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MORE ON THE EQUALITY $L^1(\mu) = L^1_{\text{loc}}(\mu)$

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In this note we characterize those measure spaces $(\Omega, \mathcal{A}, \mu)$ for which $L^1(\mu) = L^1_{\text{loc}}(\mu)$; here Ω is a topological space not necessarily metrizable.

In what follows $(\Omega, \mathcal{A}, \mu)$ will denote a measure space such that:

i) Ω is a topological space,

ii) for any compact subset K of Ω we have $K \in \mathcal{A}$ and $\mu(K) < +\infty$.

A measurable function $f : \Omega \to \mathbb{R}$ is said to be locally integrable provided $\int_K |f| d\mu < +\infty$ for any compact $K \subset \Omega$. The set of all locally integrable functions in Ω is denoted by $L^1_{\text{loc}}(\mu)$.

Clearly $L^1(\mu) \subset L^1_{\text{loc}}(\mu)$ for any measurable space but the converse inclusion generally fails. In [5] A. Villani studied the problem to characterize those measure spaces $(\Omega, \mathcal{A}, \mu)$ for which the equality

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$L^1(\mu) = L^1_{\mathrm{loc}}(\mu)$ holds. He obtained a complete answer to the problem assuming Ω to be a metrizable space. This assumption however seems to be a technical tool for the type of proof given in [5] rather than a property of Ω strictly necessary for the validity of the result. The aim of the present note is just to take care of this point. We generalize the theorem in [5] and prove that it holds for much larger classes of topological spaces. Indeed it turns out that the perfect normality of Ω together with a property which forces each countably compact subset to be compact sufficies.

Our notation follows [1] and [2]. Each topological space is assumed to be Hausdorff. A space is said to be countably compact provided that any countable open cover has a finite subcover or equivalently any closed discrete subset is finite. A space is said to be perfect provided each closed set is a G_δ, i.e. an intersection of countably many open sets. A space is $\aleph_0$-normal provided that any two disjoint closed subsets one of which is countable can be separated by open sets. The class of $\aleph_0$-normal spaces strictly lies between the class of regular spaces and the class of normal spaces. Let us also recall that, given a measure space $(\Omega, \mathcal{A}, \mu)$ an element $A \in \mathcal{A}$ is an atom provided that $\mu(A) > 0$ and for any $A' \in \mathcal{A}$ such that $A' \subset A$ we have either $\mu(A') = 0$ or $\mu(A \backslash A') = 0$.

THEOREM *If Ω is a perfect $\aleph_0$-normal space and any countable compact subset of Ω is compact then the following conditions are equivalent:*

1. *$L^1(\mu) = L^1_{\mathrm{loc}}(\mu)$;*
2. *There exists a compact subset H of Ω such that $\Omega \backslash H$ either is of measure zero or is the union of finitely many atoms of finite measure.*

Proof. The implication $2 \rightarrow 1$ is quite straightforward and it actually holds for any measure space. The only thing to be observed is that if f is a measurable function and $A \in \mathcal{A}$ is an atom then f is almost everywhere constant on A.

For the converse implication let's assume $L^1(\mu) = L^1_{loc}(\mu)$. Note first that μ must be finite, because for any compact $K \subset \Omega$ we have $\int_K 1_\Omega d\mu = \mu(K) < +\infty$ and hence $1_\Omega \in L^1(\mu)$. Let $\mathcal{V}$ be the collection of all open subsets of Ω having no compact subset of positive measure and let $V = \cup\mathcal{V}$. The set V belongs to $\mathcal{V}$ because for any compact $K \subset V$ there exist finitely many sets $V_1, \ldots, V_n \in \mathcal{V}$ such that $K \subset V_1 \cup \ldots \cup V_n$ and we can decompose K (see the Lemma in [5] the proof of Theorem 2.13 in [3]) as the union of compact sets $K_1, \ldots, K_n$ such that $K_i \subset V_i$ for any $1 \leq i \leq n$. This shows that $\mu(K) = 0$ and hence $V \in \mathcal{V}$.

