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ON THE POINT LINEAR ARBORICITY OF A GRAPH

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In a linear forest, every component is a path. The linear arboricity of a graph G is the smallest number of edge disjoint linear forests whose union is G; this concept has been much studied. We now introduce the point linear arboricity of G, defined as the smallest number of parts in a partition of $V = V(G)$ such that each part induces a linear forest. We prove an analogue to the classical theorem of Brooks for this invariant: For all r such that $2 \leq r \leq 2\Delta/2$, if G does not contain K_{2r+1}, and t is defined as $\lfloor(1+\Delta)/(2r+1)\rfloor$, then the point linear arboricity of G is at most $\lfloor(\Delta - t + 3)/2\rfloor$.

1. Introduction.

In recent literature, a variety of vertex partition problems, modeled on the chromatic number, have been considered. The general description of these problems is as follows. Suppose P is an *hereditary property*, that is, a property inherited by subgraphs. For a graph G one asks for the smallest n so that the vertices of G may be

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partitioned into n sets each of which induces a subgraph with property P. If P is the property of being independent, then n is the chromatic number. If P =acyclic, then n is the point arboricity, $a(G)$, introduced in [6] and studied further in [2]. If P = k-degenerate, i.e., for every induced subgraph H of G the minimum degree $\delta(H) \leq k$, then n is the point partition number $\rho_k(G)$, introduced in [15]. If P is the property of having maximum degree not exceeding k, then n is the k-chromatic number $\chi_k(G)$ considered in [1, 4, 13]. The concept of conditional colorability in graphs was apparently introduced first in [12] and then independently in [10]. It was studied intensively in [3] and more generally in [9, 11]. All the examples above, and several others, are special cases of conditional colorability $\chi(G: P)$, as is the new invariant we now introduce.

The *point linear arboricity* of G is denoted by $\Xi_0(G)$ as the (edge) linear arboricity, introduced in [8], is denoted by $\Xi(G)$. Our purpose is to prove the analog of Brook Theorem about this parameter.

In general we follow the notation and terminology of [7].

A *linear forest* is a graph G such that every component of G is a path. The *point linear arboricity* $\Xi_0(G)$ is the smallest integer n such that the vertices of G may be partitioned into n sets each of which induces a linear forest.

2. Bounds on the point linear arboricity.

Since a graph with maximum degree $\Delta = 1$ is certainly a linear forest, and since a linear forest has maximum degree not exceeding 2, we see that for any graph G,

$$\chi_2(G) \leq \Xi_0(G)\chi_1(G). \tag{1}$$

We use the second of these inequalities to get the following easily proved assertion whose main purpose is to motivate the Brooks-type theorem which follows it.

A simple upper bound involving the maximum degree Δ for all graphs G is given by:

$$\Xi_0(G) \leq 1 + \lfloor \Delta/2 \rfloor \tag{2}$$

Proof of (2). By Theorem 1 of [13], the vertices of G may be partitioned into $1+\lfloor \Delta/2 \rfloor$ classes so that each class induces a subgraph of maximum degree 1, which is of course a linear forest as each component is K_1 or K_2. ■

The bound in this theorem is sharp as it is achieved in cycles and complete graphs. We now show that these are the only connected extremal graphs in the case where Δ is even.

THEOREM 1. *If G is a connected graph which is neither a cycle nor a complete graph of odd order, then*

$$\Xi_0(G) \leq \lceil \delta/2 \rceil. \tag{3}$$

Proof. By Theorem 7 of [14], the vertices of G may be partitioned into $\lceil \Delta/2 \rceil$ classes, each of which induces a forest. Among such partitions, choose one which maximizes the number of external edges, that is, the number of edges which join vertices of different classes. Suppose now that some partition class C_1 has a vertex v of degree 3 or more. Then there are at most $\Delta - 3$ external edges incident with v. Hence the remaining $\lceil \Delta/2 \rceil - 1 = \lceil (\Delta - 2)/2 \rceil$ classes receive at most $\Delta - 3$ edges from v, and it follows that some class C_2 receives at most one edge from v. Moving v from C_1 to C_2 increases the number of external edges, and, since v will have degree 1 in C_2, each class still induces a forest. This contradicts the choice of partition. Hence, each partition class has maximum degree 2, and is therefore a linear forest. ■

Thus we have improved the theorem of Kronk and Mitchem [14] by showing that their upper bound remains valid with strengthened requirements on the partition classes. We remark that Maddox [17] has provided examples which show that any Brooks-type theorem for the parameter χ_k will necessarily be considerably more complicated. Our next theorem is an upper bound for $\Xi_0(G)$ in terms of the maximum degree Δ and the clique number ω, the maximum order of a complete subgraph. It is convenient to phrase the theorem in terms of excluded complete subgraphs. The proof of Theorem 2 follows a technique used by Catlin [5]. We shall need the following theorem of Lovasz [16].

THEOREM A. *Let t be a positive integer and let $k_1, k_2, \ldots, k_t$ constitute the partition:*

$$k_1 + k_2 + \ldots + k_t = \Delta - t + 1. \tag{4}$$

Then the vertices of G may be partitioned into sets $V_1, V_2, \ldots, V_t$ so that the subgraph induced by each V_i has maximum degree at most k_i.

THEOREM 2. *If G is a graph containing no subgraph K_{2r+1} for a fixed r, $2 \leq r \leq 2\Delta/2$, and we write $t = \lfloor(1+\Delta)/(2r+1)\rfloor$, then*

$$\Xi_0(G) \leq \left\lfloor \frac{\Delta - t + 3}{2} \right\rfloor. \tag{5}$$

Proof. For $1 \leq i \leq t-1$, let $k_i = 2r$. Let $k_t = \Delta - (2r+1)\ (t-1)$. Then $k_1 + k_2 + \ldots + k_t = (t-1)\ (2r) + \Delta - (2r+1)(t-1) = \Delta - t + 1$. By Theorem A, we may partition the vertex set of G into $V_1 \cup \ldots \cup V_t$ with each V_i inducing a subgraph of maximum degree at most k_i. By Theorem 1, we may partition V_i, $1 \leq i \leq t-1$, into r linear forests and by (1) we may partition V_t into $\lfloor(\Delta - (2r+1)(t-1))/2\rfloor$ linear forests. This yields a partition of G into $\lfloor(\Delta - (2r+1)(t-1))/2\rfloor = \lfloor(\Delta - t + 3)/2\rfloor$ linear forests. ■

The next inequality follows at once.

COROLLARY 2a. *If G does not contain K_{2r+1} then*

$$\Xi_0(G) \leq r + \lfloor r\Delta/(2r+1)\rfloor. \tag{6}$$

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