms P of V that leave $\vec{V}$ unchanged, i.e. to the transformations of reference frame involving only the 1,2,3 axis; in terms of the components of P this means

$$(2.3)\qquad P^{\alpha}_{\lambda} = \begin{pmatrix} 1 & 0_j \\ 0^i & P^i_j \end{pmatrix}$$

(Obviously the greek indices run from 0 to 3, while the latin ones run from 1 to 3).

In every of these frames we can use the decompositions

$$V^{\alpha} = (-\sqrt{-\vec{V}\cdot\vec{V}},\ 0^i);\ V^{\alpha}_p = (\vec{V}\cdot\vec{V}_p/\sqrt{-\vec{V}\cdot\vec{V}},\ V^i_{P_i});$$

$$A^{\alpha\beta}_n = \begin{pmatrix} \dfrac{\vec{V}\cdot A_n\vec{V}}{-\vec{V}\cdot\vec{V}} & A^{0j}_n \\ A^{i0}_n & A^{ij}_n \end{pmatrix};\quad W^{\alpha\beta}_m = \begin{pmatrix} 0 & W^{0j}_m \\ W^{i0}_m & W^{ij}_m \end{pmatrix}$$

with $A_n^{i0} = A_n^{0i}$ and $W_m^{i0} = -W_m^{0i}$, $A_n^{ij} = A_n^{ji}$, $W_m^{ij} = -W_m^{ji}$.

Moreover, the transformations (2.3) trasform V_p^i, A_n^{i0}, W_m^{i0} as vectors and A_n^{ij}, W_m^{ij} as second order symmetric and skew-symmetric tensors respectively, in the 3-dimensional euclidean vector space orthogonal to $\vec{V}$; in this way the condition $(1.1)_1$ restricted to the automorphisms (2.3) becomes the condition for a scalar-valued isotropic function f of the scalar

$$\vec{V} \cdot \vec{V}, \vec{V} \cdot \vec{V}_p, \vec{V} \cdot A_n \vec{V} \tag{2.4}$$

(that are already listed in (2.1)), the vectors V_p^i, A_n^{i0}, W_m^{i0}, the symmetric tensors A_n^{ij} and the skew-symmetric tensors W_m^{ij}, in the framework of a 3-dimensional euclidean vector space. So we can use the results found in [4, 6, 8] and say that f is a function of (2.4) and of the set S of 3-dimensional scalars obtained applying to the variables V_p^i, A_n^{i0}, W_m^{i0}, A_n^{ij}, W_m^{ij} the table for scalar-valued isotropic functions given in [4, 6, 8]. In these elements of set S we write explicitly the indices of the variables (in order not to confuse the scalar product in V with that of its 3-dimensional subspace orthogonal to $\vec{V}$) and to the contractions of the type $A^{\cdot i}B^{i \cdot}$ we substitute $A^{\cdot i}B^{j \cdot}g_{ij}$ and so on.

If we define $\mathbf{V}_p^\alpha = (0, V_p^i)$; $\mathbf{A}_n^{\alpha 0} = (0, A_n^{i0})$; $\mathbf{W}_m^{\alpha 0} = (0, W_m^{i0})$;

$$\mathcal{A}_n^{\alpha\beta} = \begin{pmatrix} 0 & 0^j \\ 0^i & A_n^{ij} \end{pmatrix}; \quad \mathcal{W}_m^{\alpha\beta} = \begin{pmatrix} 0 & 0^j \\ 0^i & W_m^{ij} \end{pmatrix},$$

we can substitute in the elements of S

$$V_p^i = \mathbf{V}_p^i; A_n^{i0} = \mathbf{A}_n^{i0}; W_m^{i0} = \mathbf{W}_m^{i0}; A_n^{ij} = \mathcal{A}_n^{ij}; W_m^{ij} = \mathcal{W}_m^{iJ};$$

then we may substitute greek indices to the latin ones, because the additional terms that we introduce in this way are all null.

Lastly we may substitute from the following identities

$$\mathbf{V}_p^\alpha = h_\mu^\alpha V_p^\mu; \mathbf{A}_n^{\alpha 0} = (-\vec{V} \cdot \vec{V})^{-\frac{1}{2}} h_\mu^\alpha A_n^{\mu\beta} V_\beta; \mathbf{W}_m^{\alpha 0} = (-\vec{V} \cdot \vec{V})^{-\frac{1}{2}} h_\mu^\alpha W_m^{\mu\beta} V_\beta;$$

$$\mathcal{A}_n^{\alpha\beta} = h_\mu^\alpha h_\nu^\beta A_n^{\mu\nu}; \mathcal{W}_m^{\alpha\beta} = h_\mu^\alpha h_\nu^\beta W^{\mu\nu};$$

where $h^\alpha_\mu = g^\alpha_\mu - (\vec{V}\cdot\vec{V})^{-1}V^\alpha V_\mu$ is the projector.

Whit this procedure we obtain that the elements of S become functions of the 4-dimensional scalars in (2.1) and of those in table 1; but in the appendix we prove that if $\vec{x}$, $\vec{y}$ are two 3-dimensional vectors, A_1 and A_2 two 3-dimensional symmetric tensors, W_1 and W_2 two 3-dimensional skew-symmetric tensors we have that

$$\vec{x}\cdot A_1A_2\vec{y} = f_1(\vec{x}\cdot\vec{x}, \vec{x}\cdot\vec{y}, \vec{y}\cdot\vec{y}, \vec{x}\cdot A_1\vec{x}, \vec{x}\cdot A_1\vec{y}, \vec{y}\cdot A_1\vec{y}, \vec{x}\cdot A_2\vec{x},$$
$$\vec{x}\cdot A_2\vec{y}, \vec{x}\cdot A_1^2\vec{x}, \vec{x}\cdot A_1^2\vec{y}, \vec{y}\cdot A_2\vec{y},$$
$$\vec{x}\cdot A_1A_2\vec{x}, \vec{x}\cdot(A_1A_2 - A_2A_1)\vec{y})$$

$$\vec{x}\cdot A_1W_1\vec{y} = f_2(\vec{x}\cdot\vec{x}, \vec{x}\cdot\vec{y}, \vec{y}\cdot\vec{y}, \vec{x}\cdot A_1\vec{x}, \vec{x}\cdot A_1\vec{y}, \vec{y}\cdot A_1\vec{y}, \vec{x}\cdot A_1^2\vec{x}, \vec{x}\cdot A_1^2\vec{y},$$
$$\vec{x}\cdot W_1\vec{y}, \vec{x}\cdot A_1W_1\vec{x}, \vec{x}\cdot(A_1W_1 - W_1A_1)\vec{y})$$
$$\vec{x}\cdot W_1W_2\vec{y} = f_3(\vec{x}\cdot\vec{x}, \vec{x}\cdot\vec{y}, \vec{y}\cdot\vec{y}, \vec{x}\cdot W_1\vec{y}, \vec{x}\cdot W_2\vec{y}, \vec{x}\cdot W_1^2\vec{x}, \vec{x}\cdot W_1^2\vec{y},$$
$$\vec{x}\cdot W_1W_2\vec{x}, \vec{x}\cdot(W_1W_2 - W_2W_1)\vec{y}, trW_1W_2)$$

If we apply these relations to V^i_p, A^{i0}_n, W^{i0}_m, A^{ij}_n, W^{ij}_m and we follow for both sides of them the procedure applied to the elements of set S, we find that the elements in table 1 becomes functions of (2.1) and (2.2). This proves that f can be written as a function of the scalars in (2.1), (2.2), i.e. these elements form a complete function basis for scalar-valued isotropic functions.

Table 1

$$\vec{V}\cdot A_{n_4}W_{m_1}W_{m_1}W_{m_2}A_{n_4}\vec{V} - (\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_4}\vec{V})(\vec{V}\cdot W_{m_1}W_{m_1}W_{m_2}A_{n_4}\vec{V}),$$

$$\vec{V}_{p_1} \cdot W_{m_1} W_{m_1} W_{m_2} \vec{V}_{p_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V}_{p_1} \cdot \vec{V})(\vec{V} \cdot W_{m_1} W_{m_1} W_{m_2} \vec{V}_{p_1}),$$

$$\vec{V} \cdot A_{n_4} W_{m_1} W_{m_2} W_{m_2} A_{n_4} \vec{V} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_4} \vec{V})(\vec{V} \cdot A_{n_4} W_{m_1} W_{m_2} W_{m_2} \vec{V}),$$

$$\vec{V} \cdot A_{n_4} W_{m_1} A_{n_1} W_{m_1} W_{m_1} A_{n_4} \vec{V} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_4} \vec{V})(\vec{V} \cdot A_{n_4} W_{m_1} A_{n_1} W_{m_1} W_{m_1} \vec{V} +$$

$$+ \vec{V} \cdot W_{m_1} A_{n_1} W_{m_1} W_{m_1} A_{n_4} \vec{V}) - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot W_{m_1} A_{n_4} \vec{V})(\vec{V} \cdot A_{n_4} W_{m_1} A_{n_1} W_{m_1} \vec{V}),$$

$$\vec{V}_{p_1} W_{m_1} W_{m_2} W_{m_2} \vec{V}_{p_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot \vec{V}_{p_1})(\vec{V}_{p_1} \cdot W_{m_1} W_{m_2} W_{m_2} \vec{V}), \vec{V} \cdot$$

$$\vec{V} \cdot W_{m_4} W_{m_1} A_{n_1} W_{m_1} W_{m_1} W_{m_4} \vec{V} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot W_{m_1} W_{m_4} \vec{V})(\vec{V} \cdot W_{m_4} W_{m_1} A_{n_1} W_{m_1} \vec{V}),$$

$$\vec{V}_{p_1} \cdot W_{m_1} A_{n_1} W_{m_1} W_{m_1} \vec{V}_{p_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot \vec{V}_{p_1})(\vec{V}_{p_1} \cdot W_{m_1} A_{n_1} W_{m_1} W_{m_1} \vec{V} +$$

$$+ \vec{V} \cdot W_{m_1} A_{n_1} W_{m_1} W_{m_1} \vec{V}_{p_1}) - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot W_{m_1} \vec{V}_{p_1})(\vec{V}_{p_1} \cdot W_{m_1} A_{n_1} W_{m_1} \vec{V}),$$

