«LE MATEMATICHE»
Vol. XLIV (1989) - Fasc. II, pp. 293-310

OSCILLATORY BEHAVIOR OF DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

G. LADAS - C. QIAN (*) (Kingston) (**)

We obtain sufficient conditions for the oscillation of all solutions of linear difference equations with positive and negative coefficients of the form

$$y_{n+1} - y_n + p_n y_{n-k} - q_n y_{n-l} = 0, \qquad n = 0, 1, 2, \dots$$

where k and l are nonnegative integers and the coefficients $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers.

1. Introduction and Preliminaries.

Consider the linear difference equation with positive and negative coefficients of the form

$$y_{n+1} - y_n + p_n y_{n-k} - q_n y_{n-l} = 0, \qquad n = 0, 1, 2, \dots \tag{1}$$

where k and l are non negative integers and the coefficients $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers which are defined

(*) On leave from Department of Mathematics, Yangzhou Teacher's College, Yangzhou, Tiangsu, P.R.C..

(**) Entrato in Redazione il 3-10-1989.

for $n \geq 0$. Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of Eq. (1).

Let $m = \max\{k, l\}$. By a *solution* of Eq. (1) we mean a sequence $\{y_n\}$ which is defined for $n \geq -m$ and which satisfies Eq. (1) for $n = 0, 1, \ldots$. Clearly, if $a_{-m}, \ldots, a_0$ are given real numbers, then Eq. (1) has a unique solutions $\{y_n\}$ satisfying the initial conditions

$$y_n = a_n \qquad \text{for } n = -m, \ldots, 0.$$

A solution $\{y_n\}$ of Eq. (1) is said to *oscillate* if for every $n_0 \geq 0$ there exists an $n \geq n_0$ such that

$$y_n y_{n+1} \leq 0.$$

Otherwise the solution is called *nonoscillatory*.

When k and l are nonnegative integers, then Eq. (1) is a linear difference equation of order $m + 1$ where $m = \max\{k, l\}$. We may also look at Eq. (1) as being a first-order linear *delay difference equation* with delay $m + 1$.

In general, if k and l are integers and if we set

$$K = \max\{0, k, l\} \quad \text{and} \quad L = \max\{1, -k, -l\},$$

then Eq. (1) is a difference equation of order $(K + L)$. When $K \geq 0$ and $L = 1$, Eq. (1) is a delay difference equation with delay $(K + L)$. When $K = 0$ and $L \geq 2$, Eq. (1) is an *advanced difference equation.* Finally when $K \geq 1$ and $L \geq 2$, then Eq. (1) is of the *mixed type.*

The oscillatory behavior of delay, advanced and mixed type difference equations with constant coefficients has been investigated in [2] and [4]. Difference equations with positive and negative coefficients which are asymptotically constant were studied in [3]. Sharp conditions for the oscillation of all solutions of difference equations with variable coefficients of the form

$$y_{n+1} - y_n + p_n y_{n-k} = 0$$

where obtained in [5]. See also [1]. The results in this paper are discrete analogues of the results in [1] for differential equations with variable coefficients.

Let $\mathbb{N}$ denote the set of nonnegative integers, $\{0,1,2,\ldots\}$. The following lemmas which are extracted from [2], [4] and [5], will be usefull in the proofs of our theorems.

LEMMA 1. *Consider the delay difference equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \qquad n = 0,1,2,\ldots \tag{2}$$

where

$$p_i \in (0,\infty) \text{ and } k_i \in \mathbb{N} \text{ for } 1 = 1,2,\ldots,m. \tag{3}$$

Assume that one of the following two conditions is satisfied:

$$(H_1) \qquad \sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 \tag{4}$$

or

$$(H_2) \qquad \left(\prod_{i=1}^{m} p_i\right)^{\frac{1}{m}} \frac{(k+1)^{k+1}}{k^k} > 1 \tag{5}$$

where

$$k = \frac{1}{m}\sum_{i=1}^{m} k_i.$$

Then every solution of Eq. (2) oscillates.

LEMMA 2. *Consider the delay difference equation*

$$x_{n+1} - x_n + p_n x_{n-k} = 0 \tag{6}$$

where $k \in \mathbb{N}$ and $\{p_n\}$ is a sequence of nonnegative real numbers. Assume that one of the following two conditions is satisfied:

$$(H_3) \qquad k \geq 1 \text{ and } \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1} \tag{7}$$

or

$$(8) \qquad (H_4) \qquad k \geq 0 \text{ and } \liminf_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1.$$

Then every solution of Eq. (6) oscillates.

