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APPLICATION OF THE MONOTONE-ITERATIVE TECHNIQUES OF V. LAKSHMIKANTHAM TO THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the paper a techniques of approximate finding of extremal quasisolutions of an initial value problem for systems of functional differential equations is justified.

1. Introduction.

The monotone-iterative techniques of V. Lakshmikantham represents a fruitful combination of the method of upper and lower solutions and a suitably chosen monotone method [1]-[9].

In the present paper by means of this techniques extremal quasi-solutions of the initial value problem for systems of functional differential equations are obtained.

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2. Statement of the problem. Preliminary notes.

Consider the initial value problem for the system of functional differential equations

$$\begin{aligned}&\overset{\circ}{x}(t) = f(t, x(t), x(h(t))) \quad \text{for} \quad t \in [0,T]\\ (1)\qquad & \\ &x(t) = \varphi(t) \quad \text{for} \quad t \in [-a, 0],\end{aligned}$$

where $x = (x_1, x_2, \dots, x_n)$, $f : [0,T] \times R^n \times R^n \to R^n$, $f = (f_1, f_2, \dots, f_n)$, $\varphi : [-a, 0] \to R^n$, $h : [0,T] \to [-a, T]$, $a = \text{const} > 0$, $T = \text{const} > 0$.

Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$. We shall say that $x \leq (\geq) y$ if for any $i = \overline{1,n}$ the inequalities $x_i \leq (\geq) y_i$ hold.

With any integer $j = \overline{1,n}$ we associate two nonnegative integers p_j and q_j such that $p_j + q_j = n - 1$ and introduce the notation

$$(x_j, [x]_{p_j}, [y]_{g_j}) = \begin{cases} (x_1, x_2, \dots, x_{p_j+1}, y_{p_j+2}, \dots, y_n) & \\ & \text{for } p_j \geq j \\ (x_1, x_2, \dots, x_{p_j}, y_{p_j+1}, \dots, y_{j-1}, x_j, y_{j+1}, \dots, y_n) & \\ & \text{for } p_j < j \end{cases}$$

According to the notation introduced problem (1) can be written down in the form $\overset{\circ}{x}_j(t) = f_j(t, x_j(t), [x(t)]_{p_j}, [x(t)]_{g_j}, x_j(h(t), [x(h(t))]_{p_j}, [x(h)(t))]_{g_j})$ for $t \in [0,T]$, $x_j(t) = \varphi_j(t)$ for $t \in [-a, 0]$, $j = \overline{1,n}$.

DEFINITION 1. *The functions* $v, w \in C([-a,T], R^n)$ $v, w \in C^1([0,T], R^n)$, $v = (v_1, v_2, \dots, v_n)$, $w = (w_1, w_2, \dots, w_n)$ *are said to be a couple of lower and upper quasisolutions of the initial value problem (1) if they satisfy the following inequalities*

$$\begin{aligned}&\overset{\circ}{v}_j(t) \leq f_j(t, v_j(t), [v(t)]_{p_j}, [w(t)]_{g_j}, v_j(h(t)), [v(h(t))]_{p_j}, [w(h(t))]_{g_j})\\ (2)\qquad &\\ &\qquad\qquad \textit{for} \quad t \in [0,T],\end{aligned}$$

$$\begin{aligned}&w_j(t) \geq f_j(t, w_j(t), [w(t)]_{p_j}, [v(t)]_{g_j}, w_j(h(t)), [w(h)(t))]_{p_j}, [v(h(t))]_{q_j}),\\ (3)\qquad &\\ &v_j(t) \leq \varphi_j(t) \leq w_j(t) \quad \textit{for } t \in [-a, 0], \ j = \overline{1,n}.\end{aligned}$$

DEFINITION 2. *In the case when (1) is an initial value problem for a scalar functional differential equation, i.e.* $n = 1$, $p_j = g_j = 0$, *the couple of lower and upper quasisolutions of problem (1) are called lower and upper solutions of the same problem [7].*