$$\vec{V} \cdot A_{n_4} A_{n_1} A_{n_1} W_{m_1} A_{n_4} \vec{V} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_4} \vec{V})(\vec{V} \cdot A_{n_1} A_{n_1} W_{m_1} A_{n_4} \vec{V}),$$

$$\vec{V}_{p_1} \cdot A_{n_1} A_{n_1} W_{m_1} \vec{V}_{p_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V}_{p_1} \cdot \vec{V})(\vec{V} \cdot A_{n_1} A_{n_1} W_{m_1} \vec{V}_{p_1}),$$

$$tr A_{n_1} A_{n_1} A_{n_2} A_{n_2} - 2(\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_1} A_{n_1} A_{n_2} A_{n_2} \vec{V}),$$

$$tr A_{n_1} A_{n_1} W_{m_1} W_{m_1} - 2(\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot W_{m_1} W_{m_1} A_{n_1} A_{n_1} \vec{V}),$$

$$tr A_{n_1} W_{m_1} W_{m_2} W_{m_2} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_1} W_{m_1} W_{m_2} W_{m_2} \vec{V} + \vec{V} \cdot W_{m_2} W_{m_2} A_{n_1} W_{m_1} \vec{V}),$$

$$tr A_{n_1} W_{m_1} W_{m_1} W_{m_2} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot W_{m_1} W_{m_1} W_{m_2} A_{n_1} \vec{V} + \vec{V} \cdot W_{m_2} A_{n_1} W_{m_1} W_{m_1} \vec{V}),$$

$$tr A_{n_1} A_{n_2} A_{n_2} W_{m_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_2} A_{n_2} W_{m_1} A_{n_1} \vec{V} + \vec{V} \cdot W_{m_1} A_{n_1} A_{n_2} A_{n_2} \vec{V}),$$

$$tr A_{n_1} A_{n_1} A_{n_2} W_{m_1} - (\vec{V} \cdot \vec{V})^{-1} (\vec{V} \cdot A_{n_2} W_{m_1} A_{n_1} A_{n_1} \vec{V} + \vec{V} \cdot A_{n_1} A_{n_1} A_{n_2} W_{m_1} \vec{V}),$$

$$tr A_{n_1} A_{n_1} W_{m_1} W_{m_1} A_{n_1} W_{m_1} - (\vec{V} \cdot V)^{-1} (\vec{V} \cdot W_{m_1} W_{m_1} A_{n_1} W_{m_1} A_{n_1} A_{n_1} \vec{V} +$$

$$+\vec{V}\cdot W_{m_1}A_{n_1}W_{m_1}A_{n_1}A_{n_1}W_{m_1}\vec{V}+\vec{V}\cdot A_{n_1}W_{m_1}A_{n_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}+$$

$$+\vec{V}\cdot W_{m_1}A_{n_1}A_{n_1}W_{m_1}W_{m_1}A_{n_1}\vec{V}+\vec{V}\cdot A_{n_1}A_{n_1}W_{m_1}W_{m_1}A_{n_1}W_{m_1}\vec{V})+$$

$$+(\vec{V}\cdot\vec{V})^{-2}(\vec{V}\cdot A_{n_1}W_{m_1}\vec{V})(\vec{V}\cdot A_{n_1}A_{n_1}W_{m_1}W_{m_1}\vec{V}).$$

3. Complete function basis for vector-valued isotropic functions.

We prove now that every vector-valued isotropic function f^α of V^α, V_p^α, $A_n^{\alpha\beta}$, $W_m^{\alpha\beta}$ $(p=1,\ldots,P; n=1,\ldots,N; m=1,\ldots,M)$ can be expressed as a linear combination through scalar coefficients of the vectors in the following set

(3.1)

$$\vec{V},\vec{V}_p,A_{n_3}\vec{V},W_{m_3}\vec{V},A_{n_1}\vec{V}_p,A_{n_1}A_{n_3}\vec{V},A_{n_1}W_{m_3}\vec{V},A_{n_1}A_{n_1}\vec{V}_p,A_{n_1}A_{n_1}A_{m_3}\vec{V},$$

$$A_{n_1}A_{n_1}W_{m_3}\vec{V},W_{m_1}\vec{V}_p,W_{m_1}A_{n_3}\vec{V},W_{m_1}W_{m_3}\vec{V},W_{m_1}W_{m_1}\vec{V}_p,$$

$$W_{m_1}W_{m_1}A_{n_3}\vec{V},W_{m_1}W_{m_1}W_{m_3}\vec{V},(A_{n_1}A_{n_2}-A_{n_2}A_{n_1})\vec{V}_p,$$

$$(A_{n_1}A_{n_2}-A_{n_2}A_{n_1})A_{n_3}\vec{V},(A_{n_1}A_{n_2}-A_{n_2}A_{n_1})W_{m_3}\vec{V},$$

$$(W_{m_1}W_{m_2}-W_{m_2}W_{m_1})\vec{V}_p,(W_{m_1}W_{m_2}-W_{m_2}W_{m_1})A_{n_3}\vec{V},$$

$$(W_{m_1}W_{m_2}-W_{m_2}W_{m_1})W_{m_3}\vec{V},(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})\vec{V}_p,$$

$$(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})A_{n_3}\vec{V},(A_{n_1}W_{m_1}-W_{m_1}A_{n_1})W_{m_3}\vec{V}.$$

with $p=1,\ldots,P$

$$n_1,n_2,n_3=1,\ldots,N \quad \text{and} \quad n_1<n_2,$$

$$m_1, m_2, m_3 = 1, \ldots, M \quad \text{and} \quad m_1 < m_2.$$

To this end let us consider the scalar function $f^\alpha V_\alpha$ and the vector-valued function $\mathbf{f}^\alpha = h^\alpha_\mu f^\mu$; we have then

$$f^\alpha = \mathbf{f}^\alpha + (\vec{V} \cdot \vec{V})^{-1}(f^\mu V_\mu)V^\alpha; \tag{3.2}$$

if we find the representation for $\mathbf{f}^\alpha$, substituting it and that for $f^\mu \ V_\mu$ (obtained in the previous section), in (3.2) we obtain the representation for f^α.

Let us take $(-\vec{V} \cdot \vec{V})^{-\frac{1}{2}}\vec{V}$ as 0-axis and complete it to an orthonormal base of the vector space. In this reference frame we have

$$\mathbf{f}^0 = 0; \ \mathbf{f}^i = f^i.$$

If we consider the restriction of condition $(1.1)_2$ to the automorphisms P of V that leave $\vec{V}$ unchanged, i.e. to the transformations of the reference frame involving only the 1,2,3 axis, we obtain that they leave $\mathbf{f}^0$ unchanged while transform $f^i = \mathbf{f}^i$ as a 3-dimensional euclidean vector. So, as we have done for the scalar-valued functions in these particular reference frames, we can use the results found in [4,6,8] and say that $\mathbf{f}^i$ is a linear combination of the set of 3-dimensional vectors obtained applying to the variables $V^i_p, A^{i0}_n, W^{i0}_m$, A^{ij}_n, W^{ij}_m the table for vector-valued isotropic functions given in [4,6,8]. Proceeding as in section 2, we find the representation of $\mathbf{f}^\alpha$ and then, by (3.2), the abovementioned expression for f^α.

4. Complete representation for second order symmetric tensor-valued isotropic functions.

Evey second order symmetric tensor-valued isotropic function $f^{\alpha\beta}$ of V^α, V^α_p, $A^{\alpha\beta}_n$, $W^{\alpha\beta}_m$ $(p = 1, \ldots, P; n = 1, \ldots, N; m = 1, \ldots, M)$ can be expressed as a linear combination through scalar coefficients of the tensors in (4.1) and (4.2)

$$(4.1) \qquad I, A_{n_1}, A_{n_1} A_{n_1}, A_{n_1} A_{n_2} + A_{n_2} A_{n_1},$$

$$\vec{V}_{p_1} \otimes \vec{V}_{p_1}, A_{n_1} \vec{V} \otimes A_{n_1} \vec{V}, W_{m_1} \vec{V} \otimes W_{m_1} \vec{V},$$

$$A_{n_1} \vec{V} \otimes A_{n_2} \vec{V} + A_{n_2} \vec{V} \otimes A_{n_1} \vec{V}, W_{m_1} \vec{V} \otimes W_{m_2} \vec{V} + W_{m_2} \vec{V} \otimes W_{m_1} \vec{V}, W_{m_1} W_{m_1},$$

$$\vec{V}_{p_1} \otimes \vec{V}_{p_2} + \vec{V}_{p_2} \otimes \vec{V}_{p_1}, \vec{V}_{p_1} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes \vec{V}_{p_1}, \vec{V}_{p_1} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V} \otimes \vec{V}_{p_1},$$

$$A_{n_1} \vec{V} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes A_{n_1} \vec{V}, W_{m_1} W_{m_2} + W_{m_2} W_{m_1},$$

$$\vec{V}_{p_1} \otimes W_{m_1} \vec{V}_{p_1} + W_{m_1} \vec{V}_{p_1} \otimes \vec{V}_{p_1}, \vec{V}_{p_1} \otimes A_{n_1} \vec{V}_{p_1} + A_{n_1} \vec{V}_{p_1} \otimes \vec{V}_{p_1},$$

$$A_{n_3} \vec{V} \otimes A_{n_1} A_{n_3} \vec{V} + A_{n_1} A_{n_3} \vec{V} \otimes A_{n_3} \vec{V}, W_{m_3} \vec{V} \otimes A_{n_1} W_{m_3} \vec{V} + A_{n_1} W_{m_3} \vec{V} \otimes W_{m_3} \vec{V},$$

$$A_{n_1} W_{m_1} - W_{m_1} A_{n_1}, W_{m_3} \vec{V} \otimes A_{n_1} A_{n_1} W_{m_3} \vec{V} + A_{n_1} A_{n_1} W_{m_3} \vec{V} \otimes W_{m_3} \vec{V},$$

$$A_{n_3} \vec{V}_{p_1} \otimes \vec{V}_{p_2} + \vec{V}_{p_2} \otimes A_{n_3} \vec{V}_{p_1} - (A_{n_3} \vec{V}_{p_2} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_3} \vec{V}_{p_2}),$$

$$A_{n_3} A_{n_1} \vec{V} \otimes A_{n_2} \vec{V} + A_{n_2} \vec{V} \otimes A_{n_3} A_{n_1} \vec{V} - (A_{n_3} A_{n_2} \vec{V} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V} \otimes A_{n_3} A_{n_2} \vec{V}),$$