LEMMA 3. *Let k be a positive integer and let $\{p_n\}$ be a sequence of nonnegative real numbers such that*

$$(9) \qquad \sum_{i=n-k}^{n-1} p_i > 0 \text{ for all large } n.$$

Then the delay difference inequality

$$(10) \qquad x_{n+1} - x_n + p_n x_{n-k} \leq 0, \qquad n = 0, 1, 2, \ldots$$

has an eventually positive solution if and only if the delay difference equation

$$(11) \qquad y_{n+1} - y_n + p_n y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots$$

has an eventually positive solution.

Remark 1. Lemma 1 when (H_1) holds and Lemma 2 when (H_4) holds are from [1]. Lemma 1 when (H_2) holds is from [4]. Lemma 2 when (H_3) holds and Lemma 3 are from [5].

2. Asymptotic Behavior of Nonoscillatory Solutions.

In this section we study the asymptotic behavior of the nonoscillatory solutions of Eq. (1) where

$$(12) \qquad \begin{cases} k,\ l \in \mathbb{N} \text{ and } p_n,\ q_n \text{ are sequences of nonnegative} \\ \text{real numbers defined for } n = 0, 1, 2, \ldots \end{cases}$$

LEMMA 4. *Assume that (12) holds,*

$$k > l, \tag{13}$$

$$p_n - q_{n-k+l} \geq 0 \text{ but } p_n - q_{n-k+l} \not\equiv 0 \qquad \text{for } n \geq k - l \tag{14}$$

and

$$\sum_{j=l+1}^{k} q_{n-j} \leq 1 \qquad \text{for } n \geq k. \tag{15}$$

Let $\{y_n\}$ be an eventually positive solution of Eq. (1) and set

$$z_n = y_n - \sum_{j=l+1}^{k} q_{n+l-j}\, y_{n-j} \qquad \text{for } n \geq k - l. \tag{16}$$

Then $\{z_n\}$ is an eventually positive and decreasing sequence.

Proof. Assume that $n_0 \geq 0$ is such that

$$y_n > 0 \qquad \text{for } n \geq n_0.$$

Then for $n \geq n_0 + k$,

$$\begin{aligned} z_{n+1} - z_n &= (y_{n+1} - y_n) - (q_n y_{n-l} - q_{n+l-k}\, y_{n-k}) \\ &= -(p_n - q_{n+l-k})y_{n-k} \leq 0. \end{aligned}$$

Hence $\{z_n\}$ is a decreasing sequnce for $n \geq n_0 + k$ and so either

$$\lim_{n\to\infty} z_n = -\infty \tag{17}$$

or

$$\lim_{n\to\infty} z_n = L \in \mathbb{R}. \tag{18}$$

First assume that $\{y_n\}$ is an unbounded sequence. Then there exists a subsequence $\{y_{n_m}\}$ such that

$$y_{n_m} = \max\{y_n : n \leq n_m\} \quad \text{for } m = 1, 2, \ldots.$$

From (16) we find,

$$\begin{aligned} z_{n_m} &= y_{n_m} - \sum_{j=l+1}^{k} q_{n_m+l-j}\, y_{n_m-j} \\ &\geq \left(1 - \sum_{j=l+1}^{k} q_{n_m+l-j}\right) y_{n_m} \\ &\geq 0. \end{aligned}$$

As $\{z_n\}$ is decreasing, it follows that $z_n > 0$ and the proof is complete when $\{y_n\}$ is unbounded.

Next assume that $\{y_n\}$ is a bounded sequence. Then (18) holds and also

$$\mu = \limsup_{n\to\infty} y_n$$

exists and is finite. Let $\{y_{n_s}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{s\to\infty} y_{n_s} = \mu.$$

Then for $\varepsilon > 0$ and sufficiently small and for n sufficiently large,

$$\begin{aligned} z_{n_s} &= y_{n_s} - \sum_{j=l+1}^{k} q_{n_s+l-k}\, y_{n_s-j} \\ &\geq y_{n_s} - (\mu + \varepsilon). \end{aligned}$$

By taking limits as $s \to \infty$ we see that $L \geq -\varepsilon$. As $\varepsilon > 0$ is arbitrary, it follows that $L \geq 0$ and so $z_n > 0$. The proof is complete.

The next result provides sufficient conditions so that every nonoscillatory solution of Eq. (1) tends to zero as $n \to \infty$.