DEFINITION 3. *The couple of functions* $v, w \in C[(-a, T], R^n)$, $v, w \in C^1([0, T], R^n)$ *is said to be a couple of quasisolutions of the inital value problem (1) if (2) and (3) are satisfied as equalities.*

DEFINITION 4. *The couple of functions* $v, w \in C([-a, T], R^n)$ $v, w \in C^1([0, T], R^n)$ *is said to be a couple of minimal and maximal quasisolutions of the initial value problem (1) if it is a couple of quasisolutions of the same problem and for any couple of quasisolutions* (u, z) *of (1) the inequalities* $v(t) \leq u(t) \leq w(t)$ *and* $v(t) \leq z(t) \leq w(t)$ *hold for* $t \in [-a, T]$.

Remark 1. We shall note that in general for the couple of minimal and maximal quasisolutions (v, w) of problem (1) the inequality $v(t) \leq w(t)$ holds for $t \in [-a, T]$ while for an arbitrary couple of quasisolutions (u, z) of (1) an analogous inequality relating the functions $u(t)$ and $z(t)$ may be not valid.

Remark 2. If for any $j = \overline{1, n}$ the equalities $p_j = j - 1$, $g_j = n - j$ hold and the functions $v(t)$, $w(t)$ are a couple of quasisolutions of the initial value problem (1), then they are solutions of the same problem. If in this case problem (1) has a unique solution $u(t)$, then the couple of functions (u, u) is a couple of minimal and maximal quasisolutions of (1).

For any couple of functions $v, w \in C([-a, T], R^n)$, $v, w \in C^1([0, T], R^n)$ such that $v(t) \leq w(t)$ for $t \in [-a, T]$ define the set of functions

$$S(v, w) = \{u \in C([-a, T], R^n),\ u \in C^1([0, T], R^n) :$$
$$v(t) \leq u(t) \leq w(t) \quad \text{for} \quad t \in [-a, T]\}.$$

3. Main Results.

LEMMA 1. *Let the following conditions hold:*

1. *The function* $h \in C([0,T],[-a,T])$ *and satisfies the inequalities* $t-a \leq h(t) \leq t$ *for* $t \in [-a,T]$.
2. *The function* $m \in C([-a,T],R)$, $m \in C^1([0,T],R)$ *satisfies the inequalities*

(4) $$\overset{\circ}{m}(t) \leq -Mm(t) - Nm(h(t)) \quad \textit{for} \quad t \in [0,T]$$

(5) $$m(0) \leq m(t) \leq 0 \quad \textit{for} \quad t \in [-a,0],$$

where M *and* N *are positive constants.*

3. *The following inequality holds*

(6) $$NTe^{MT} \leq 1.$$

Then the inequality $m(t) \leq 0$ *holds for* $t \in [-a,T]$.

Proof. Consider the auxiliary function $g(t) = m(t)e^{MT}$. The function $g(t)$ satisfies the inequalities

(7) $$\overset{\circ}{g}(t) \leq -Ne^{M(t-h(t))}g(h(t)) \quad \text{for} \quad t \in [0,T]$$

(8) $$g(0) \leq g(t) \leq 0 \quad \text{for} \quad t \in [-a,0].$$

We shall prove that $g(t) \leq 0$ for $t \in [-a,T]$. Suppose that this is not true, i.e. that there exists a point $\xi \in (0,T]$ such that $g(\xi) > 0$. Consider the following three cases:

Case 1. Let $g(0) = 0$ and $g(t) \geq 0$, $g(t) \not\equiv 0$ for $t \in [0,\varepsilon]$ where $\varepsilon > 0$ is a sufficiently small number. From inequality (8) it follows that $g(t) \equiv 0$ for $t \in [-a,0]$. Then by assumption there exist points $\xi_1, \xi_2 \in [0,T]$, $\xi_1 < \xi_2$ such that $g(t) \equiv 0$ for $t \in [-a,\xi_1]$ and $g(t) > 0$ for $t \in (\xi_1,\xi_2]$. From inequality (7) it follows that $\overset{\circ}{g}(t) \leq 0$ for $t \in (\xi_1,\xi_2]$

wich shows that the function $g(t)$ is monotone nonincreasing in the interval $(\xi_1, \xi_2]$, i.e. $g(t) \leq g(\xi_1) = 0$ for $t \in (\xi_1, \xi_2]$. Last inequality contradicts the choice of the point ξ_2.

Case 2. Let $g(0) < 0$. By assumption there exists a point $\eta \in (0, T]$ such that $g(t) \leq 0$, $g(t) \not\equiv 0$ for $t \in [-a, \eta)$, $g(\eta) = 0$ and $g(t) > 0$ for $t \in (\eta, \eta + \varepsilon)$, where $\varepsilon > 0$ is a sufficiently small number. Introduce the notation

$$\inf\{g(t) : t \in [-a, \eta]\} = -\lambda, \ \lambda = \text{const} > 0.$$

From the continuity of the function $g(t)$ it follows that there exists a point $\zeta \in (0, \eta)$ such that $g(\zeta) = -\lambda$. Then

$$\mathring{g}(\bar{t}) = \frac{g(\eta) - g(\zeta)}{\eta - \zeta} = \frac{\lambda}{\eta - \zeta} > \frac{\lambda}{T}, \tag{9}$$

where $\bar{t} \in (\zeta, \eta)$.

From (7) it follows that the following inequality holds

$$\begin{aligned} \mathring{g}(\bar{t}) &\leq -Ne^{M(\bar{t} - h(\bar{t}))} g(h(t)) \leq \\ &\leq N\lambda e^{M(\bar{t} - h(\bar{t}))} \leq N\lambda e^{MT}. \end{aligned} \tag{10}$$

From inequalities (9) and (10) we obtain

$$TNe^{MT} > 1. \tag{11}$$

Inequality (11) contradicts inequality (6).

Case 3. Let $g(0) = 0$ and $g(t) \leq 0$, $g(t) \not\equiv 0$ for $t \in [0, b]$ where $0 < b \leq T$ is a sufficiently smal number. By arguments analogous to those in Case 2 we again get to a contradiction.

Hence $g(t) \leq 0$ for $t \in [-a, T]$ which shows that $m(t) \leq 0$ for $t \in [-a, T]$.

This completes the proof of Lemma 1.

LEMMA 2. *Let the following conditions be fulfilled:*

1. Conditions 1 and 2 of Lemma 1 hold.

2. The following inequality holds

$$(M+N)T \leq 1.$$

Then $m(t) \leq 0$ for $t \in [-a,T]$.

The proof of Lemma 2 is analogous to the proof of Lemma 1 but we do not introduce the auxiliary function $g(t)$ and consider instead the function $m(t)$ itself.

THEOREM 1. *Let the following conditions hold:*

1. The functions $v,w \in C([-a,T],R^n)$, $v,w \in C^1([0,T],R^n)$ are a couple of lower and upper quasisolutions of the initial value problem (1) and satisfy the inequalities $v(t) \leq w(t)$ for $t \in [-a,T]$ and $v(0)-\varphi(0) \leq v(t)-\varphi(t)$, $w(0)-\varphi(0) \geq w(t)-\varphi(t)$ for $t \in [-a,0]$.

