«LE MATEMATICHE»
Vol. XLIV (1989) - Fasc. II, pp. 259-280

ON THE EXISTENCE OF WEAK SOLUTIONS OF THE STATIONARY SEMICONDUCTOR EQUATIONS WITH VELOCITY SATURATION

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In the present paper we prove the existence and differentiability of weak solutions to the stationary semiconductor equations where the mobility coefficients model the effect of saturation of drift velocity. The proof of existence relies on approximation by uniformly bounded mobility coefficients.

1. Introduction.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain which is assumed to represent the cross-section $(N = 2)$ or the spatial region $(N = 3)$ occupied by a semiconductor device(1). The stationary distribution of carriers in Ω can be described by the following systems of PDE's:

$$\operatorname{div}(\mu(\nabla\psi)e^{\psi}\nabla u) = R(e^{\psi}u, e^{-\psi}v)(uv - 1), \tag{1.1}$$

(*) Entrato in Redazione il 18 dicembre 1989

(1) Physically, of course, $N = 2$ and $N = 3$ make sense only. However, most of our discussion in valid for any dimension of space $N \geq 2$.

$$\operatorname{div}(\nu(\nabla\psi)e^{-\psi}\nabla v) = R(e^{\psi}u, e^{-\psi}v)(uv-1), \tag{1.2}$$

$$\Delta\psi = e^{\psi}u - e^{-\psi}v - f \tag{1.3}$$

where $\varphi_n = -\log u$ and $\varphi_p = \log v$ are the electron and hole quasi-Fermi potentials, respectively, μ and ν are the electron and hole mobilities, respectively, ψ is the electrostatic potential, R is a given function on $[0,+\infty)\times[0,+\infty)$ [$R(e^{\psi}u, e^{-\psi}v)$ $(uv-1)$ represents the rate of carrier recombination-generation], f is the net impurity (doping) concentration(2).

The above system of equations involves the assumption that Boltzmann statistics can be used to model the semiconductor behavior; accordingly, the electron and hole densities are given by

$$n = e^{\psi-\varphi_n}, \quad p = e^{\varphi_p-\psi},$$

respectively (cf. e.g. [2], [13], [17]).

Let $\partial\Omega$ denote the boundary of Ω. Without further reference, throughout we suppose:

$$\partial\Omega \text{ is Lipschitzian,}$$

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \text{ with } \Gamma_0 \cap \Gamma_1 = \emptyset, \text{ meas } \Gamma_0 > 0.$$

Then we complete (1.1)-(1.3) by the following boundary conditions:

$$u = u_0, \quad v = v_0, \quad \psi = \psi_0 \text{ on } \Gamma_0, \tag{1.4}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial \psi}{\partial n} = 0 \text{ on } \Gamma_1, \tag{1.5}$$

where u_0, v_0 and ψ_0 are given functions on Γ_0 (n =unit outward normal along Γ_1 (cf. [13], [15], [17]). ■

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(2) For notational simplicity, units are choosen such that the elementary charge and the dielectric constant are equal to one.

In semiconductor physics it is well-known that the linear proportionality of the electric field $E = -\nabla\psi$ to the drift velocities $|v_n^d| = \mu|E|$ (electrons) and $|v_p^d| = \nu|E|$ (holes) is only true at low electric field strength. At high electric field strength, however, the drift velocities saturate:

$$(*) \qquad \lim_{|E|\to\infty} |v_n^d| = v_{ns}\,, \quad \lim_{|E|\to 0} |v_p^d| = v_{ps};$$

here v_{ns} and v_{ps} are the so-called saturation velocities (cf. [2], [13; pp. 18-19], [17; p. 93], [20]).

The aim of the present paper is to study (1.1)-(1.3) under the following conditions on μ and ν:

$$(1.6) \qquad \mu, \nu \in C(\mathbb{R}^N),$$

$$(1.7) \qquad \begin{cases} 0 < \mu_1 \leq \mu(\xi)(1+|\xi|^2)^{1/2} \leq \mu_2 < \infty & \forall \xi \in \mathbb{R}^N, \\ 0 < \nu_1 \leq \nu(\xi)(1+|\xi|^2)^{1/2} \leq \nu_2 < \infty & \forall \xi \in \mathbb{R}^N, \end{cases}$$

(μ_i, ν_i =const; $i = 1, 2$).

Simple examples which obey (1.6), (1.7) and model the effect of velocity saturation (*), are given by

$$\mu(\xi) = \frac{\mu_0}{(1+|\xi|^\sigma)^{1/\sigma}} \quad (\sigma = \text{const} > 0),$$

$$\mu(\xi) = \frac{\mu_0}{(1+|\xi|^2(1+|\xi|)^{-1}+|\xi|^2)^{1/2}}$$

($\xi \in \mathbb{R}^N$; μ_0 =const> 0) (cf. [2; p. 121], [17], [20]).

Remark. Let μ, ν satisfy (1.6). Then (1.7) is readily seen to be equivalent to the following system of conditions:

$$(1.8_1) \qquad \mu(\xi) > 0,\ \nu(\xi) > 0\ \forall \xi \in \mathbb{R}^N,$$

$$(1.8_2) \qquad \mu(\xi)|\xi| \leq \mu_0,\ \nu(\xi)|\xi| \leq \nu_0\ \forall \xi \in \mathbb{R}^N,$$

$$(1.8_3) \qquad \liminf_{|\xi|\to\infty} \mu(\xi)|\xi| > 0, \ \liminf_{|\xi|\to\infty} \nu(\xi)|\xi| > 0$$

(μ_0, ν_0 =const>0).

We note that (1.6), (1.7) are essential for our mathematical analysis below; these conditions need not, however, imply (*). ■

The paper is organized as follows. Section 2 presents our main results on the existence (Theorem 1) and interior differentiability (Theorem 2) of weak solutions to (1.1)-(1.5) with mobility coefficients satisfying (1.6), (1.7). In the following section we prove the existence of weak solutions to (1.1)-(1.5), the mobilities of which being assumed continuous and uniformly bounded on $\mathbb{R}^N$ from above and below by positive constants. Although of preparatory character to our subsequent discussion, this existence result may be of interest in itself. In Section 4, we prove Theorem 1 by replacing μ and ν by $\varepsilon+\mu$ and $\varepsilon+\nu$ ($\varepsilon > 0$), respectively, solving (1.1)-(1.5) with the mobilities $\varepsilon+\mu$ and $\varepsilon+\nu$, establishing estimates on the solutions u_ε, v_ε, ψ_ε and then letting tend $\varepsilon \to 0$. Finally, Section 5 is devoted to the proof of the interior differentiability of u and v.