Now put $H = \Omega \backslash V$ and assume that H is not countably compact. There exists then a sequence $\{x_n : n \in \mathbb{N}\}$ of distinct points of H such that the set $\{x_n : n \in \mathbb{N}\}$ is closed discrete in H and hence in Ω.

By the regularity of Ω we can choose on open set U_1 such that $x_1 \in U_1$ and $x_n \notin \bar{U}_1$ for any $n > 1$. Next we choose an open set U_2 such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, $x_2 \in U_2$ and $x_n \notin \bar{U}_2$ for any $n > 2$ and so on. In this way we can construct a sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint open sets such that $x_n \in U_n$ for any $n \in \mathbb{N}$.

Since Ω is $\aleph_0$-normal there exists an open set A such that $\{x_n : n \in \mathbb{N}\} \subset A \subset \bar{A} \subset \bigcup_{n \in \mathbb{N}} U_n$.

It is easy to check that the family $\{U_n \cap A : n \in \mathbb{N}\}$ is discrete and moreover for any $n \in \mathbb{N}$ there exists a compact set $K_n \subset U_n \cap A$ such that $\mu(K_n) > 0$ (otherwise we would get the contradiction $x_n \in V$).

Since every compact subset of Ω meets only finitely many members of the family $\{K_n : n \in \mathbb{N}\}$, it turns out that the function $f : \Omega \to \mathbb{R}$ defined by $f = \sum_{n \in \mathbb{N}} (\mu(K_n))^{-1} 1_{K_n}$ is locally integrable. On the other end $\int_\Omega f d\mu = +\infty$ which is in contradiction with the assumption $L^1(\mu) = L^1_{loc}(\mu)$.

Consequently H is countably compact and hence compact.

Let us show now that $\mu(K \cap V) = 0$ for any compact $K \subset \Omega$.

Since Ω is perfect, the set V can be expressed as the union of countably many closed sets, say $F_1, \dots, F_n, \dots$

We have $K \cap V = \bigcup_{n \in \mathbb{N}} (K \cap F_n)$ and each $K \cap F_n$ is a compact subset of V. By the choice of V this implies that $\mu(K \cap V) = 0$.

Finally if $\mu(V) > 0$ then V must be the union of finitely many atoms, because otherwise there exists in V a sequence $\{A_n : n \in \mathbb{N}\}$ of disjoint measurable sets of positive measure and we can define a function $f_n = \sum_{n \in \mathbb{N}} (\mu(A_n))^{-1} 1_{A_n}$ which belongs to $L^1_{\text{loc}}(\mu) \setminus L^1(\mu)$.

This completes the proof.

Many classes of spaces have the property that each countably compact subset is compact (see [4]).

In particular the hypotheses of the theorem are fullfilled in the following well known cases:

– Ω meta-compact and perfectly normal;

– Ω paracompact and perfect;

– Ω normal σ-space;

– Ω perfectly normal and with G_δ diagonal;

– Ω stratifiable;

– Ω closed continuous image of a metrizable space.

All the above mentioned classes of spaces are strictly larger than the class of metrizable spaces.

If in the theorem we assume also that the measure μ satisfies the following tightness condition:

(*) for any open set $V \in \mathcal{A}$ we have $\mu(V) = \sup\{\mu(K) : K \subset V, K$ compact$\}$, then the assumption that Ω is perfect can be omitted and we get by the same proof a simpler statement as follows:

THEOREM' *Let μ satisfy condition (*). If Ω is $\aleph_0$+normal and each countably subset is compact then the following facts are equivalent:*

1) $L^1(\mu) = L^1_{loc}(\mu)$

2) Ω *is «essentially» compact, i.e. there exists a compact set* H *such that* $\mu(\Omega \backslash H) = 0$.

It is worthwhile noticing that the topological assumptions on Ω in the previous statement are satisfied whenever Ω is a paracompact space.

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