THEOREM 1. *Assume that (12)-(15) hold and that one of the following two conditions is satisfied.*

(H_5) *there exist a positive constant* α *such that*

$$p_n - q_{n+l-k} \geq \alpha \quad \textit{for } n \geq k - l \tag{19}$$

or

(H_6) *There exist a positive constant* $\beta \in (0,1)$ *such that*

$$\sum_{j=l+1}^{k} q_{n-j} \leq 1-\beta \quad \textit{for } n \geq k \tag{20}$$

and

$$\sum_{i=k+l}^{\infty} (p_i - q_{i+l-k}) = \infty. \tag{21}$$

Then every nonoscillatory solution of Eq. (1) tends to zero as $n \to \infty$.

Proof. It sufficies to show that every eventually positive solution $\{y_n\}$ of Eq. (1) tends to zero as $n \to \infty$. By Lemma 4 it follows that the sequence $\{z_n\}$, which is defined by (16), is eventually decreasing and positive. Hence

$$\lim_{n\to\infty} z_n = L \in \mathbb{R}^+. \tag{22}$$

Summing *up* both sides of

$$z_{n+1} - z_n = -(p_n - q_{n+l-k})y_{n-k}$$

from n_1 to ∞, for n_1 sufficiently large, we find

$$L - z_{n_1} = -\sum_{i=n_1}^{\infty} (p_i - q_{i+l-k})y_{i-k}. \tag{23}$$

First assume that (19) holds. Then (23) implies that

$$\sum_{i=n_1}^{\infty} y_{i-k} < \infty.$$

Hence

$$\lim_{n\to\infty} y_n = 0$$

and the proof is complete when (19) holds.

Next assume that (20) and (21) hold. From (23) it follows that

$$\liminf_{n\to\infty} y_n = 0.$$

Also (16) implies that $z_n \leq y_n$ and in view of (22), $L = 0$. Now we claim that $\{y_n\}$ is a bounded sequence. Otherwise, there exists a subsequnce $\{y_{n_r}\}$ of $\{y_n\}$ such that

$$y_{n_r} = \max\{y_n : n \leq n_r\} \qquad \text{for } r = 1, 2, \ldots \text{ and } \lim_{r\to\infty} y_{n_r} = \infty.$$

Then by (16) and (20),

$$\begin{aligned} z_{n_r} &= y_{n_r} - \sum_{j=l+1}^{k} q_{n_r+l-j}\, y_{n_r-j} \\ &\geq \left(1 - \sum_{j=l+1}^{k} q_{n_r+l-j}\right) y_{n_r-j} \\ &\geq \beta y_{n_r} \to \infty \text{ as } r \to \infty \end{aligned}$$

which contradicts the fact that $L = 0$ and establishes our claim that $\{y_n\}$ is bounded. Set

$$\mu = \limsup_{n\to\infty} y_n$$

and let $\{y_{n_s}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{s\to\infty} y_{n_s} = \mu.$$

Then for $\varepsilon > 0$ and sufficiently small and for s sufficiently large, it follows from (16) and (20) that

$$\begin{aligned} z_{n_s} &= y_{n_s} - \sum_{j=l+1}^{k} q_{n_s+l-j}\, y_{n_s-j} \\ &\geq y_{n_s} - (\mu + \varepsilon)(1 - \beta). \end{aligned}$$

By taking limits as $s \to \infty$ and by using the fact that $L = 0$ we obtain

$$0 \leq \mu_s - (\mu + \varepsilon)(1 - \beta).$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\mu = 0$ and the proof is complete.

3. Sufficient Conditions for Oscillation.

In this section we will establish sufficient conditions for the oscillation of all solutions of Eq. (1).

THEOREM 2. *Assume that (12)-(15) hold and that one of the following two conditions is satisfied:*

(24)
$$(H_7) \quad \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} (p_i - q_{i-k+l}) > \left(\frac{k}{k+1}\right)^{k+1}$$

or

(25)
$$\begin{cases} (H_8) \quad \displaystyle\sum_{i=n-k}^{n-1} (p_i - q_{i-k+l}) > 0 \text{ for all large } n \text{ and} \\ \displaystyle\limsup_{n\to\infty} \sum_{i=n-k}^{n} (p_i - q_{i-k+l}) > 1. \end{cases}$$

Then every solution of Eq. (1) oscillates.