The existence of weak solutions to (1.1)-(1.5) with constant or uniformly bounded mobilities has been proved in [4], [5], [9]-[11], and [13]-[15]. Similar existence results may be found in [16]. The case of mobility coefficients which model velocity saturation, is briefly discussed in [11; pp. 586-587].

2. Statement of main results.

Let $W_p^m(\Omega)$ ($m = 1, 2, \ldots, 1 \le p < \infty$) denote the usual Sobolev space of all functions in $L^p(\Omega)$ having their generalized derivatives up to order m in $L^p(\Omega)$. Define

$$V = \{u \in W_2^1(\Omega) : u = 0 \text{ a.e. on } \Gamma_0\}.$$

Then

$$\left(\int_\Omega |u|^{2^*} dx\right)^{1/2^*} \le c_0 \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \quad \forall u \in V \tag{2.1}$$

where 2^* denotes the Sobolev imbedding exponent (i.e. $1 \le 2^* < \infty$ if $N = 2$, $2^* = \dfrac{2N}{N-2}$ if $N \ge 3$) (c_0 =const> 0).

We consider the following conditions on the data:

$$\begin{cases} u_0, v_0 \in W_2^1(\Omega), \\ 0 < \operatorname{ess\,inf}_{\Gamma_0} u_0 \le \operatorname{ess\,sup}_{\Gamma_0} u_0 < \infty, \\ 0 < \operatorname{ess\,inf}_{\Gamma_0} v_0 \le \operatorname{ess\,sup}_{\Gamma_0} v_0 < \infty, \end{cases} \tag{2.2}$$

$$\psi_0 \in W_2^1(\Omega),\ \operatorname{ess\,sup}_{\Gamma_0} |\psi_0| < \infty, \tag{2.3}$$

$$R \in C([0,\infty) \times [0,\infty)),\ R(s,t) \ge 0\ \forall s,t \in [0,\infty), \tag{2.4}$$

$$f \in L^2_{\rm loc}(\Omega) \cap L^p(\Omega) \left(p > \frac{N}{2}\right). \tag{2.5}$$

With the data (2.2) we associate the positive reals

$$\begin{cases} \delta_1 = \min\{\operatorname{ess\,inf}_{\Gamma_0} u_0,\ (\operatorname{ess\,sup}_{\Gamma_0} v_0)^{-1}\}, \\ \delta_2 = \min\{\operatorname{ess\,inf}_{\Gamma_0} v_0,\ (\operatorname{ess\,sup}_{\Gamma_0} u_0)^{-1}\}. \end{cases} \tag{2.6}$$

Obviously, $\delta_1 \le \dfrac{1}{\delta_2}$.

We then have

THEOREM 1. *Let (1.6), (1.7) and (2.2) -(2.5) be satisfied.*

Then there exist functions $u, v \in W^1_{2,\rm loc}(\Omega)$ *and* $\psi \in W_2^1(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\delta_1 \le u \le \frac{1}{\delta_2},\ \delta_2 \le v \le \frac{1}{\delta_1} \text{ a.e. in } \Omega, \tag{2.7}$$

(2.8) $$\psi = \psi_0 \text{ a.e. on } \Gamma_0,$$

(2.9) $$\mu(\nabla\psi)\nabla u,\ \nu(\nabla\psi)\nabla v \in [L^2(\Omega)]^N,\ \nabla^2\psi \in [L^2_{\mathrm{loc}}(\Omega)]^{N^2}\ (^3),$$

(2.10) $$\int_\Omega \mu(\nabla\psi)e^{\psi}\nabla u, \nabla\varphi dx = \int_\Omega R(e^{\psi}u, e^{-\psi}v)(1-uv)\varphi dx \ \forall\varphi \in V,$$

(2.11) $$\int_\Omega \nu(\nabla\psi)e^{-\psi}\nabla v \cdot \nabla\varphi dx = \int_\Omega R(e^{\psi}u, e^{-\psi}v)(1-uv)\varphi dx \ \forall\varphi \in V,$$

(2.12) $$\Delta\psi = e^{\psi}u - e^{-\psi}v - f \qquad \text{a.e. in } \Omega.$$

Remarks. 1. Condition (2.4) is satisfied by the well-known Shockley-Read-Hall and Auger recombination-generation terms

$$R_{SRH}(s,t) = \frac{a_0}{a_1 + a_2 s + a_3 t} \qquad (a_i = \text{const} > 0;\ i = 0,1,2,3),$$

$$R_A(s,t) = b_1 s + b_2 t \quad (b_i = \text{const} > 0;\ i = 1,2)$$

($s, t \geq 0$; cf. [2], [13], [17] for further details).

2. Integral identities (2.10), (2.11) represent the weak formulation of (1.1), (1.2), respectively. Indeed, let u, v, ψ be a sufficiently regular solution to (1.1)-(1.5). Multiplying (1.1), (1.2) by $\varphi \in V$, integrating over Ω and integrating by parts the term on the left we get (2.10), (2.11).

3. Theorem 1 does not yield any information on the boundary behavior of u and v. A discussion of the regularity of u, v and ψ near the boundary will be given in a forthcoming paper. ■

(3) $\nabla\varphi = \{\varphi_{x_1}, \dots, \varphi_{x_N}\}$, $\nabla^2\varphi = \{\varphi_{x_i x_j}\}$ (=matrix of second derivatives).

The natural question arises whether (2.10), (2.11) do imply the local square integrability in Ω of the second derivatives of u and v (provided that μ and ν possess appropriate differentiability properties). Notice that once $u, v \in W^2_{2,\mathrm{loc}}(\Omega)$ is established then by a standard argument from (2.10), (2.11) we conclude (1.1), (1.2) to hold a.e. in Ω.