$$A_{n_3} W_{m_1} \vec{V} \otimes W_{m_2} \vec{V} + W_{m_2} \vec{V} \otimes A_{n_3} W_{m_1} \vec{V} -$$

$$-(A_{n_3} W_{m_2} \vec{V} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes A_{n_3} W_{m_2} \vec{V}),$$

$$A_{n_3} \vec{V}_{p_1} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V} \otimes A_{n_3} \vec{V}_{p_1} - (A_{n_3} A_{n_1} \vec{V} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_3} A_{n_1} \vec{V}),$$

$$A_{n_3} \vec{V}_{p_1} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes A_{n_3} \vec{V}_{p_1} - (A_{n_3} W_{m_1} \vec{V} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_3} W_{m_1} \vec{V}),$$

$$A_{n_3} A_{n_1} \vec{V} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes A_{n_3} A_{n_1} \vec{V} -$$

$$-(A_{n_3} W_{m_1} \vec{V} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V} \otimes A_{n_3} W_{m_1} \vec{V}),$$

$$W_{m_1}A_{n_1}W_{m_1}W_{m_1} - W_{m_1}W_{m_1}A_{n_1}W_{m_1} - (\vec{V}\cdot\vec{V})^{-1}(W_{m_1}\vec{V}\otimes W_{m_1}W_{m_1}A_{n_1}\vec{V}+$$

$$+W_{m_1}W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V} - W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}W_{m_1}\vec{V} - W_{m_1}A_{n_1}W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}+$$

$$+W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V} + W_{m_1}W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}),$$

$$W_{m_1}W_{m_3}\vec{V}\otimes W_{m_1}W_{m_3}\vec{V},\ A_{n_1}\vec{V}\otimes W_{m_1}A_{n_1} + W_{m_1}A_{n_1}\otimes A_{n_1}\vec{V},$$

$$W_{m_3}\vec{V}\otimes W_{m_1}W_{m_3}\vec{V} + W_{m_1}W_{m_3}\vec{V}\otimes W_{m_3}\vec{V},$$

$$W_{m_3}\vec{V}_{p1}\otimes\vec{V}_{p2} + \vec{V}_{p2}\otimes W_{m_3}\vec{V}_{p1} - W_{m_3}\vec{V}_{p2}\otimes\vec{V}_{p1} - \vec{V}_{p1}\otimes W_{m_3}\vec{V}_{p2},$$

$$W_{m_3}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V} + A_{n_2}\vec{V}\otimes W_{m_3}A_{n_1}\vec{V} -$$

$$-W_{m_3}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V} - A_{n_1}\vec{V}\otimes W_{m_3}A_{n_2}\vec{V},$$

$$W_{m_3}W_{m_1}\vec{V}\otimes W_{m_2}\vec{V} + W_{m_2}\vec{V}\otimes W_{m_3}W_{m_1}\vec{V} -$$

$$-W_{m_3}W_{m_2}\vec{V}\otimes W_{m_1}\vec{V} - W_{m_1}\vec{V}\otimes W_{m_3}W_{m_2}\vec{V},$$

$$W_{m_3}\vec{V}_{p1}\otimes A_{n_1}\vec{V} + A_{n_1}\vec{V}\otimes W_{m_3}\vec{V}_{p1} - W_{m_3}A_{n_1}\vec{V}\otimes\vec{V}_{p1} - \vec{V}_{p1}\otimes W_{m_3}A_{n_1}\vec{V},$$

$$W_{m_3}\vec{V}_{p1}\otimes W_{m_1}\vec{V} + W_{m_1}\vec{V}\otimes W_{m_3}\vec{V}_{p1} - W_{m_3}W_{m_1}\vec{V}\otimes\vec{V}_{p1} - \vec{V}_{p1}\otimes W_{m_3}W_{m_1}\vec{V},$$

$$W_{m_3}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V} + W_{m_1}\vec{V}\otimes W_{m_3}A_{n_1}\vec{V} -$$

$$-W_{m_3}W_{m_1}\vec{V}\otimes A_{n_1}\vec{V} - A_{n_1}\vec{V}\otimes W_{m_3}W_{m_1}\vec{V},$$

$$\vec{V}\otimes\vec{f} + \vec{f}\otimes\vec{V}.$$

(4.2) $$A_{n_1}A_{n_2}A_{n_2} + A_{n_2}A_{n_2}A_{n_1},\ A_{n_2}A_{n_1}A_{n_1} + A_{n_1}A_{n_1}A_{n_2},$$

$$W_{m_1}W_{m_2}W_{m_2} - W_{m_2}W_{m_2}W_{m_1},$$

$$W_{m_2}W_{m_1}W_{m_1} - W_{m_1}W_{m_1}W_{m_2},\ \vec{V}_{p_1} \otimes A_{n_1}A_{n_1}\vec{V}_{p_1} + A_{n_1}A_{n_1}\vec{V}_{p_1} \otimes \vec{V}_{p_1},$$

$$W_{m_1}\vec{V}_{p_1} \otimes W_{m_1}W_{m_1}\vec{V}_{p_1} + W_{m_1}W_{m_1}\vec{V}_{p_1} \otimes W_{m_1}\vec{V}_{p_1} -$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(W_{m_1}\vec{V} \otimes W_{m_1}W_{m_1}\vec{V}_{p_1} + W_{m_1}W_{m_1}\vec{V}_{p_1} \otimes W_{m_1}\vec{V} +$$

$$+W_{m_1}\vec{V}_{p_1} \otimes W_{m_1}W_{m_1}\vec{V} + W_{m_1}W_{m_1}\vec{V} \otimes W_{m_1}\vec{V}_{p_1}),$$

$$A_{n_3}\vec{V} \otimes A_{n_1}A_{n_1}A_{n_3}\vec{V} + A_{n_1}A_{n_1}A_{n_3}\vec{V} \otimes A_{n_3}\vec{V},$$

$$W_{m_1}A_{n_1}W_{m_1},\ A_{n_1}A_{n_1}W_{m_1} - W_{m_1}A_{n_1}A_{n_1},$$

$$W_{m_1}\vec{V}_{p_1} \otimes W_{m_1}\vec{V}_{p_1},\ W_{m_1}A_{n_1}\vec{V} \otimes W_{m_1}A_{n_1}\vec{V},$$

$$W_{m_1}A_{n_1}\vec{V} \otimes W_{m_1}W_{m_1}A_{n_1}\vec{V} + W_{m_1}W_{m_1}A_{n_1}\vec{V} \otimes W_{m_1}A_{n_1}\vec{V} -$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}\vec{V})(W_{m_1}\vec{V} \otimes W_{m_1}W_{m_1}A_{n_1}\vec{V} + W_{m_1}W_{m_1}A_{n_1}\vec{V} \otimes W_{m_1}\vec{V} +$$

$$+W_{m_1}A_{n_1}\vec{V} \otimes W_{m_1}W_{m_1}\vec{V} + W_{m_1}W_{m_1}\vec{V} \otimes W_{m_1}A_{n_1}\vec{V}),$$

$$W_{m_1}W_{m_3}\vec{V} \otimes W_{m_1}W_{m_1}W_{m_3}\vec{V} + W_{m_1}W_{m_1}W_{m_3}\vec{V} \otimes W_{m_1}W_{m_3}\vec{V},$$

where $\vec{f}$ is the generic element of set (3.1),

$$P_1, P_2 = 1, \ldots, P \quad \text{and} \quad P_1 < P_2,$$

$$n_1, n_2, n_3 = 1, \ldots, N \quad \text{and} \quad n_1 < n_2,$$

$$m_1, m_2, m_3 = 1, \ldots, M \quad \text{and} \quad m_1 < m_2.$$

Infact, if we define

$$f^\alpha = f^{\alpha\beta}V_\beta(\vec{V}\cdot\vec{V})^{-1} - \frac{1}{2}f^{\mu\nu}V_\mu V_\nu V^\alpha(\vec{V}\cdot\vec{V})^{-2}, \mathbf{f}^{\alpha\beta} = f^{\mu\nu}h^\alpha_\mu h^\beta_\nu, \tag{4.3}$$

we have $f^{\alpha\beta} = \mathbf{f}^{\alpha\beta} + 2f^{(\alpha}V^{\beta)}$.

Then we can use for f^{α} the representation found in section 3 and it remains to find only that for $\mathbf{f}^{\alpha\beta}$. Let us take, as before, $(-\vec{V}\cdot\vec{V})^{-\frac{1}{2}}\vec{V}$ as 0-axis and complete it to an orthonormal base of the vector space. In this reference frame we have

$$\mathbf{f}^{\alpha 0} = 0, \quad \mathbf{f}^{ij} = f^{ij}.$$

If we consider the restriction of condition $(1.1)_3$ to the transformations of the reference frame involving only the 1,2,3 axis, we obtain that they leave $\mathbf{f}^{\alpha 0} = 0$ while transform $\mathbf{f}^{ij} = f^{ij}$ as a 3-dimensional euclidean symmetric second order tensor of the variables $V^i_p, A^{i0}_n, W^{i0}_m, A^{ij}_n, W^{ij}_m$; applying the corresponding table found in [4,6,8] and proceeding as in section 2 we find the expression for $\mathbf{f}^{\alpha\beta}$ and than, from (4.3) that $f^{\alpha\beta}$ is a linear combination through scalar coefficients of the elements in (4.1) and table 2; but in the appendix, proposition 4, we prove that if $\vec{x}, \vec{y}$ are two 3-dimensional vectors, A and W are two 3-dimensional tensors, symmetric the first and skew-symmetric the other, we have

$$\begin{aligned}\vec{x}\otimes A\vec{y} + A\vec{y}\otimes\vec{x} = {} & a_1\vec{x}\otimes\vec{x} + a_2(\vec{x}\otimes\vec{y} + \vec{y}\otimes\vec{x}) + a_3\vec{y}\otimes\vec{y} + \\ & + a_4(\vec{x}\otimes A\vec{x} + A\vec{x}\otimes\vec{x}) + \\ & + a_5(\vec{x}\otimes A\vec{y} + A\vec{y}\otimes\vec{x} - \vec{y}\otimes A\vec{x} - A\vec{x}\otimes\vec{y}),\end{aligned}$$

$$\begin{aligned}\vec{x}\otimes W\vec{y} + W\vec{y}\otimes\vec{x} = {} & b_1\vec{x}\otimes\vec{x} + b_2(\vec{x}\otimes\vec{y} + \vec{y}\otimes\vec{x}) + b_3\vec{y}\otimes\vec{y} + \\ & + b_4(\vec{x}\otimes W\vec{x} + W\vec{x}\otimes\vec{x}) + \\ & + b_5(\vec{x}\otimes W\vec{y} + W\vec{y}\otimes\vec{x} - \vec{y}\otimes W\vec{x} - W\vec{x} - W\vec{x}\otimes\vec{y}).\end{aligned}$$

with a_i, b_i scalar coefficients.