2. The function $h \in C([0,T],[-a,T])$ satisfies the inequalities $t-a \leq h(t) \leq t$ for $t \in [0,T]$.

3. The function $f \in C([0,T]\times R^n \times R^n, R^n)$, $f=(f_1,f_2,\dots,f_n)$, $f_j(t,x,y)=f_j(t,x_j,[x]_{p_j},[x]_{g_j},y_j,[y]_{p_j},[y]_{g_j})$, is monotone nondecreasing with respect to $[x]_{p_j}$ and $[y]_{p_j}$ and monotone nonincreasing with respet to $[x]_{g_j}$ and $[y]_{g_j}$ and for $x,y \in S(v,w)$, $x(t) \leq y(t)$ satisfies the inequalities $f_j(t,x_j,[x]_{p_j}, [x]_{g_j}, x_j(h(t)), [x(h(t))]_{p_j}, [x(h(t))]_{g_j}) - f_j(t,y_j,[x]_{p_j},[x]_{g_j}, y_j(h(t)), [x(h(t))]_{p_j}, [x(h(t))]_{g_j}) \geq -M_j(x_j-y_j) - N_j(x_j(h(t))- y_j(h(t)))$, $j=\overline{1,n}$, $t \in [0,T]$ where M_j and $N_j (j=\overline{1,n})$ are positive constants such that

$$N_j T e^{M_j T} \leq 1.$$

Then there exist monotone sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$, $v^{(0)}(t) \equiv v(t)$, $w^{(0)}(t) \equiv w(t)$ which are uniformly convergent for $t \in [-a,T]$ and their limits $\bar{v}(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim_{k\to\infty} w^{(k)}(t)$ are a couple of minimal and maximal quasisolutions of the initial value problem (1).

Moreover, if $u(t)$ is a solution of the initial value problem (1) such that $u \in S(v,w)$, then the inequalities $\bar{v}(t) \le u(t) \le \bar{w}(t)$ hold for $t \in [-a,T]$.

Proof. Let $\eta, \mu \in S(v,w)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be arbitrary functions. Consider the initial value problems for the linear scalar functional differential equations

$$\begin{aligned} &\overset{\circ}{x}_j + M_j x_j + N_j x_j(h(t)) = \sigma_j(t,\eta,\mu) \quad \text{for} \quad t \in [0,T] \\ &x_j(t) = \varphi_j(t) \quad \text{for} \quad t \in [-a,T], \end{aligned} \tag{12}$$

where

$$\sigma_j(t,\eta,\mu) = f_j(t, \eta_j(t), [\eta(t)]_{p_j}, [\mu(t)]_{q_j}, \eta_j(h(t)),$$

$$[\eta(h(t))]_{p_j}, [\mu(h(t))]_{q_j}) + M_j \eta_j(t) + N_j \eta_j(h(t)), \; j = \overline{1,n}.$$

The initial value problems (12) have a unique solution for each fixed couple of functions $\eta, \mu \in S(v,w)$.

Define the map $A : S(v,w) \times S(v,w) \to S(v,w)$ by the equality $x = A(\eta,\mu)$, where $x = (x_1, \dots, x_n)$, $x_j(t)$ is the unique solution of the initial value problems (12) for the couple of functions $\eta, \mu \in S(v,w)$.

We shall prove that $v \le A(v,w)$. Introduce the notations $g(t) = v(t) - v^{(1)}(t)$ and $v^{(1)} = A(v,w)$. Then the following inequalities hold

$$\begin{aligned} &\overset{\circ}{g}_j(t) \le -M_j g_j(t) - N_j g_j(h(t)) \quad \text{for} \quad t \in [0,T] \\ &g(0) \le g(t) \le 0 \quad \text{for} \quad t \in [-a,0]. \end{aligned} \tag{13}$$

From Lemma 1 it follows that the functions $g_j(t)$, $j = \overline{1,n}$ are nonpositive, i.e. $v \le A(v,w)$. In an analogous way it is proved that $w \ge A(w,v)$.

Let $\eta, \mu \in S(v,w)$ be such that $\eta(t) \le \mu(t)$ for $t \in [-a,T]$.

Set $x^{(1)} = A(\eta,\mu)$, $x^{(2)} = A(\mu,\eta)$ and $g(t) = x^{(1)} - x^2$, $g = (g_1, g_2, \dots, g_n)$.