Let $N = 2$. We then have

THEOREM 2. *Let $\mu, \nu \in C^1(\mathbb{R}^2)$ satisfy (1.7) and*

$$\left|\frac{\partial \mu}{\partial \xi_i}(\xi)\right|, \left|\frac{\partial \nu}{\partial \xi_i}(\xi)\right| \le c = \text{const} < \infty \quad \forall \xi \in \mathbb{R}^2 \tag{2.13}$$

$(i = 1, 2)$. Assume (2.4) and $f \in L^4_{\mathrm{loc}}(\Omega) \cap L^{p_0}(\Omega)$ $(p_0 > 1)$.

Let $u, v \in W^1_{2,\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ and $\psi \in W^1_2(\Omega) \cap L^\infty(\Omega)$ satisfy (2.10)-(2.12). Then:

$$u, v \in W^2_{2,\mathrm{loc}}(\Omega). \tag{2.14}$$

■

3. Uniformly bounded mobilities.

This section is concerned with the existence of weak solutions of (1.1)-(1.5) with mobility coefficients μ and ν which are continuous and bounded from above and below by positive constants on the whole $\mathbb{R}^N$.

To begin with, we note a well-known result on the boundedness from above of weak subsolutions of elliptic equations([4])

LEMMA 1. (Stampacchia [18], [19]). *Let $f \in L^p(\Omega)$ $\left(p > \frac{N}{2}\right)$. Let $u \in W^1_2(\Omega)$ satisfy*

$$0 \le \operatorname*{ess\,sup}_{\Gamma_0} u < \infty,$$

([4]) For the sake of simplicity, we present this result in a form already specialized for our later purposes.

$$\int_\Omega \nabla u \cdot \nabla(u-k)^+ dx \leq \int_\Omega f(u-k)^+ dx \quad \forall k \geq k_0 \geq \operatorname{ess\,sup}_{\Gamma_0} u.$$ [5]

Then:

$$u(x) \leq k_0 + 2^{\frac{\alpha}{\alpha-1}} c_0^2 \|f\|_{L^p(\Omega)} (meas\ \Omega)^{\alpha-1} \text{ for a.a. } x \in \Omega,$$

where

$$\begin{cases} \alpha = \dfrac{2q-1}{p} \left(\dfrac{p+1}{2} < q < p \text{ arbitrary}\right) & \text{if } N = 2, \\ \alpha = \dfrac{N(p-1)}{pN} & \text{if } N \geq 3 \end{cases}$$

(c_0 according to (2.1)). ■

We begin by solving (1.3) under the mixed boundary conditions (1.4), (1.5) on ψ. To this end, let δ_1, δ_2 be arbitrary (fixed) positive reals such that $\delta_1 \leq \frac{1}{\delta_2}$. Suppose we are given:

(3.2)
$$\begin{cases} u, v \text{ measurable in } \Omega, \\ \delta_1 \leq u \leq \dfrac{1}{\delta_2},\ \delta_2 \leq v \leq \dfrac{1}{\delta_1} \text{ a.e. in } \Omega, \end{cases}$$

(3.3)
$$\psi_0 \in W_2^1(\Omega), \operatorname{ess\,sup}_{\Gamma_0} |\psi_0| < \infty,$$

(3.4)
$$f \in L^p(\Omega) \left(p > \frac{N}{2}\right).$$

We have

PROPOSITION 1. *Let (3.2)-(3.4) be satisfied. Then there exists exactly one $\psi \in W_2^1(\Omega)$ such that*

(3.5)
$$\psi = \psi_0 \quad \text{a.e. on } \Gamma_0,$$

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(5) $t^+ = \max\{0, t\}$ $(t \in \mathbb{R})$.

(3.6) $$|\psi| \leq \lambda \quad a.e.\ in\ \Omega,$$

(3.7) $$\int_\Omega \nabla\psi \cdot \nabla\varphi dx = \int_\Omega (e^{-\psi}v - e^{\psi}u + f)\varphi dx \quad \forall \varphi \in V,$$

where

$$\lambda = \max\{-\log\delta_1, -\log\delta_2, -\operatorname*{ess\,inf}_{\Gamma_0}\psi_0, \operatorname*{ess\,sup}_{\Gamma_0}\psi_0\}$$
$$+ 2^{\frac{\alpha}{\alpha-1}} c_0^2 ||f||_{L^p(\Omega)} (\text{meas } \Omega)^{\alpha-1}$$

(c_0 and α according to (2.1) and (3.1), respectively).

This result can be easily obtained by using the cut-off procedure as in [5], [13], establishing the existence of a weak solution via the theory of pseudo-monotone operators (cf. e.g. [12]) and then proving the bound (3.6) by applying Lemma 1. ■

The following proposition is fundamental to our subsequent discussion. We note that a similar existence result has been proved in [13; Chap. 3.2] under stronger hypotheses on the data and by a technically slightly different reasoning (cf. also [4], [6], [10]). Therefore we omit the proof.

PROPOSITION 2. *Let $\mu, \nu \in C(\mathbb{R}^N)$ satisfy*

(3.8) $$\begin{cases} 0 < \mu_1 \leq \mu(\xi) \leq \mu_2 < \infty & \forall \xi \in \mathbb{R}^N, \\ 0 < \nu_1 \leq \nu(\xi) \leq \nu_2 < \infty & \forall \xi \in \mathbb{R}^N, \end{cases}$$

($\mu_i, \nu_i = \text{const};\ i = 1, 2$). *Let (2.2)-(2.4) and (3.4) be fulfilled.*

Then there exist functions $u, v, \psi \in W_2^1(\Omega)$ such that

(3.9) $$u = u_0,\ v = v_0,\ \psi = \psi_0 \quad a.e.\ on\ \Gamma_0,$$

(3.10) $$\delta_1 \leq u \leq \frac{1}{\delta_2},\ \delta_2 \leq v \leq \frac{1}{\delta_1},\ |\psi| \leq \lambda \quad a.e.\ in \Omega.$$