If we apply these relations to V^i_p, A^{i0}_n, W^{i0}_m, A^{ij}_n, W^{ij}_m and follow for both sides of them the abovementioned procedure, we find that the tensors in table 2 are linear combinations of those in (4.1) and (4.2); then they constitute a complete set of generators for second order tensor-valued symmetric isotropic functions.

Table 2

$$A_{n_1}A_{n_2}A_{n_2}+A_{n_2}A_{n_2}A_{n_1}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_1}\vec{V}\otimes A_{n_2}A_{n_2}\vec{V}+A_{n_2}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}),$$

$$A_{n_2}A_{n_1}A_{n_1}+A_{n_1}A_{n_1}A_{n_2}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_2}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}+A_{n_1}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}),$$

$$W_{m_1}W_{m_2}W_{m_2}-W_{m_2}W_{m_2}W_{m_1}-(\vec{V}\cdot\vec{V})^{-1}(W_{m_1}\vec{V}\otimes W_{m_2}W_{m_2}\vec{V}+W_{m_2}W_{m_2}\vec{V}\otimes W_{m_1}\vec{V}),$$

$$W_{m_2}W_{m_1}W_{m_1}-W_{m_1}W_{m_1}W_{m_2}-(\vec{V}\cdot\vec{V})^{-1}(W_{m_2}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V}+$$

$$+W_{m_1}W_{m_1}\vec{V}\otimes W_{m_2}\vec{V}),\ \vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}_{p_1}+A_{n_1}A_{n_1}\vec{V}_{p_1}\otimes\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(\vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}+A_{n_1}A_{n_1}\vec{V}\otimes\vec{V}_{p_1}),$$

$$A_{n_3}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_3}\vec{V}+A_{n_1}A_{n_1}A_{n_3}\vec{V}\otimes A_{n_3}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_3}\vec{V})(A_{n_3}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}+A_{n_1}A_{n_1}\vec{V}\otimes A_{n_3}\vec{V}),$$

$$W_{m_1}A_{n_1}W_{m_1}+(\vec{V}\cdot\vec{V})^{-1}(W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}),$$

$$A_{n_1}A_{n_1}W_{m_1}-W_{m_1}A_{n_1}A_{n_1}+(\vec{V}\cdot\vec{V})^{-1}(W_{m_1}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}+A_{n_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}),$$

$$W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}\vec{V}_{p_1}-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}_{p_1}+W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}\vec{V}),$$

$$W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}\vec{V})\cdot$$

$$\cdot(W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}+W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}),$$

$$W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}W_{m_1}\vec{V}_{p_1}+W_{m_1}W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot W_{m_1}\vec{V}_{p_1})(W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}_{p_1})-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(W_{m_1}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V}_{p_1}+W_{m_1}W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}\vec{V}+$$

$$+W_{m_1}\vec{V}_{p_1}\otimes W_{m_1}W_{m_1}\vec{V}+W_{m_1}W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}_{p_1}),$$

$$W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}A_{n_1}\vec{V}+W_{m_1}W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}[(\vec{V}\cdot W_{m_1}A_{n_1}\vec{V})(W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V})+$$

$$+(\vec{V}\cdot A_{n_1}\vec{V})(W_{m_1}\vec{V}\otimes W_{m_1}W_{m_1}A_{n_1}\vec{V}+W_{m_1}W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}+$$

$$+W_{m_1}A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V}+W_{m_1}W_{m_1}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V})],$$

$$W_{m_1}W_{m_3}\vec{V}\otimes W_{m_1}W_{m_1}W_{m_3}\vec{V}+W_{m_1}W_{m_1}W_{m_3}\vec{V}\otimes W_{m_1}W_{m_3}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot W_{m_1}W_{m_3}\vec{V})(W_{m_1}W_{m_3}\vec{V}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}\otimes W_{m_1}W_{m_3}\vec{V}).$$

5. Complete representation for second order skew-symmetric tensor-valued isotropic functions.

If $f^{\alpha\beta}$ is a second order skew-symmetric tensor-valued isotropic function of $V^\alpha, V_p^\alpha, A_n^{\alpha\beta}, W_m^{\alpha\beta}$ $(p=1,\dots,P; n=1,\dots,N; m=1,\dots,M)$ we can use the same considerations of section 4, but with (5.1), (5.2) instead of (4.1), (4.2), $f^{\alpha\beta}=\mathbf{f}^{\alpha\beta}+2f^{[\alpha}V^{\beta]}$ instead of (4.3), table 3 instead of table 2, proposition 5 of the appendix instead of proposition 4 and with the word «skew-symmetric»instead of «symmetric»; in this way we obtain that $f^{\alpha\beta}$ can be expressed as a linear combination, through scalar coefficients, of the tensors in (5.1), (5.2).

$$W_{m_1},\, W_{m_1}W_{m_2}-W_{m_2}W_{m_1},\, A_{n_1}A_{n_2}-A_{n_2}A_{n_1}, \tag{5.1}$$

$$\vec{V}_{p_1} \otimes \vec{V}_{p_2} - \vec{V}_{p_2} \otimes \vec{V}_{p_1}, A_{n_1} W_{m_1} + W_{m_1} A_{n_1},$$

$$A_{n_1} \vec{V} \otimes A_{n_2} \vec{V} - A_{n_2} \vec{V} \otimes A_{n_1} \vec{V}, W_{m_1} \vec{V} \otimes W_{m_2} \vec{V} - W_{m_2} \vec{V} \otimes W_{m_1} \vec{V},$$

$$\vec{V}_{p_1} \otimes A_{n_1} \vec{V} - A_{n_1} \vec{V} \otimes \vec{V}_{p_1}, \vec{V}_{p_1} \otimes W_{m_1} \vec{V} - W_{m_1} \vec{V} \otimes \vec{V}_{p_1},$$

$$A_{n_1} \vec{V} \otimes W_{m_1} \vec{V} - W_{m_1} \vec{V} \otimes A_{n_1} \vec{V}, \vec{V}_{p_1} \otimes A_{n_1} \vec{V}_{p_1} - A_{n_1} \vec{V}_{p_1} \otimes \vec{V}_{p_1},$$

$$A_{n_1} \vec{V} \otimes A_{n_3} A_{n_1} \vec{V} - A_{n_3} A_{n_1} \vec{V} \otimes A_{n_1} \vec{V}, W_{m_1} \vec{V} \otimes A_{n_1} W_{m_1} \vec{V} - A_{n_1} W_{m_1} \vec{V} \otimes W_{m_1} \vec{V},$$

$$W_{m_1} \vec{V} \otimes A_{n_1} A_{n_1} W_{m_1} \vec{V} - A_{n_1} A_{n_1} W_{m_1} \vec{V} \otimes W_{m_1} \vec{V},$$

$$A_{n_1} \vec{V}_{p_1} \otimes \vec{V}_{p_2} - \vec{V}_{p_2} \otimes A_{n_1} \vec{V}_{p_1} - A_{n_1} \vec{V}_{p_2} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_1} \vec{V}_{p_2},$$

$$A_{n_4} A_{n_1} \vec{V} \otimes A_{n_2} \vec{V} - A_{n_2} \vec{V} \otimes A_{n_4} A_{n_1} \vec{V} - A_{n_4} A_{n_2} \vec{V} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V} \otimes A_{n_4} A_{n_2} \vec{V},$$

$$A_{n_1} W_{m_1} \vec{V} \otimes W_{m_2} \vec{V} - W_{m_2} \vec{V} \otimes A_{n_1} W_{m_1} \vec{V} - A_{n_1} W_{m_2} \vec{V} \otimes W_{m_1} \vec{V} + W_{m_1} \vec{V} \otimes A_{n_1} W_{m_2} \vec{V},$$

$$A_{n_1} \vec{V}_{p_1} \otimes A_{n_4} \vec{V} - A_{n_4} \vec{V} \otimes A_{n_1} \vec{V}_{p_1} - A_{n_1} A_{n_4} \vec{V} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_1} A_{n_4} \vec{V},$$

$$A_{n_1} \vec{V}_{p_1} \otimes W_{m_1} \vec{V} - W_{m_1} \vec{V} \otimes A_{n_1} \vec{V}_{p_1} - A_{n_1} W_{m_1} \vec{V} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes A_{n_1} W_{m_1} \vec{V},$$

$$A_{n_1} A_{n_4} \vec{V} \otimes W_{m_1} \vec{V} - W_{m_1} \vec{V} \otimes A_{n_1} A_{n_4} \vec{V} -$$

$$- A_{n_1} W_{m_1} \vec{V} \otimes A_{n_4} \vec{V} + A_{n_4} \vec{V} \otimes A_{n_1} W_{m_1} \vec{V},$$

$$\vec{V}_{p_1} \otimes W_{m_1} \vec{V}_{p_1} - W_{m_1} \vec{V}_{p_1} \otimes \vec{V}_{p_1}, A_{n_1} \vec{V} \otimes W_{m_1} A_{n_1} \vec{V} - W_{m_1} A_{n_1} \vec{V} \otimes A_{n_1} \vec{V},$$

$$W_{m_3} \vec{V} \otimes W_{m_1} W_{m_3} \vec{V} - W_{m_1} W_{m_3} \vec{V} \otimes W_{m_3} \vec{V},$$

$$W_{m_3} \vec{V} \otimes W_{m_1} W_{m_1} W_{m_3} \vec{V} - W_{m_1} W_{m_1} W_{m_3} \vec{V} \otimes W_{m_3} \vec{V},$$

$$W_{m_1} \vec{V}_{p_1} \otimes \vec{V}_{p_2} - \vec{V}_{p_2} \otimes W_{m_1} \vec{V}_{p_1} - W_{m_1} \vec{V}_{p_2} \otimes \vec{V}_{p_1} + \vec{V}_{p_1} \otimes W_{m_1} \vec{V}_{p_2},$$