Proof. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $\{y_n\}$. By Lemma 4 it follows that the sequence $\{z_n\}$ which is defined by (16) is eventually positive and

$$z_{n+1} - z_n + (p_n - q_{n+l-k})y_{n-k} = 0.$$

Also eventually,

(26)
$$0 < z_n \le y_n$$

and so,

(27)
$$z_{n+1} - z_n + (p_n - q_{n+l-k})z_{n-k} \le 0.$$

However, by Lemma 3 and 2, inequality (27) cannot have an eventually positive solution. This contradicts (26) and the proof is complete.

Before we can establish the next oscillation theorem we need the following result about difference inequalities which is interesting in its own right.

LEMMA 5. *Assume that for* $i = 1, 2, \ldots, N$, $k_i \in \mathbb{N}$ *and* $\{r_n^{(i)}\}$ *are sequences of nonnegative real numbers such that for every* $n_0 \in \mathbb{N}$ *there exists an* $i_0 \in \{0, 1, \ldots, \mathbb{N}\}$ *with the property that*

$$\sum_{j=n_0}^{n_0+k_{i_0}} r_j(i_0) > 0. \tag{28}$$

Let $k = \max\{k_0, k_1, \ldots, k_N\}$ *and assume that the inequality*

$$\sum_{i=0}^{N} \sum_{j=n}^{\infty} r_j^{(i)} b_{j-k_i} \leq b_n \quad \text{for } n \geq n_1 \tag{29}$$

has a positive solution $b = \{b_n\}_{n_1-k}^{\infty}$ *such that*

$$b_{n_1} < b_n \quad \text{for } n_1 - k \leq n < n_1\,. \tag{30}$$

Then there exist a positive solution $c = \{c_n\}_{n_1-k}^{\infty}$ *of corresponding equation*

$$\sum_{i=0}^{N} \sum_{j=n}^{\infty} r_j^{(i)} c_{j-k_i} = c_n \quad \text{for } n \geq n_1. \tag{31}$$

Proof. Define the set of nonnegative sequences

$$\Lambda = \{\tilde{c} = \{\tilde{c}_n\}\infty_{n=n_1} : 0 \leq \tilde{c}_n \leq b_n \quad \text{for } n \geq n_1\}.$$

For every $\tilde{c} \in \Lambda$ define the sequence $c = \{c_n\}_{n=n_1-k}^{\infty}$ by

$$c_n = \begin{cases} \tilde{c}, & n \geq n_1 \\ \tilde{c}_{n_1} + b_n - b_{n_1}, & n_1 - k \leq n < n_1. \end{cases}$$

Clearly

$$0 \le c_n \le b_n \quad \text{for } n \ge n_1 - k$$

and in view of (30)

(32) $$c_n > 0 \quad \text{for } n_1 - k \le n < n_1.$$

Now define the mapping T on Λ as follows: For every $\tilde{c} = \{\tilde{c}_n\} \in \lambda$, let the nth term of the sequence $T\tilde{c}$ be

$$\sum_{i=0}^{N} \sum_{j=n}^{\infty} r_j^{(i)} c_{j-k_i}.$$

Then one can see that T is monotone in the sense that if $\tilde{c}^{(1)}, \tilde{c}^{(2)} \in \Lambda$ and $\tilde{c}^{(1)} \le \tilde{c}^{(2)}$ (that is, $\tilde{c}_n^{(1)} \le \tilde{c}_n^{(2)}$ for $n \ge n_1$) then $T\tilde{c}^{(1)} \le T\tilde{c}^{(2)}$. From (29), $Tb \le b$, from which it follows that

$$T\colon \Lambda \to \Lambda.$$

Set

$$\tilde{c}^{(0)} = \{b_n\}_{n=n_1}^{\infty} \quad \text{and} \quad \tilde{c}^{(m)} = T\tilde{c}^{(m-1)} \quad \text{for } m = 1, 2, \ldots.$$

Then one can see by induction that the sequence $\{\tilde{c}(m)\}$ of elements of Λ is such that

$$0 \le \tilde{c}_n^{(m+1)} \le \tilde{c}_n^{(m)} \le b_n \quad \text{for } n \ge n_1.$$

Thus,

$$\tilde{c}_n = \lim_{m\to\infty} \tilde{c}_n^{(m)}, \quad n \ge n_1$$

exists and $\tilde{c} = \{\tilde{c}_n\}_{n=n_1}^{\infty}$ belongs to Λ. Also $T\tilde{c} = \tilde{c}$ and so c is a solution of Eq. (31). It remains to show that