(3.11) $$\int_\Omega \mu(\nabla\psi) e^{\psi} \nabla u \cdot \nabla\varphi dx = \int_\Omega R(e^{\psi}u, e^{-\psi}v)(1 - uv)\varphi dx,$$

$$\int_\Omega \nu(\nabla\psi)e^{\psi}\nabla v\cdot\nabla\varphi dx = \int_\Omega R(e^{\psi}u, e^{-\psi}v)(1-uv)\varphi dx, \tag{3.12}$$

$$\int_\Omega \nabla\psi\cdot\nabla\varphi dx = \int_\Omega (e^{-\psi}v - e^{\psi}u + f)\varphi dx \tag{3.13}$$

for all $\varphi \in V$ *(*δ_1*,* δ_2 *according to (2.6),* λ *according to Prop. 1).* ■

4. Proof of Theorem 1.

First of all, we note that (1.6), (1.7) imply

$$0 < \mu(\xi), \nu(\xi) \le \mu^* = \text{const} < \infty \ \forall \xi \in \mathbb{R}^N. \tag{4.1}$$

1. Approximation.

Let $0 < \varepsilon \le 1$. By Prop. 2. there exist functions $u_\varepsilon, v_\varepsilon, \psi_\varepsilon \in W_2^1(\Omega)$ such that

$$u_\varepsilon = u_0, v_\varepsilon = v_0, \psi_\varepsilon = \psi_0 \quad \text{a.e. on } \Gamma_0, \tag{4.2}$$

$$\delta_1 \le u_\varepsilon \le \frac{1}{\delta_2},\ \delta_2 \le v_\varepsilon \le \frac{1}{\delta_1},\ |\psi_\varepsilon| \le \lambda \quad \text{a.e. in } \Omega, \tag{4.3}$$

$$\int_\Omega (\varepsilon + \mu(\nabla\psi_\varepsilon))e^{\psi_\varepsilon}\nabla u_\varepsilon\cdot\nabla\varphi dx = \int_\Omega R(e^{\psi_\varepsilon}u_\varepsilon, e^{-\psi_\varepsilon}v_\varepsilon)(1-u_\varepsilon v_\varepsilon)\varphi dx, \tag{4.4}$$

$$\int_\Omega (\varepsilon + \nu(\nabla\psi_\varepsilon))e^{\psi_\varepsilon}\nabla v_\varepsilon\cdot\nabla\varphi dx = \int_\Omega R(e^{\psi_\varepsilon}u_\varepsilon, e^{-\psi_\varepsilon}v_\varepsilon)(1-u_\varepsilon v_\varepsilon)\varphi dx, \tag{4.5}$$

$$\int_\Omega \nabla\psi_\varepsilon\cdot\nabla\varphi dx = \int_\Omega (e^{-\psi_\varepsilon}v_\varepsilon - e^{\psi_\varepsilon}u_\varepsilon + f)\varphi dx \tag{4.6}$$

for all $\varphi \in V$ (δ and λ according to Prop. 2).

2. Estimates.

Inserting $\varphi = \psi_\varepsilon - \psi_0$ into (4.6) and observing (4.3) we get

$$(4.7)\quad \int_\Omega |\nabla\psi_\varepsilon|^2 dx \leq 2\int_\Omega \left(e^\lambda\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)+|f|\right)(\lambda+|\psi_0|)+\int_\Omega |\nabla\psi_0|^2 dx$$

for all $0 < \varepsilon \leq 1$.

Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$(6). Then the method of difference quotient gives

$$(4.8)\quad \int_{\Omega'} |\nabla^2\psi_\varepsilon|^2 dx \leq c\int_{\Omega''}\left[|\nabla\psi_\varepsilon|^2+\left(e^\lambda\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)+|f|\right)^2\right]dx \leq c(^7)$$

(cf. e.g. [7], [8]) for all $0 < \varepsilon \leq 1$, where the constants tend to $+\infty$ if dist $(\Omega', \partial\Omega'') \to 0$ or $\mathrm{dist}(\Omega'', \partial\Omega) \to 0$.

Next, taking $\varphi = u_\varepsilon - u_0$ in (4.4) and making use of (2.4) and (4.1), (4.3) we find

$$\int_\Omega (\varepsilon+\mu(\nabla\psi_\varepsilon))e^{\psi_\varepsilon}|\nabla u_\varepsilon|^2 dx =$$

$$= \int_\Omega (\varepsilon+\mu(\psi_\varepsilon))e^{\psi_\varepsilon}\nabla u_\varepsilon\cdot\nabla u_0 dx$$

$$+\int_\Omega R(e^{\psi_\varepsilon}u_\varepsilon, e^{-\psi_\varepsilon}v_\varepsilon)(1-u_\varepsilon v_\varepsilon)(u_\varepsilon-u_0)dx$$

$$\leq \sqrt{1+\mu^*}e^{\lambda/2}\int_\Omega \sqrt{\varepsilon+\mu(\nabla\psi_\varepsilon)}e^{\psi_\varepsilon/2}|\nabla u_\varepsilon|\,|\nabla u_0|dx$$

$$+(\max_{\substack{0\leq s\leq e^\lambda\delta_2^{-1}\\ 0\leq t\leq e^\lambda\delta_1^{-1}}}, R(s,t))((\mathrm{meas}\Omega)|\lambda|\int_\Omega |u_0|dx).$$

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(6) $\Omega' \subset\subset \Omega$ means: Ω' open, $\bar{\Omega}' \subset \Omega$.

(7) In what follows, by c we denote positive constants which possibly change their numerical value from line to line, but do not depend on ε.