$$W_{m_1}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}-A_{n_2}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}-W_{m_1}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}+$$

$$+A_{n_1}\vec{V}\otimes W_{m_1}A_{n_2}\vec{V},$$

$$W_{m_3}W_{m_1}\vec{V}\otimes W_{m_2}\vec{V}-W_{m_2}\vec{V}\otimes W_{m_3}W_{m_1}\vec{V}-$$

$$-W_{m_3}W_{m_2}\vec{V}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}\otimes W_{m_3}W_{m_2}\vec{V},$$

$$W_{m_1}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes W_{m_1}\vec{V}_{p_1}-W_{m_1}A_{n_1}\vec{V}\otimes\vec{V}_{p_1}+$$

$$+\vec{V}_{p_1}\otimes W_{m_1}A_{n_1}\vec{V},$$

$$W_{m_1}\vec{V}_{p_1}\otimes W_{m_3}\vec{V}-W_{m_3}\vec{V}\otimes W_{m_1}\vec{V}_{p_1}-W_{m_1}W_{m_3}\vec{V}\otimes\vec{V}_{p_1}+$$

$$+\vec{V}_{p_1}\otimes W_{m_1}W_{m_3}\vec{V},$$

$$W_{m_1}A_{n_1}\vec{V}\otimes W_{m_3}\vec{V}-W_{m_3}\vec{V}\otimes W_{m_1}A_{n_1}\vec{V}-W_{m_1}W_{m_3}\vec{V}\otimes A_{n_1}\vec{V}+$$

$$+A_{n_1}\vec{V}\otimes W_{m_1}W_{m_3}\vec{V},$$

$$A_{n_1}W_{m_1}\vec{V}\otimes A_{n_2}W_{m_1}\vec{V}-A_{n_2}W_{m_1}\vec{V}\otimes A_{n_1}W_{m_1}\vec{V}+$$

$$+W_{m_1}\vec{V}\otimes A_{n_1}A_{n_2}W_{m_1}\vec{V}-$$

$$-A_{n_1}A_{n_2}W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}-W_{m_1}\vec{V}\otimes A_{n_2}A_{n_1}W_{m_1}\vec{V}+A_{n_2}A_{n_1}W_{m_1}\vec{V}\otimes W_{m_1}\vec{V},$$

$$A_{n_1}A_{n_4}\vec{V}\otimes A_{n_2}A_{n_4}\vec{V}-A_{n_2}A_{n_4}\vec{V}\otimes A_{n_1}A_{n_4}\vec{V}+A_{n_4}\vec{V}\otimes A_{n_1}A_{n_2}A_{n_4}\vec{V}-$$

$$-A_{n_1}A_{n_2}A_{n_4}\vec{V}\otimes A_{n_4}\vec{V}-A_{n_4}\vec{V}\otimes A_{n_2}A_{n_1}A_{n_4}\vec{V}+A_{n_2}A_{n_1}A_{n_4}\vec{V}\otimes A_{n_4}\vec{V},$$

$$A_{n_1}\vec{V}_{p_1}\otimes A_{n_2}\vec{V}_{p_1}-A_{n_2}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}_{p_1}+\vec{V}_{p_1}\otimes A_{n_1}A_{n_2}\vec{V}_{p_1}-A_{n_1}A_{n_2}\vec{V}_{p_1}\otimes\vec{V}_{p_1}-$$

$$-\vec{V}_{p_1}\otimes A_{n_2}A_{n_1}\vec{V}_{p_1}+A_{n_2}A_{n_1}\vec{V}_{p_1}\otimes\vec{V}_{p_1},$$

$$A_{n_1} A_{n_2} A_{n_3} + A_{n_2} A_{n_3} A_{n_1} + A_{n_3} A_{n_1} A_{n_2} - A_{n_2} A_{n_1} A_{n_3} - A_{n_1} A_{n_3} A_{n_2} - A_{n_3} A_{n_2} A_{n_1},$$

$$\vec{V} \otimes \vec{f} - \vec{f} \otimes \vec{V}.$$

(5.2)
$$A_{n_1} A_{n_1} A_{n_2} - A_{n_2} A_{n_1} A_{n_1},\ A_{n_2} A_{n_2} A_{n_1} - A_{n_1} A_{n_2} A_{n_2},$$

$$\vec{V}_{p_1} \otimes A_{n_1} A_{n_1} V_{p_1} - A_{n_1} A_{n_1} V_{p_1} \otimes V_{p_1},$$

$$A_{n_4} \vec{V} \otimes A_{n_1} A_{n_1} A_{n_4} \vec{V} - A_{n_1} A_{n_1} A_{n_4} \vec{V} \otimes A_{n_4} \vec{V},$$

$$A_{n_1} W_{m_1} \vec{V} \otimes A_{n_1} A_{n_1} W_{m_1} \vec{V} - A_{n_1} A_{n_1} W_{m_1} \vec{V} \otimes A_{n_1} W_{m_1} \vec{V},$$

$$A_{n_1} \vec{V} \otimes W_{m_1} W_{m_1} A_{n_1} \vec{V} - W_{m_1} W_{m_1} A_{n_1} \vec{V} \otimes A_{n_1} \vec{V},$$

$$\vec{V}_{p_1} \otimes W_{m_1} W_{m_1} \vec{V}_{p_1} - W_{m_1} W_{m_1} \vec{V}_{p_1} \otimes \vec{V}_{p_1},$$

$$A_{n_1} W_{m_1} W_{m_1} - W_{m_1} W_{m_1} A_{n_1} +$$

$$+(\vec{V} \cdot \vec{V})^{-1}(+A_{n_1} W_{m_1} \vec{V} \otimes W_{m_1} \vec{V} - W_{m_1} \vec{V} \otimes A_{n_1} W_{m_1} \vec{V}),$$

$$A_{n_1} A_{n_4} \vec{V} \otimes A_{n_1} A_{n_1} A_{n_4} \vec{V} - A_{n_1} A_{n_1} A_{n_4} \vec{V} \otimes A_{n_1} A_{n_4} \vec{V} -$$

$$-(\vec{V} \cdot \vec{V})^{-1}(\vec{V} \cdot A_{n_4} \vec{V})(A_{n_1} \vec{V} \otimes A_{n_1} A_{n_1} A_{n_4} \vec{V} - A_{n_1} A_{n_1} A_{n_4} \vec{V} \otimes A_{n_1} +$$

$$+A_{n_1} A_{n_4} \vec{V} \otimes A_{n_1} A_{n_1} \vec{V} - A_{n_1} A_{n_1} \vec{V} \otimes A_{n_1} A_{n_4} \vec{V}),$$

$$A_{n_1} \vec{V}_{p_1} \otimes A_{n_1} A_{n_1} \vec{V}_{p_1} - A_{n_1} A_{n_1} \vec{V}_{p_1} \otimes A_{n_1} \vec{V}_{p_1} -$$

$$-(\vec{V} \cdot \vec{V})^{-1}(\vec{V} \cdot \vec{V}_{p_1})(A_{n_1} \vec{V} \otimes A_{n_1} A_{n_1} \vec{V}_{p_1} -$$

$$-A_{n_1} A_{n_1} \vec{V}_{p_1} \otimes A_{n_1} \vec{V} + A_{n_1} \vec{V}_{p_1} \otimes A_{n_1} A_{n_1} \vec{V} - A_{n_1} A_{n_1} \vec{V} \otimes A_{n_1} \vec{V}_{p_1}),$$

$$A_{n_2} A_{n_1} A_{n_2} A_{n_2} - A_{n_2} A_{n_2} A_{n_1} A_{n_2} - (\vec{V} \cdot \vec{V})^{-1}(A_{n_2} A_{n_1} \vec{V} \otimes A_{n_2} A_{n_2} \vec{V} -$$

$$-A_{n_2}A_{n_2}\vec{V}\otimes A_{n_2}A_{n_1}\vec{V}+A_{n_2}\vec{V}\otimes A_{n_2}A_{n_2}A_{n_1}\vec{V}-A_{n_2}A_{n_2}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}+$$

$$+A_{n_2}A_{n_1}A_{n_2}\vec{V}\otimes A_{n_2}\vec{V}-A_{n_2}\vec{V}\otimes A_{n_2}A_{n_1}A_{n_2}\vec{V}),$$

$$A_{n_1}A_{n_2}A_{n_1}A_{n_1}-A_{n_1}A_{n_1}A_{n_2}A_{n_1}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_1}A_{n_2}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}-$$

$$-A_{n_1}A_{n_1}\vec{V}\otimes A_{n_1}A_{n_2}\vec{V}+A_{n_1}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_2}\vec{V}-A_{n_1}A_{n_1}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}+$$

$$+A_{n_1}A_{n_2}A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}A_{n_2}A_{n_1}\vec{V}).$$

where $\vec{f}$ is the generic element of set (3.1) but without $\vec{V}$,

$$p_1,p_2=1,\dots,P \quad \text{and} \quad p_1<p_2,$$

$$n_1,n_2,n_3,n_4=1,\dots,N \quad \text{and} \quad n_1<n_2<n_3,$$

$$m_1,m_2,m_3=1,\dots,M \quad \text{and} \quad m_1<m_2.$$

Table 3

$$A_{n_1}A_{n_1}A_{n_2}-A_{n_2}A_{n_1}A_{n_1}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_1}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}-A_{n_2}\otimes A_{n_1}A_{n_1}\vec{V}),$$

$$A_{n_2}A_{n_2}A_{n_1}-A_{n_1}A_{n_2}A_{n_2}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_2}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_2}A_{n_2}\vec{V}),$$

$$\vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}_{p_1}-A_{n_1}A_{n_1}\vec{V}_{p_1}\otimes\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(\vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}-A_{n_1}A_{n_1}\vec{V}\otimes\vec{V}_{p_1}),$$

$$A_{n_4}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_4}\vec{V}-A_{n_1}A_{n_1}A_{n_4}\vec{V}\otimes A_{n_4}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_4}\vec{V})(A_{n_4}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}-A_{n_1}A_{n_1}\vec{V}\otimes A_{n_4}\vec{V}),$$

$$A_{n_1}W_{m_1}\vec{V}\otimes A_{n_1}A_{n_1}W_{m_1}\vec{V}-A_{n_1}A_{n_1}W_{m_1}\vec{V}\otimes A_{n_1}W_{m_1}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}W_{m_1}\vec{V})(A_{n_1}W_{m_1}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}W_{m_1}\vec{V}),$$