By Lemma 1 the functions $g_j(t)$, $j = \overline{1,n}$ are nonpositive, i.e. $A(\eta,\mu) \le A(\mu,\eta)$.

Define the sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)}(t) \equiv v(t), \qquad w^{(0)}(t) \equiv w(t),$$

$$v^{(k+1)} = A(v^{(k)}, w^{(k)}), \quad w^{(k+1)} = A(w^{(k)}, v^{(k)}), \quad k \ge 0$$

The functions $v^{(k)}(t)$ and $w^{(k)}(t)$ for $k \ge 0$ and $t \in [-a,T]$ satisfy the inequalities

$$\begin{aligned} &v^{(0)}(t) \le v^{(1)}(t) \le \dots \le v^{(k)}(t) \le \dots \le \\ &\le \dots \le w^{(k)}(t) \le \dots \le w^{(1)}(t) \le w^{(0)}(t). \end{aligned} \tag{14}$$

Hence the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are uniformly convergent for $t \in [-a,T]$. Introduce the notations $\bar{v}(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim_{k\to\infty} w^{(k)}(t)$. We shall show that $(\bar{v},\bar{w})$ are a couple of minimal and maximal quasisolutions of the initial value problem (1). From the definition of the functions $v^{(k)}(t)$ and $w^{(k)}(t)$ it follows that the functions $(\bar{v},\bar{w})$ are a couple of quasisolutions of the initial value problem (1). Let $u_1, u_2 \in S(v,w)$ be a couple of quasisolutions of (1). From inequalities (14) it follows that there exists a positive integer k such that $v^{(k-1)}(t) \le u_1(t) \le w^{(k-1)}(t)$ and $v^{(k-1)}(t) \le u_2(t) \le w^{(k-1)}(t)$ for $t \in [-a,T]$. Introduce the notation $g_j(t) = v_j^{(k)} - u_{1j}$, $j = \overline{1,n}$. By Lemma 1 the functions $g_j(t)$, $j = \overline{1,n}$ are nonpositive, i.e. $v^{(k)}(t) \le u_1(t)$ for $t \in [-a,T]$.

In an analogous way it is proved that the inequalities $u_1(t) \le w^{(k)}(t)$ and $v^{(k)}(t) \le u_2(t) \le w^{(k)}(t)$ hold for $t \in [-a,T]$, which shows that the couple of functions $(\bar{v},\bar{w})$ is a couple of minimal and maximal quasisolutions of the initial value problem (1).

Let $u(t)$ be a solution of (1) for which $u \in S(v,w)$. Consider the couple of functions (u,u) which is a couple of quasisolutions of the initial value problem (1). From the fact that the couple $(\bar{v},\bar{w})$ is a couple of minimal and maximal quasisolutions of (1) it follows that the inequalities $\bar{v}(t) \le u(t) \le \bar{w}(t)$ hold for $t \in [-a,T]$.

THEOREM 2. *Let the following conditions be satisfied:*

1. *Conditions 1, 2 and 3 of Theorem 1 hold.*
2. *The following inequalities hold*

$$(M_j + N_j)T \leq 1, \quad j = \overline{1, n}.$$

Then there exist monotone sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$, $v^{(0)}(t) \equiv v(t)$, $w^{(0)}(t) \equiv w(t)$ which are uniformly convergent in the interval $[-a, T]$ and their limits $\bar{v}(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim_{k\to\infty} w^{(k)}(t)$ are a couple of minimal and maximal quasisolutions of the initial value problem (1). Moreover, if $u(t)$ is a solution of (1) such that $u \in S(v, w)$, then the inequalities $\bar{v}(t) \leq u(t) \leq \bar{w}(t)$ hold for $t \in [-a, T]$.

The proof of Theorem 2 is analogous to the proof of Theorem 1 using Lemma 2 instead of Lemma 1.

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