An analogous reasoning applies to (4.5). Thus,

$$(4.9)\qquad \begin{cases} \displaystyle\int_\Omega (\varepsilon + \mu(\nabla\psi_\varepsilon))|\nabla u_\varepsilon|^2 dx \le c \quad \forall 0 < \varepsilon \le 1, \\ \displaystyle\int_\Omega (\varepsilon + \nu(\nabla\psi_\varepsilon))|\nabla v_\varepsilon|^2 dx \le c \quad \forall 0 < \varepsilon \le 1. \end{cases}$$

Let $\Omega' \subset\subset \Omega$, and let $\zeta \in C^1(\mathbb{R}^N)$ satisfy $\zeta = 1$ on Ω', $0 \le \zeta \le 1$ in $\mathbb{R}^N$ and $\operatorname{supp}(\zeta) \subset \Omega$. Then $\varphi = u_\varepsilon(1 + |\nabla\psi_\varepsilon|^2)^{1/2}\zeta^2$ is admissible in (4.4). By (1.7) and (4.3),

$$\begin{aligned} \mu_1 e^{-\lambda} \int_\Omega |\nabla u_\varepsilon|^2 \zeta^2 dx \le \\ &\le \int_\Omega (\varepsilon + \mu(\nabla\psi_\varepsilon)) e^{\psi_\varepsilon} |\nabla u_\varepsilon|^2 (1 + |\nabla\psi_\varepsilon|^2)^{1/2} \zeta^2 dx = \\ &= -\int_\Omega (\varepsilon + \mu(\nabla\psi_\varepsilon)) e^{\psi_\varepsilon} u_{\varepsilon x_i} u_\varepsilon (1 + |\nabla\psi_\varepsilon|^2)^{-1/2} \psi_{\varepsilon x_j} \psi_{\varepsilon x_i x_j} \zeta^2 dx (^8) \\ &\quad - 2\int_\Omega (\varepsilon + \mu(\nabla\psi_\varepsilon)) e^{\psi_\varepsilon} u_{\varepsilon x_i} u_\varepsilon (1 + \nabla\psi_\varepsilon|^2)^{1/2} \zeta \zeta_{x_i} dx \\ &\quad + \int_\Omega R(e^{\psi_\varepsilon} u_\varepsilon, e^{-\psi_\varepsilon} v_\varepsilon)(1 - u_\varepsilon v_\varepsilon)(1 + |\nabla\psi_\varepsilon|^2)^{1/2} \zeta^2 dx \\ &\le \frac{1}{2} \mu_1 e^{-\lambda} \int_\Omega |\nabla u_\varepsilon|^2 \zeta^2 dx + c; \end{aligned}$$

here we have used (2.4), (4.1), (4.3), (4.7) and (4.8). Inserting $\varphi = v_\varepsilon(1 + |\nabla\psi_\varepsilon|^2)^{1/2}\zeta^2$ into (4.5) we obtain an analogous estimate on ∇v_ε.

Thus,

$$(4.10)\qquad \int_{\Omega'} (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \le c \quad \forall 0 < \varepsilon \le 1.$$

(8) Throughout, a repeated index implies summation from $1, \ldots, N$.

3. Passage to limit.

By passing to a subsequence if necessary, from (4.3), (4.7), (4.9) we may infer that

$$u_\varepsilon \to u, v_\varepsilon \to v \text{ weakly in } L^q(\Omega)\ (\forall 1 \leq q < \infty), \tag{4.11}$$

$$\psi_\varepsilon \to \psi \text{ weakly in } W_2^1(\Omega), \psi_\varepsilon(x) \to \psi(x) \quad \text{for a.a. } x \in \Omega, \tag{4.12}$$

$$\varepsilon u_{\varepsilon x_i} \to 0,\ \varepsilon v_{\varepsilon x_i} \to 0 \text{ strongly in } L^2(\Omega), \tag{4.13}$$

$$\sqrt{\mu(\nabla\psi_\varepsilon)}u_{\varepsilon x_i} \to \chi_i, \sqrt{\nu(\nabla\psi_\varepsilon)}v_{\varepsilon x_i} \to \omega_i \quad \text{weakly in } L^2(\Omega) \tag{4.14}$$

as $\varepsilon \to 0$ ($i = 1,\ldots,N$). Then (2.7) and (2.8) are straightforward; in addition, $|\psi| \leq \lambda$ a.e. in Ω. By a standard argument, (4.10) and (4.11) imply $u, v \in W^1_{2,\text{loc}}(\Omega)$.

Next, let $\{\Omega_k\}$ ($k = 1, 2, \ldots$) be a sequence of domains such that $\Omega_k \Subset \Omega_{k+1}, \partial\Omega_k$ is smooth and $\bigcup_{k=1}^{\infty}\Omega_k = \Omega$. Therefore the imbedding $W_2^1(\Omega_k) \subset L^2(\Omega_k)$ is compact. Using repeatedly (4.8) and (4.10) with Ω_k in place of Ω', we get by the aid of a diagonal procedure the existence of a subsequence (not relabelled) such that

$$\nabla\psi_\varepsilon(x) \to \nabla\psi(x) \quad \text{for a.a. } x \in \Omega, \tag{4.15}$$

$$u_\varepsilon(x) \to u(x),\ v_\varepsilon(x) \to v(x) \quad \text{for a.a. } x \in \Omega \tag{4.16}$$

as $\varepsilon \to 0$.

We have:

$$\begin{cases} \mu(\nabla\psi_\varepsilon)\nabla u_\varepsilon \to \mu(\nabla\psi)\nabla u & \text{weakly in } [L^2(\Omega)]^N, \\ \nu(\nabla\psi_\varepsilon)\nabla v_\varepsilon \to \nu(\nabla\psi)\nabla v & \text{weakly in } [L^2(\Omega)]^N \end{cases} \tag{4.17}$$

as $\varepsilon \to 0$. It suffices to prove the first claim. Let $\Omega' \subset\subset \Omega$. By (1.6) and (4.15),

$$\sqrt{\mu(\nabla\psi_\varepsilon)} \to \sqrt{\mu(\nabla\psi)} \quad \text{strongly in } L^2(\Omega'). \tag{4.18}$$

Hence, by (4.10), (4.11), $\sqrt{\mu(\nabla\psi_\varepsilon)}u_{\varepsilon x_i} \to \sqrt{\mu(\nabla\psi)}u_{x_i}$ weakly in $L^2(\Omega')$ as $\varepsilon \to 0$ ($i = 1, \ldots, N$). Then (4.14) implies $\chi_i = \sqrt{\mu(\nabla\psi)}u_{x_i}$ a.e. in Ω', and thus a.e. in Ω. Combining (4.14) and (4.15) gives the first claim in (4.17).