$$A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}A_{n_1}\vec{V}-W_{m_1}W_{m_1}A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}\vec{V})(A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V}-W_{m_1}W_{m_1}\vec{V}\otimes A_{n_1}\vec{V}),$$

$$A_{n_1}W_{m_1}W_{m_1}-W_{m_1}W_{m_1}A_{n_1}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_1}\vec{V}\otimes W_{m_1}W_{m_1}\vec{V}-$$

$$-W_{m_1}W_{m_1}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}W_{m_1}\vec{V}\otimes W_{m_1}\vec{V}+W_{m_1}\vec{V}+A_{n_1}W_{m_1}\vec{V}),$$

$$A_{n_1}A_{n_4}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_4}\vec{V}-A_{n_1}A_{n_1}A_{n_4}\vec{V}\otimes A_{n_1}A_{n_4}\vec{V}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_4}\vec{V})(A_{n_1}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_4}\vec{V}-A_{n_1}A_{n_1}A_{n_4}\vec{V}\otimes A_{n_1}\vec{V}+$$

$$+A_{n_1}A_{n_4}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}-A_{n_1}A_{n_1}\vec{V}\otimes A_{n_1}A_{n_4}\vec{V})+[-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}A_{n_4}\vec{V})+$$

$$+(\vec{V}\cdot\vec{V})^{-2}(\vec{V}\cdot A_{n_4}\vec{V})(\vec{V}\cdot A_{n_1}\vec{V})](A_{n_1}A_{n_4}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}A_{n_4}\vec{V}),$$

$$V_{p_1}\otimes W_{m_1}W_{m_1}\vec{V}_{p_1}-W_{m_1}W_{m_1}\vec{V}_{p_1}\otimes\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(\vec{V}_{p_1}\otimes W_{m_1}W_{m_1}\vec{V}-W_{m_1}W_{m_1}\vec{V}\otimes\vec{V}_{p_1}),$$

$$A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}_{p_1}-A_{n_1}A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}_{p_1}-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot\vec{V}_{p_1})(A_{n_1}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}_{p_1}-$$

$$-A_{n_1}A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}+A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}A_{n_1}\vec{V}-A_{n_1}A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}_{p_1})-$$

$$-(\vec{V}\cdot\vec{V})^{-1}(\vec{V}\cdot A_{n_1}\vec{V}_{p_1})(A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}_{p_1})+$$

$$+(\vec{V}\cdot\vec{V})^{-2}(\vec{V}\cdot\vec{V}_{p_1})(\vec{V}\cdot A_{n_1}\vec{V})(A_{n_1}\vec{V}_{p_1}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}_{p_1}),$$

$$A_{n_2}A_{n_1}A_{n_2}A_{n_2}-A_{n_2}A_{n_2}A_{n_1}A_{n_2}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_2}A_{n_1}\vec{V}\otimes A_{n_2}A_{n_2}\vec{V}-$$

$$-A_{n_2}A_{n_2}\vec{V}\otimes A_{n_2}A_{n_1}\vec{V}+A_{n_2}\vec{V}\otimes A_{n_2}A_{n_2}A_{n_1}\vec{V}-A_{n_2}A_{n_2}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}+$$

$$+A_{n_2}A_{n_1}A_{n_2}\vec{V}\otimes A_{n_2}\vec{V}-A_{n_2}\vec{V}\otimes A_{n_2}A_{n_1}A_{n_2}\vec{V})+$$

$$+(\vec{V}\cdot\vec{V})^{-2}(\vec{V}\cdot A_{n_2}\vec{V})(A_{n_2}A_{n_1}\vec{V}\otimes A_{n_2}\vec{V}-A_{n_2}\vec{V}\otimes A_{n_2}A_{n_1}\vec{V}),$$

$$A_{n_1}A_{n_2}A_{n_1}A_{n_1}-A_{n_1}A_{n_1}A_{n_2}A_{n_1}-(\vec{V}\cdot\vec{V})^{-1}(A_{n_1}A_{n_2}\vec{V}\otimes A_{n_1}A_{n_1}\vec{V}-$$

$$-A_{n_1}A_{n_1}\vec{V}\otimes A_{n_1}A_{n_2}\vec{V}+A_{n_1}\vec{V}\otimes A_{n_1}A_{n_1}A_{n_2}\vec{V}-A_{n_1}A_{n_1}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}+$$

$$+A_{n_1}A_{n_2}A_{n_1}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}A_{n_2}A_{n_1}\vec{V})+$$

$$+(\vec{V}\cdot\vec{V})^{-2}(\vec{V}\cdot A_{n_1}\vec{V})(A_{n_1}A_{n_2}\vec{V}\otimes A_{n_1}\vec{V}-A_{n_1}\vec{V}\otimes A_{n_1}A_{n_2}\vec{V}),$$

6. Conclusion.

In this way we have obtained the complete representations for scalar-valued, vector-valued and second order tensor-valued isotropic functions with an arbitrary number of scalar, vectorial and second order tensorial independent variables.

We hope that they are irreducible too (in the sense that no proper their subset suffices to give a complete representation), because the method we have used is based on the corresponding 3-dimensional representations [4,6] which we proved to be irreducible [8]. A direct prove of irreducibility will be treated in a future work.

We have proved these representations in a pseudo-euclidean vector space with signature $-+++$, but it is obvious that they hold also in a 4-dimensional euclidean vector space if the hypotesis that the 4-vector $\vec{V}$ is time-like is substituted with the corresponding one that it is not null.

Appendix.

In Section 2 we have used the properties that if $\vec{x}$, $\vec{y}$ are two 3-dimensional vectors, A_1 and A_2 two 3-dimensional symmetric tensors, W_1 and W_2 two 3-dimensional skew-symmetric tensors we have that

1) $\vec{x}\cdot A_1A_2\vec{y}$ can be expressed as a function f_1 of the scalars

$$\vec{x}\cdot\vec{x},\vec{x}\cdot\vec{y},\vec{y}\cdot\vec{y},\vec{x}\cdot A_1\vec{x},\vec{x}\cdot A_1\vec{y},\vec{y}\cdot A_1\vec{y},\vec{x}\cdot A_2\vec{x},\vec{x}\cdot A_2\vec{y},\vec{x}\cdot A_1^2\vec{x},$$

$$\vec{x}\cdot A_1^2\vec{y},\vec{y}\cdot A_2\vec{y},\ \vec{x}\cdot A_1A_2\vec{x},\vec{x}\cdot(A_1A_2-A_2A_1)\vec{y},$$

2) $\vec{x}\cdot A_1W_1\vec{y}$ as a function f_2 of

$$\vec{x}\cdot\vec{x},\vec{x}\cdot\vec{y},\vec{y}\cdot\vec{y},\vec{x}\cdot A_1\vec{x},\vec{x}\cdot A_1\vec{y},$$

$$\vec{y}\cdot A_1\vec{y},\vec{x}\cdot A_1^2\vec{x},\vec{x}\cdot A_1^2\vec{y},\vec{x}\cdot W_1\vec{y},$$

$$\vec{x}\cdot A_1W_1\vec{x},\vec{x}\cdot(A_1W_1-W_1A_1)\vec{y},$$

3) $\vec{x}\cdot W_1W_2\vec{y}$ as a function f_3 of

$$\vec{x}\cdot\vec{x},\vec{x}\cdot\vec{y},\vec{y}\cdot\vec{y},\vec{x}\cdot W_1\vec{y},\vec{x}\cdot W_2\vec{y},\vec{x}\cdot W_1^2\vec{x},$$

$$\vec{x}\cdot W_1^2\vec{y},\vec{x}\cdot W_1W_2\vec{x},\vec{x}\cdot(W_1W_2-W_2W_1)\vec{y},trW_1W_2.$$

Infact, if $\vec{x}=\vec{0}$, the propositions are verified with $f_1=0$, $f_2=0$, $f_3=0$; if $\vec{x}\neq\vec{0}$ but $\vec{y}=(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$ then we have

$$\vec{x}\cdot A_1A_2\vec{y}=(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot A_1A_2\vec{x}),$$

$$\vec{x}\cdot A_1W_1\vec{y}=(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot A_1W_1\vec{x}),$$

$$\vec{x}\cdot W_1W_2\vec{y}=(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot W_1W_2\vec{x});$$

lastly, if $\vec{x}\neq 0$ and $\vec{y}\neq(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$ we can define

$$\vec{y}' = \vec{y} - (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$$

and our propositions become equivalent to the following ones

1') $\vec{x}\cdot A_1A_2\vec{y}'$ can be expressed as a function of the scalars

$$\vec{x}\cdot\vec{x}, \vec{x}\cdot A_1\vec{x}, \vec{x}\cdot A_2\vec{x}, \vec{x}\cdot A_1A_1\vec{x}, \vec{x}\cdot A_1A_2\vec{x},$$

$$\vec{y}'\cdot\vec{y}', \vec{x}\cdot A_1\vec{y}', \vec{y}'\cdot A_1\vec{y}', \vec{x}\cdot A_2\vec{y}', \vec{y}'\cdot A_2\vec{y}',$$

$$\vec{x}\cdot A_1A_1\vec{y}', \vec{x}\cdot(A_1A_2 - A_2A_1)\vec{y}',$$

2') $\vec{x}\cdot A_1W_1\vec{y}'$ as a function of $\vec{x}\cdot\vec{x}$, $\vec{x}\cdot A_1\vec{x}$, $\vec{x}\cdot A_1A_1\vec{x}$,

$$\vec{x}\cdot A_1W_1\vec{x}, \vec{y}'\cdot\vec{y}', \vec{x}\cdot W_1y', \vec{x}\cdot A_1\vec{y}', \vec{y}'\cdot A_1\vec{y}',$$

$$\vec{x}\cdot A_1A_1\vec{y}', \vec{x}\cdot(A_1W_1 - W_1A_1)\vec{y}',$$

3') $\vec{x}\cdot W_1W_2\vec{y}'$ as a function of the scalars $\vec{x}\cdot\vec{x}, \vec{x}\cdot W_1W_1\vec{x}$,

$$\vec{x}\cdot W_1W_2\vec{x}, \vec{y}'\cdot\vec{y}', \vec{x}\cdot W_1\vec{y}', \vec{x}\cdot W_2\vec{y}', \vec{x}\cdot W_1W_1y',$$

$$\vec{x}\cdot(W_1W_2 - W_2W_1)\vec{y}', trW_1W_2.$$

Infact we have

$$\vec{x}\cdot A_1A_2\vec{y} = (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot A_1A_2\vec{x}) + \vec{x}\cdot A_1A_2\vec{y}',$$

$$\vec{x}\cdot A_1W_1\vec{y} = (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot A_1W_1\vec{x}) + \vec{x}\cdot A_1W_1\vec{y}',$$

$$\vec{x}\cdot W_1W_2\vec{y} = (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})(\vec{x}\cdot W_1W_2\vec{x}) + \vec{x}\cdot W_1W_2\vec{y}',$$

and the scalars in the hypothesis of 1'), 2'), 3') can be found from the corresponding ones in 1), 2), 3).