Finally, using (4.12), (4.13), (4.16) and (4.17) we may let tend $\varepsilon \to 0$ in (4.4), (4.5), (4.6) to obtain (2.10), (2.11) and

$$\int_\Omega \nabla\psi \cdot \nabla\varphi dx = \int_\Omega (e^{-\psi}v - e^{\psi}u + f)\varphi dx \quad \forall\varphi \in V.$$

The latter equation implies $\psi \in W^2_{2,\text{loc}}(\Omega)$ (independently of (4.8); cf. [7], [8]) and (2.12). ■

5. Proof of Theorem 2.

We begin by introducing some notations. Let $\Omega \subset \mathbb{R}^N$ be any bounded domain, and let $0 < \theta < 1$. define

$$W_2^\theta(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\theta}} dxdy < \infty \right\}.$$

Set

$$B_r = B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\},$$
$$\Delta_{h,i}u(x) = u(x + he_i) - u(x)$$

($e_i = \{0, \ldots, 0, 1, 0, \ldots 0\}$ with 1 at the i-th place; $i = 1, \ldots, N$).

For our discussion below, we need the following

LEMMA 2. *Let $u \in L^2(B_{2r})$ satisfy*

$$\int_{-r}^{r} \frac{1}{|h|^{1+2\theta}} \left(\int_{B_r} |\Delta_{h,i}u|^2 dx \right) dh < \infty \quad (i = 1, \ldots, N).$$

Then $u \in W_2^\theta(B_r)$ *and*

$$\int_{B_r}\int_{B_r}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\theta}}dxdy \le c\sum_{i=1}^{N}\int_{-r}^{r}\frac{1}{|h|^{1+2\theta}}\left(\int_{B_r}|\Delta_{h,i}u|^2dx\right)dh$$

with c = const > 0 *independent of* r *(cf. e.g. [1; pp. 209-210], [3; Lemma II.3]).* ■

We divide the proof of Theorem 2 into four steps.

1° Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ with $\partial\Omega'' \in C^2$. Fix $\zeta \in C^\infty(\mathbb{R}^2)$ with $\zeta = 1$ on Ω', $0 \le \zeta \le 1$ in $\mathbb{R}^2$ and $\zeta = 0$ in $\mathbb{R}^2\backslash\Omega''$(9).

Set $f_1 = (e^{-\psi}v - e^{\psi}u + f)\zeta - \nabla\psi\cdot\nabla\zeta - \psi\Delta\zeta$ a.e. in Ω''. We obtain $\psi\zeta \in W_2^1(\Omega'')$, $\psi\zeta = 0$ a.e. on $\partial\Omega''$ and

$$\int_{\Omega''}\nabla(\psi\zeta)\cdot\nabla\varphi dx = \int_{\Omega''} f_1\varphi dx$$

for all $\varphi \in W_2^1(\Omega'')$ with $\varphi = 0$ a.e. on $\partial\Omega''$ (cf. (2.12)). The theory of linear elliptic boundary value problems implies $\psi\zeta \in W_4^2(\Omega'')$ and

$$||\psi\zeta||_{W_4^2(\Omega'')} \le c(||f_1||_{L^4(\Omega'')} + ||\psi\zeta||_{L^4(\Omega'')})$$

(cf. e.g. [7], [8]). Thus, $\psi \in W_4^2(\Omega')$ and

$$||\psi||^4_{W_4^2(\Omega')} \le c\left\{\int_{\Omega''}(|e^{-\psi}v - e^{\psi}u + f|^4 + |\psi_0|^4dx + \left(\int_{\Omega''}|\nabla\psi_0|^2dx\right)^2\right\}.$$

This estimate can be used in (5.5) below in order to evaluate ∇^2u, ∇^2v in terms of bounds on u, v, ψ and integrals of $f, \psi_0, \nabla\psi_0$; we shall, however, dispense with these details. ■

---

(9) In order to simplify the presentation, from now on we suppose $N = 2$. We note that the reasoning in step 1° continues to hold for any $N \ge 2$ (with $\psi\zeta \in W_s^2(\Omega'')$ ($s = 4$ if $N = 2, 3, 4$; $s = \dfrac{2N}{N-2}$ if $N \ge 5$)), while the discussion both in step 2° and 3° remains true for $N = 2, 3$.

2° Set $Q = R(e^{\psi}u, e^{-\psi}v)\,(1-uv)$ a.e. in Ω. Then (2.10), (2.11) take the form

$$\int_\Omega \mu(\nabla\psi)e^{\psi}\nabla u \cdot \nabla\varphi dx = \int_\Omega Q\varphi dx \quad \forall\varphi \in V, \tag{5.1}$$

$$\int_\Omega \nu(\nabla\psi)e^{\psi}\nabla v \cdot \nabla\varphi dx = \int_\Omega Q\varphi dx \quad \forall\varphi \in V, \tag{5.2}$$

respectively.

From the preceding step we obtain by virtue of Sobolev's imbedding theorem $\nabla\psi \in [C^{1/2}(\Omega)]^2$. Hence, given $\Omega' \Subset \Omega$, there exist positive constants μ_1', ν_1' such that $\mu(\nabla\psi) \geq \mu_1'$ $\nu(\nabla\psi) \geq \nu_1'$ for all $x \in \bar{\Omega}'$.

Then (5.1), (5.2) imply higher integrability of $\nabla u, \nabla v$ via reserve Hölder inequality, i.e. there exists a $q_0 > 2$ such that

$$\nabla u, \nabla v \in [L^{q_0}_{\text{loc}}(\Omega)]^2 \tag{5.3}$$

(cf. [6; pp. 136-150]). ■

3° If $q_0 \geq 4$ in (5.3) we may apply the well-known difference quotient method to (5.1), (5.2) to obtain (2.14) (cf. e.g. [7], [8]).

Otherwise suppose that $\nabla u, \nabla v \in [L^q_{\text{loc}}(\Omega)]^2$ for a certain $2 < q < 4$. We are going to prove that ∇u, $\nabla v \in [W^\theta_{2,\text{loc}}(\Omega)]^2$ for all $0 < \theta < \dfrac{q-2}{2}$ (obviously, it suffices to consider ∇u only, for an analogous reasoning applies word by word to ∇v).

Let $\Omega' \Subset \Omega'' \Subset \Omega$ be arbitrary. Consider $B_r = B_r(x_0)$ where $0 < r \leq \frac{1}{3}\text{dist}(\Omega', \partial\Omega'')$ and $x_0 \in \Omega'$. Let $\zeta \in C^\infty(\mathbb{R}^2)$ be a cut-off function for B_{2r} : $\zeta = 1$ on B_r, $\zeta = 0$ in $\mathbb{R}^2 \setminus B_{2r}$ and $0 \leq \zeta \leq 1$, $|\nabla\zeta| \leq \dfrac{c_0}{r}$ in $\mathbb{R}^2$ ($c_0 = \text{const} > 0$ independent of r).

Set $\lambda = ||\psi||_{L^\infty(\Omega)}$. The function $\varphi = \Delta_{-h,i}(\zeta^2\Delta_{h,i}u)$ ($|h| \leq r; i = 1, 2$) is an admissible test function in (5.1). Observing that $\mu(\nabla\psi) \geq \mu_1'' = \text{const} > 0$ for all $x \in \Omega''$ (cf. step 1° with Ω'' in place of Ω') we find

$$
\begin{aligned}
\mu_1'' e^{-\lambda} \int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx &\leq \int_{B_{2r}} \mu(\nabla \psi) e^{\psi} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx \\
&= -\int_{B_{2r}} (\Delta_{h,i} \mu(\nabla \psi)) e^{\psi(x+he_i)} \nabla u(x+he_i) \cdot (\Delta_{h,i} \nabla u) \zeta^2 dx \\
&\quad - \int_{B_{2r}} \mu(\nabla \psi)(\Delta_{h,i} e^{\psi}) \nabla u(x-he_i) \cdot (\Delta_{h,i} \nabla u) \zeta^2 dx \\
&\quad - 2\int_{B_{2r}} [\Delta_{h,i}(\mu(\nabla \psi) e^{\psi} \nabla u] \cdot (\nabla \zeta)(\Delta_{h,i} u) \zeta \, dx \\
&\quad - \int_{B_{2r}} Q \Delta_{-h,i}(\zeta^2 \Delta_{h,i} u) dx = I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{5.4}
$$

To estimate I_1 we note that (2.13) gives

$$
\begin{aligned}
|\Delta_{h,i} \mu(\nabla \psi(x))| &\leq \int_0^1 \left| \frac{\partial \mu}{\partial \xi_j} (\nabla \psi(x) + \tau \Delta_{h,i} \nabla \psi(x)) \right| d\tau |\Delta_{h,i} \psi_{x_j}(x)| \\
&\leq c |\Delta_{h,i} \nabla \psi(x)|
\end{aligned}
$$

($x \in B_{2r}$). Taking into account (4.1) and $\dfrac{2q}{q-2} > q$, we obtain by the aid of Hölder's inequality

$$
\begin{aligned}
I_1 &\leq e^{\lambda} \left(\int_{B_{2r}} |\Delta_{h,i} \mu(\nabla \psi)|^{2q/(q-2)} dx \right)^{(q-2)/2q} \times \\
&\quad \times \left(\int_{B_{2r}} |\nabla u(x+he_i)|^q dx \right)^{1/q} \left(\int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx \right)^{1/2} \\
&\leq \frac{1}{5} \mu_1'' e^{-\lambda} \int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx \\
&\quad + c|h|^{q-2} \left(\int_{B_{3r}} |\nabla^2 \psi|^q dx \right)^{(q-2)/q} \left(\int_{B_{3r}} |\nabla u|^q dx \right)^{2/q}.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
I_2 &\leq \mu^* \int_{B_{2r}} |\Delta_{h,i} e^{\psi}|\, |\nabla u(x + he_i)|\, |\Delta_{h,i} \nabla u| \zeta^2 dx \\
&\leq \frac{1}{5} \mu_1'' e^{-\lambda} \int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx \\
&+ c|h|^{q-2} \left(\int_{B_{3r}} |\nabla \psi|^q dx \right)^{(q-2)/q} \left(\int_{B_{3r}} |\nabla u|^q dx \right)^{2/q},
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\leq \frac{2c_0}{r} \int_{B_{2r}} |\Delta_{h,i}(\mu(\nabla\psi) e^{\psi} \nabla u)|\, |\Delta_{h,i} u| \zeta\, dx \\
&\leq \frac{1}{5} \mu_1'' e^{-\lambda} \int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx + ch^2 r^{-2} \int_{B_{3r}} |\nabla u|^2 dx \\
&+ c|h|^{q-2} r^{-2/q} \left(\int_{B_{3r}} (|\nabla \psi|^q + |\nabla^2 \psi|^q) dx \right)^{(q-2)/2q} \\
&\times \left(\int_{B_{3r}} |\nabla u|^q dx \right)^{2/q}
\end{aligned}
$$

(without loss of generality, we may assume that $r \leq 1$).

Finally,

$$
\begin{aligned}
I_4 &\leq cr\|Q\|_{L^\infty(\Omega)} \left(\int_{B_{2r}} |\Delta_{-h,i}(\zeta^2 \Delta_{h,i} u)|^2 dx \right)^{1/2} \\
&\leq cr|h|\, \|Q\|_{L^\infty(\Omega)} \left(\int_{B_{3r}} |(\zeta^2 \Delta_{h,i} u)_{x_i}|^2 dx \right)^{1/2} \\
&\leq \frac{1}{5} \mu_1'' e^{-\lambda} \int_{B_{2r}} |\Delta_{h,i} \nabla u|^2 \zeta^2 dx \\
&+ ch^2 \|Q\|_{L^\infty(\Omega)} \left(r^2 + r^{-2} \int_{B_{3r}} |\nabla u|^2 dx \right).
\end{aligned}
$$