Let us prove 1'). In the reference frame in which $\vec{x}\equiv(x,0,0)$, $\vec{y}'\equiv(0,y'0)$ with $x>0$, $y'>0$, we obtain x and y' from $\vec{x}\cdot\vec{x}$ and $\vec{y}'\cdot\vec{y}'$,

$$A_1^{11}, A_1^{12}, A_1^{22}, A_2^{11}, A_2^{12}, A_2^{22} \quad \text{from} \quad \vec{x}\cdot A_1\vec{x}, \vec{x}\cdot A_1\vec{y}', \vec{y}'\cdot A_1\vec{y}',$$

$$\vec{x} \cdot A_2\vec{x}, \vec{x} \cdot A_2\vec{y}', \vec{y}' \cdot A_2\vec{y}',$$

$(A_1^{13})^2$ and $A_1^{13}A_1^{23}$ from $\vec{x} \cdot A_1A_1\vec{x}$, $\vec{x} \cdot A_1A_1\vec{y}'$;

If $A_1^{13} = 0$ then $\vec{x} \cdot A_1A_2\vec{y}' = xy'(A_1^{11}A_2^{12} + A_1^{12}A_2^{22})$ and then it is known as function of the scalars already used; if $A_1^{13} \neq 0$ we choose the sense of the 3-axis such that $A_1^{13} > 0$ and obtain A_1^{13}, $A_1^{23}, A_1^{13}, A_2^{23}$ from $(A_1^{13})^2$, A_1^{13} A_1^{23}, $\vec{x} \cdot A_1A_2\vec{x}$, and $\vec{x} \cdot (A_1A_2 - A_2A_1)\vec{y}'$.

But

$$\vec{x} \cdot A_1A_2\vec{y}' = xy'(A_1^{11}A_2^{12} + A_1^{12}A_2^{22} + A_1^{13}A_2^{23})$$

and then is determined as function of the scalars we used.

Similarly to prove 2') let us use the reference frame in which $\vec{x} \equiv (x, 0, 0), \vec{y}' \equiv (0, y', 0)$ with $x > 0$, $y' > 0$; we obtain x and y' from $\vec{x} \cdot \vec{x}$ and $\vec{y}' \cdot \vec{y}'$,

$$A_1^{11}, A_1^{12}, A_1^{22}, W_1^{12} \quad \text{from} \quad \vec{x} \cdot A_1\vec{x}, \vec{x} \cdot A_1\vec{y}', \vec{y}' \cdot A_1\vec{y}', \vec{x} \cdot W_1\vec{y}',$$

$(A_1^{13})^2$ and $A_1^{13}A_1^{23}$ from $\vec{x} \cdot A_1A_1\vec{x}, \vec{x} \cdot A_1A_1\vec{y}'$;

If $A_1^{13} = 0$ then $\vec{x} \cdot A_1W_1\vec{y}' = xy'A_1^{11}W_1^{12}$ and then it is known as function of the scalars already used; if $A_1^{13} \neq 0$ we choose the sense of the 3-axis such that $A_1^{13} > 0$ and obtain

$$A_1^{13}, A_1^{23}, W_1^{13}, W_1^{23} \quad \text{from} \quad (A_1^{13})^2, A_1^{13}A_1^{23}, \vec{x} \cdot A_1W_1\vec{x},$$

and $\vec{x} \cdot (A_1W_1 - W_1A_1)\vec{y}'$.

But

$$\vec{x} \cdot A_1W_1\vec{y}' = xy'(A_1^{11}W_1^{12} + A_1^{13}W_1^{32})$$

and then is determined as function of the scalars we used.

It remains to prove 3'). In the reference frame in which $\vec{x} \equiv (x, 0, 0)$, $\vec{y}' \equiv (0, y', 0)$ with $x > 0$, $y' > 0$, we obtain x and y' from $\vec{x} \cdot \vec{x}$ and $\vec{y}' \cdot \vec{y}'$,

$$W_1^{12}, W_2^{12}, (W_1^{13})^2 \quad \text{from} \quad \vec{x} \cdot W_1\vec{y}', \vec{x} \cdot W_2\vec{y}', \vec{x} \cdot W_1W_1\vec{x};$$

If $W_1^{13} = 0$ then $\vec{x} \cdot W_1W_2\vec{y}' = 0$ and then it is known.

If $W_1^{13}\neq 0$ we choose the sense of the 3-axis such that $W_1^{13} > 0$ and obtain W_1^{13}, W_1^{23} from $(W_1^{13})^2$, $\vec{x}\cdot W_1 W_1 \vec{y}'$; moreover $W_1^{13}W_2^{31}+W_1^{23}W_2^{32}$ and $W_1^{13}W_2^{32} - W_2^{13}W_1^{32}$ can be found from trW_1W_2 and $\vec{x}\cdot(W_1W_2 - W_2W_1)\vec{y}'$; from them W_2^{13} and W_2^{23} can be obtained because $(W_1^{13})^2 + (W_1^{23})^2\neq 0$. But $\vec{x}\cdot W_1 W_2 \vec{y}' = xy'$, $W_1^{13}W_2^{32}$ and then is determined as function of the scalars we used.

In section 4 we have also used the properties that if $\vec{x}$, $\vec{y}$ are two 3-dimensional vectors, A and W are two 3-dimesional tensors, symmetric the first and skew-symmetric the other, then we have

PROPOSITION 4.

$$\begin{aligned}\vec{x}\otimes A\vec{y} + A\vec{y}\otimes\vec{x} = {} & a_1\vec{x}\otimes\vec{x} + a_2(\vec{x}\otimes\vec{y}+\vec{y}\otimes\vec{x}) + a_3\vec{y}\otimes\vec{y}+ \\ & + a_4(\vec{x}\otimes A\vec{x} + A\vec{x}\otimes\vec{x})+ \\ & + a_5(\vec{x}\otimes A\vec{y} + A\vec{y}\otimes\vec{x} - \vec{y}\otimes A\vec{x} - A\vec{x}\otimes\vec{y}),\end{aligned} \tag{5.1}$$

$$\begin{aligned}\vec{x}\otimes W\vec{y} + W\vec{y}\otimes\vec{x} = {} & b_1\vec{x}\otimes\vec{x} + b_2(\vec{x}\otimes\vec{y}+\vec{y}\otimes\vec{x}) + b_3\vec{y}\otimes\vec{y}+ \\ & + b_4(\vec{x}\otimes W\vec{x} + W\vec{x}\otimes\vec{x})+ \\ & + b_5(\vec{x}\otimes W\vec{y} + W\vec{y}\otimes\vec{x} - \vec{y}\otimes W\vec{x} - W\vec{x}\otimes\vec{y}).\end{aligned} \tag{5.2}$$

with a_i, b_i scalar coefficients.

Let us prove it. If $\vec{x} = \bar{0}$, then the proposition is true with $a_i = b_i = 0$; if $\vec{x}\neq\bar{0}$, but $\vec{y} = (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$, then the proposition is verified with

$$a_1 = a_2 = a_3 = a_5 = b_1 = b_2 = b_3 = b_5 = 0, a_4 = b_4 = (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y}).$$

If $\vec{x}\neq\bar{0}$ and $\vec{y}\neq(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$, we define

$$\vec{u}_1 = \vec{x}(\vec{x}\cdot\vec{x})^{-\frac{1}{2}}, \tag{5.3}$$

$$\vec{u}_2 = [\vec{y}\cdot\vec{y} - (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})^2]^{-\frac{1}{2}}[\vec{y} - (\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}] \tag{5.4}$$

and see that

$$\begin{aligned}\vec{u}_1 \otimes A\vec{u}_2 + A\vec{u}_2 \otimes \vec{u}_1 = {} & p_1\vec{u}_1 \otimes \vec{u}_1 + p_2(\vec{u}_1 \otimes \vec{u}_2 + \vec{u}_2 \otimes \vec{u}_1)+ \\ & + p_3\vec{u}_2 \otimes \vec{u}_2 + p_4(\vec{u}_1 \otimes A\vec{u}_1 + A\vec{u}_1 \otimes \vec{u}_1)+ \\ & + p_5(\vec{u}_1 \otimes A\vec{u}_2 + A\vec{u}_2 \otimes \vec{u}_1 - \\ & - \vec{u}_2 \otimes A\vec{u}_1 - A\vec{u}_1 \otimes \vec{u}_2),\end{aligned} \tag{5.5}$$

$$\begin{aligned}\vec{u}_1 \otimes W\vec{u}_2 + W\vec{u}_2 \otimes \vec{u}_1 = {} & q_1\vec{u}_1 \otimes \vec{u}_1 + q_2(\vec{u}_1 \otimes \vec{u}_2 + \vec{u}_2 \otimes \vec{u}_1)+ \\ & + q_3\vec{u}_2 \otimes \vec{u}_2 + q_4(\vec{u}_1 \otimes W\vec{u}_1 + W\vec{u}_1 \otimes \vec{u}_1)+ \\ & + q_5(\vec{u}_1 \otimes W\vec{u}_2 + W\vec{u}_2 \otimes \vec{u}_1 - \\ & - \vec{u}_2 \otimes W\vec{u}_1 - W\vec{u}_1 \otimes \vec{u}_2),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}p_1 = {} & 2\vec{u}_1 \cdot A\vec{u}_2 - 2(\vec{u}_1 \cdot A\vec{u}_1)[\vec{u}_1 \cdot AA\vec{u}_1 - (\vec{u}_1 \cdot A\vec{u}_1)^2 - (\vec{u}_1 \cdot A\vec{u}_2)^2]^{-1} \times \\ & \times [\vec{u}_1 \cdot AA\vec{u}_2 - (\vec{u}_1 \cdot A\vec{u}_2)(\vec{u}_1 \cdot A\vec{u}_1 + \vec{u}_2 \cdot A\vec{u}_2)],\end{aligned}$$