Inserting these estimates into (5.4) gives

$$\int_{B_r} |\Delta_{h,i}\nabla u|^2 dx \le c|h|^{q-2}\Bigg\{ \left(\left(\int_{B_{3r}} (|\nabla\psi|^q + |\nabla^2\psi|^q)dx\right)^{(q-2)/q} + r^{-2/q}\left(\int_{B_{3r}} (|\nabla\psi|^q + |\nabla^2\psi|^q)dx\right)^{(q-2)/2q}\right\} \times \left(\int_{B_{3r}} |\nabla u|^q dx\right)^{2/q} + ch^2\left(r^2 + r^{-2}\int_{B_{3r}} |\nabla|^2 dx\right) \tag{5.5}$$

for all $|h| \le r$, where the constant c depends neither on r nor on h. Thus,

$$\int_{-r}^{r} \frac{1}{|h|^{1+2\theta}}\left(\int_{B_r} |\Delta_{h,i}\nabla u|^2 dx\right) dh \le c < \infty \quad \forall 0 < \theta < \frac{q-2}{2} \tag{5.6}$$

$(i = 1, 2)$. ■

4° From (5.6) we obtain $\nabla u \in [W_2^\theta(B_r)]^2$ (cf. Lemma 2 above). Hence, by Sobolev's imbedding theorem, $\nabla u \in [L^{2/(1-\theta)}(B_r)]^2$. Observing that $q > 2$ $(= N)$ one can choose $\dfrac{q-2}{q} < \theta < \dfrac{q-2}{2}$ to get $\dfrac{2}{1-\theta} > q$.

We repeat this argument. To begin with, define $\omega(t) = \dfrac{t^2-4}{4t}$ $(2 < t < 4)$. The function ω is strictly increasing.

We start by considering (5.5) with $q = q_0$ (q_0 according to (5.3); $2 < q_0 < 4$). There follows (5.6) with $\theta_0 = \omega(q_0)$, hence $\nabla u \in [L^{q_1}(B_r)]^2$ with $q_1 = \dfrac{2}{1-\omega(q_0)} > q_0$.

If $q_1 \ge 4$, we have finished. If, however, $q_1 < 4$ we again consider (5.5) with $q = q_1$ to obtain $\nabla u \in [L^{q_2}(B_r)]^2$ with $q_2 = \dfrac{2}{1-\omega(q_1)} > q - 1$.

We may therefore define reals q_k $(k = 1, 2, \dots)$ as follows: if $2 < q_k < 4$ set

$$q_{k+1} = \frac{2}{1-\omega(q_k)} \; (k = 0, 1, 2, \dots).$$

It is readily seen that

$$q_k > q_{k-1} \Rightarrow q_{k+1} - q_k > q_k - q_{k-1}$$

($k = 1, 2, \ldots$). Hence, $q_{k+1} \geq q_0 + (k+1)(q_1 - q_0)$.

Let k_0 be the positive integer such that $q_{k_0} < 4$ and $q_{k_0+1} \geq 4$. Then we consider (5.5) with $q = q_{k_0}$ and obtain $\nabla u \in [L^{q_{k_0}+1}(B_r)]^2$. Whence $\nabla u \in [L^{q_{k_0}+1}_{\mathrm{loc}}(\Omega)]^2$. ∎

REFERENCES

[1] Adams R.A., *Sobolev spaces,* Academic Press, New York 1975.

[2] Baccarani G., Rudan M., Guerrieri R., Ciampolini P., *Physical models for numerical device simulation,* In: process and device modeling; W. L. Engl (ed.), Elsevier Sci. Publ. B. V., North-Holland, 1986; pp. 107-158.

[3] Campanato S., *Sulla regolarità delle soluzioni di equazioni differenziali di tipo ellittico,* Editrice Tecnico Scientifica, Pisa 1963.

[4] Gajewski H., *On the existence of steady - state carrier distributions in semiconductors,* In: Probleme u. Meth. Math. Phys.; Teubner - Texte Math. 63, Leipzig 1984; pp. 76-82.

[5] Gajewski H., Gröger K., *On the basic equations for carrier transport in semiconductors,* J. Math. Anal. Appl. **113** (1986), 12-35.

[6] Giaquinta M., *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Princeton Univ. Press, Princeton, New Jersey 1983.

[7] Gilbarg D., Trudinger N.S., *Elliptic partial differential equations of second order,* (2nd ed). Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1983.

[8] Grisvard P., *Elliptic problems in nonsmooth domains,* Pitman Advanced Publ. Progr., Boston, London, Melbourne 1985.

[9] Gröger K., *On the boundedness of solutions to the basic equations in semiconductor theory,* Math. Nachr. **129** (1986), 167-174.

[10] Gröger K., *On steady-state carrier distributions in semiconductor devices,* Aplikace Mat. **32** (1987), 49-56.

[11] Jerome J.W., *Consistency of semiconductor modeling: an existence / stability analysis for the stationary van Roosbroeck system,* SIAM J. Appl. Math. **45** (1985), 565-590.

[12] Lions J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Gauthier-Villars, Paris 1969.

[13] Markowich P., *The stationary semiconductor device equations,* Springer-Verlag, Wien, New York 1986.

[14] Mock M., *On equations describing steady-state carrier distributions in a semiconductor device,* Comm. Pure Appl. Math. **25** (1972), 781-792.

[15] Mock M., *Analysis of mathematical models of semiconductor devices,* Boole Press, Dublin 1983.

[16] Seidman T.I., *Steady state solutions of diffusion-reaction systems with electrostatic convection,* Nonlinear Anal., Theory, Meth. & Appl. **4** (1980), 623-637.

[17] Selberherr S., *Analysis and simulation of semiconductor devices,* Springer-Verlag, Wien, New York 1984.

[18] Stampacchia G., *Le problème de Dirichlet pour les equations elliptiques du second ordre à coefficients discontinus,* Ann. Inst. Fourier, Grenoble, **15** (1965), 189-258.

[19] Stampacchia G., *Equations elliptiques du second ordre à coefficients discontinus,* Les presses de l'univ. de Montréal, 1966.

[20] Thornber K.K., *Relation of drift velocity to low-field mobility and high-field saturation velocity,* J. Apl. Phys. **51** (1980), 2127-2136.

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