$$\begin{aligned}p_2 = {} & \vec{u}_2 \cdot A\vec{u}_2 - \vec{u}_2 \cdot A\vec{u}_1[\vec{u}_1 \cdot AA\vec{u}_1 - (\vec{u}_1 \cdot A\vec{u}_1)^2 - (\vec{u}_1 \cdot A\vec{u}_2)^2]^{-1} \times \\ & \times [\vec{u}_1 \cdot AA\vec{u}_2 - (\vec{u}_1 \cdot A\vec{u}_2)(\vec{u}_1 \cdot A\vec{u}_1 + \vec{u}_2 \cdot A\vec{u}_2)],\end{aligned}$$

$$\begin{aligned}p_4 = {} & [\vec{u}_1 \cdot AA\vec{u}_1 - (\vec{u}_1 \cdot A\vec{u}_1)^2 - (\vec{u}_1 A\vec{u}_2)^2]^{-1} \times \\ & \times [u_1 \cdot AA\vec{u}_2 - (\vec{u}_1 \cdot A\vec{u}_2)(\vec{u}_1 \cdot A\vec{u}_1 + \vec{u}_2 \cdot A\vec{u}_2)],\end{aligned}$$

$$p_3 = p_5 = 0$$

if

$$[\vec{u}_1 \cdot AA\vec{u}_1 - (\vec{u}_1 \cdot A\vec{u}_1)^2 - (\vec{u}_1 \cdot A\vec{u}_2)^2] \neq 0$$

while

$$p_1 = 0,\ p_2 = \vec{u}_1 \cdot A\vec{u}_1,\ p_3 = 2\vec{u}_2 \cdot A\vec{u}_1,\ p_4 = 0,\ p_5 = 1$$

if

$$[\vec{u}_1 \cdot AA\vec{u}_1 - (\vec{u}_1 \cdot A\vec{u}_1)^2 - (\vec{u}_1 \cdot A\vec{u}_2)^2] = 0.$$

Similarly

$$q_1 = 2\vec{u}_1 \cdot W\vec{u}_2,$$

$$q_2 = (\vec{u}_1 \cdot W\vec{u}_2)[\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2]^{-1} \cdot (\vec{u}_1 \cdot WW\vec{u}_2),$$

$$q_3 = 0, q_4 = [\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2]^{-1} \cdot (\vec{u}_1 \cdot WW\vec{u}_2), q_5 = 0$$

if

$$\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2 \neq 0,$$

while

$$q_1 = q_2 = 0, q_3 = 2\vec{u}_2 \cdot W\vec{u}_1, q_4 = 0, q_5 = 1$$

if

$$\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2 = 0.$$

This can be seen easier in the reference frame in which $\vec{u}_1 \equiv (1,0,0)$, $\vec{u}_2 \equiv (0,1,0)$ simply confronting the components of the first members with those of the second ones.

Substituting (3) and (4) in (5) and (6), we find (1) and (2).

Lastly, in section 5 we have used the properties that if $\vec{x}$, $\vec{y}$ are two 3-dimensional vectors, A and W are two 3-dimensional tensors, symmetric the first and skew-symmetric the other, then we have

PROPOSITION 5.

$$\begin{aligned}\vec{x} \otimes A\vec{y} - A\vec{y} \otimes \vec{x} = c_1(\vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}) + c_2(\vec{x} \otimes A\vec{x} - A\vec{x} \otimes \vec{x}) + \\ + c_3(\vec{x} \otimes A\vec{y} - A\vec{y} \otimes \vec{x} - \vec{y} \otimes A\vec{x} + A\vec{x} \otimes \vec{y}),\end{aligned} \tag{5.7}$$

$$\begin{aligned}\vec{x} \otimes W\vec{y} - W\vec{y} \otimes \vec{x} = d_1(\vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}) + d_2(\vec{x} \otimes W\vec{x} - W\vec{x} \otimes \vec{x}) + \\ + d_3(\vec{x} \otimes W\vec{y} - W\vec{y} \otimes \vec{x} - \vec{y} \otimes W\vec{x} + W\vec{x} \otimes \vec{y}).\end{aligned} \tag{5.8}$$

with c_i, d_i scalar coefficients.

Let us prove it. If $\vec{x} = \bar{0}$, then the proposition is true with $c_i = d_i = 0$; if $\vec{x} \neq \bar{0}$, but $\vec{y} = (\vec{x} \cdot \vec{x})^{-1}(\vec{x} \cdot \vec{y})\vec{x}$, then the proposition is verified with

$$c_1 = c_3 = d_1 = d_3 = 0, \ c_2 = d_2 = (\vec{x} \cdot \vec{x})^{-1}(\vec{x} \cdot \vec{y}).$$

If $\vec{x}\neq\vec{0}$ and $\vec{y}\neq(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}$, we define

$$(5.9)\quad \vec{u}_1=\vec{x}(\vec{x}\cdot\vec{x})^{-\frac{1}{2}},\ \vec{u}_2=[\vec{y}\cdot\vec{y}-(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})^2]^{-\frac{1}{2}}[\vec{y}-(\vec{x}\cdot\vec{x})^{-1}(\vec{x}\cdot\vec{y})\vec{x}]$$

and see that

$$(5.10)\quad \begin{aligned}\vec{u}_1\otimes A\vec{u}_2-A\vec{u}_2\otimes\vec{u}_1 = {} & r_1(\vec{u}_1\otimes\vec{u}_2-\vec{u}_2\otimes\vec{u}_1)+\\ & +r_2(\vec{u}_1\otimes A\vec{u}_1-A\vec{u}_1\otimes\vec{u}_1)+\\ & +r_3(\vec{u}_1\otimes A\vec{u}_2-A\vec{u}_2\otimes\vec{u}_1-\vec{u}_2\otimes A\vec{u}_1+\\ & +A\vec{u}_1\otimes\vec{u}_2),\end{aligned}$$

$$(5.11)\quad \begin{aligned}\vec{u}_1\otimes W\vec{u}_2-W\vec{u}_2\otimes\vec{u}_1 = {} & s_1(\vec{u}_1\otimes\vec{u}_2-\vec{u}_2\otimes\vec{u}_1)+\\ & +s_2(\vec{u}_1\otimes W\vec{u}_1-W\vec{u}_1\otimes\vec{u}_1)+\\ & +s_3(\vec{u}_1\otimes W\vec{u}_2-W\vec{u}_2\otimes\vec{u}_1-\vec{u}_2\otimes W\vec{u}_1+\\ & +W\vec{u}_1\otimes\vec{u}_2).\end{aligned}$$

where

$$\begin{aligned}r_1 = {} & \vec{u}_2\cdot A\vec{u}_2-(\vec{u}_2\cdot A\vec{u}_1)[\vec{u}_1\cdot AA\vec{u}_1-(\vec{u}_1\cdot A\vec{u}_1)^2-(\vec{u}_1\cdot A\vec{u}_2)^2]^{-1}\times\\ & \times[\vec{u}_1\cdot AA\vec{u}_2-(\vec{u}_1\cdot A\vec{u}_2)(\vec{u}_1\cdot A\vec{u}_1+\vec{u}_2\cdot A\vec{u}_2)],\end{aligned}$$

$$\begin{aligned}r_2 = {} & [\vec{u}_1\cdot AA\vec{u}_1-(\vec{u}_1\cdot A\vec{u}_1)^2-(\vec{u}_1 A\vec{u}_2)^2]^{-1}\times\\ & \times[\vec{u}_1\cdot AA\vec{u}_2-(\vec{u}_1\cdot A\vec{u}_2)(\vec{u}_1\cdot A\vec{u}_1+\vec{u}_2\cdot A\vec{u}_2)],\end{aligned}$$

$$r_3=0$$

if

$$\vec{u}_1\cdot AA\vec{u}_1-(\vec{u}_1\cdot A\vec{u}_1)^2-(\vec{u}_1\cdot A\vec{u}_2)^2\neq 0$$

while

$$r_1=-\vec{u}_1\cdot A\vec{u}_1,\ r_2=0,\ r_3=1$$

if

$$\vec{u}_1\cdot AA\vec{u}_1-(\vec{u}_1\cdot A\vec{u}_1)^2-(\vec{u}_1\cdot A\vec{u}^2)^2=0.$$

Similarly

$$s_1 = -(\vec{u}_2 \cdot W\vec{u}_1)[\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2]^{-1} \times (\vec{u}_1 \cdot WW\vec{u}_2),$$

$$s_2 = [\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2]^{-1} \times (\vec{u}_1 \cdot WW\vec{u}_2), s_3 = 0$$

if

$$\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2 \neq 0,$$

while

$$s_1 = s_2 = 0, s_3 = 1$$

if

$$\vec{u}_1 \cdot WW\vec{u}_1 + (\vec{u}_1 \cdot W\vec{u}_2)^2 = 0.$$

This can be seen easier in the reference in which $\vec{u}_1$ is the first axis and $\vec{u}_2$ the second.

Substituting (9) in (10) and (11) we obtain (7) and (8).

REFERENCES

[1] Liu I., Müller I., Ruggeri T., *Annals of Physics,* **159**, (1986), 191.

[2] Amendt P., Weitzner H., Phys. Fluids, **28**, (1985), 949.

[3] Anile A.M., Pennisi S., Published in A.M. Anile and Y. Choquet-Bruhat (Eds.), *Relativistic Fluid Dynamics,* Proceedings of the C.I.M.E. Symposium on Relativistic Fluid Dynamics, (Noto 1987).

[4] Smith G.F., Int. J. Engng Sci. **9**, (1971) 899-916.

[5] Wang C.C., Arch. Ratl; Mech. Anal. **43**, (1971) 392-395.

[6] Boehler J.P., Zamm **57**, (1977) 323-327.

[7] Telega J.J., Arch. Mech. Warsaw **36**, 147-162.

[8] Pennisi S., Trovato M., Int. J. Engng Sci. **25**, (1987) 1059-1065.

[9] Wang C.C., Arch. Ratl. Mech. Anal. **33**, (1969), 286.

[10] Pennisi S., Suppl. B.U.M.I. **5**, (1986), 191-214.

Università di Catania
Dipartimento di Matematica
Catania