(33) $$c_n > 0 \quad \text{for } n \ge n_1 - k.$$

In view of (32), if (33) were false there would exist some $n_2 \ge n_1$ such that

$$c_{n_2} = 0 \quad \text{and} \quad c_n > 0 \quad \text{for } n_1 - k \le n < n_2.$$

Then from (31)

$$\sum_{i=0}^{N}\sum_{j=n_2}^{\infty} r_j^{(i)} c_{j-k_i} = 0. \tag{34}$$

But by (28) there exists an $i_2 \in \{0, 1, \ldots, N\}$ such that

$$\sum_{j=n_2}^{n_2+k_{i_2}} r_j^{(i_2)} > 0.$$

Hence

$$\sum_{i=0}^{N}\sum_{j=n_2}^{\infty} r_j^{(i)} c_{j-k_i} \geq \sum_{j=n_2}^{n_2+k_{i_2}} r_j^{(i_2)} c_{j-k_{i_2}} > 0$$

which contradicts (34) and completes the proof.

THEOREM 3. *Assume that (12), (13), (15), and (19) hold and that there exists a nonnegative number Q such that*

$$\sum_{j=l+1}^{k} q_{n-j} \geq Q \quad \textit{for } n \textit{ large.} \tag{35}$$

Suppose also that there exists a nonnegative integer N such that every solution of the delay difference equation

$$B_{n+1} - B_n + \sum_{i=0}^{N} (p_n - q_{n+l-k}) Q^i B_{n-k-il} = 0 \tag{36}$$

oscillates. Then every solution of Eq. (1) also oscillates.

Proof. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $\{y_n\}$. By Lemma 4 and Theorem 1 it follows for m sufficiently large the sequence $\{z_n\}$ which is defined by (16) satisfies

$$0 < z_n \leq y_n \tag{37}$$

and

$$z_{n+1} \leq z_n. \tag{38}$$

Also

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n = 0. \tag{39}$$

From (1) and (16) we see that

$$z_{n+1} - z_n + (p_n - q_{n+l-k})z_{n-k} + (p_n - q_{n+k-l}) \sum_{j=l+k+1}^{2k} q_{n+l-j}\, y_{n-j} = 0, \ n \geq 2k$$

and by induction we find that for $m = 1, 2, \ldots, N$,

$$\begin{aligned} &z_{n+1} - z_n + (p_n - q_{n+l-k})z_{n-k} + \\ &+(p_n - q_{n+l-k}) \sum_{i=1}^{m} \left[\sum_{j=l+k+1}^{2k} q_{n+l-j} \sum_{j_1=l+i+1}^{k+j} q_{n+l-j_1} \sum_{j_i=l+j_{i-1}+1}^{k+1_{i-1}} q_{n+l-j_i} z_{n-j_i} \right] \\ &+(p_n - q_{n+l-k}) \sum_{j=l+k+1}^{2k} q_{n+l-j} \sum_{j_1=l+i+1}^{k+j} q_{n+l-j_1} \sum_{j_{m+1}=l+j_m+1}^{k+j_m} q_{n+l-j_{m+1}}\, y_{n-j_{m+1}} \\ &= 0, \qquad n \geq (N+2)k. \end{aligned} \tag{40}$$

In view of (35) and the decreasing character of $\{z_n\}$, by replacing z_{n-j_i} by $z_{n-l-j_{i-1}}$ in the sum we find from (40) that $\{z_n\}$ satisfies the delay difference inequality

$$z_{n+1} - z_n + \sum_{i=1}^{N} (p_n - q_{n+l-k}) Q^i z_{n-k-il} \leq 0. \tag{41}$$

By summing up from n to ∞ both sides of (41) and by using (39) we obtain

$$-z_n + \sum_{i=1}^{N} \sum_{j=n}^{\infty} (p_j - q_{j+l-k}) Q^i z_{j-k-il} \leq 0$$

or

$$\sum_{i=0}^{N}\sum_{j=n}^{\infty}(p_j - q_{j+l-k})Q^i z_{j-k-il} \leq z_n. \tag{42}$$

In view of (12), (37) and (38) it is easy to see that the hypotheses of Lemma 5 are satisfied. Then the equation

$$\sum_{i=0}^{N}\sum_{j=n}^{\infty}(p_j - q_{j+l-k})Q^i B_{j-k-il} = B_n$$

has a positive solution $\{B_n\}$. Clearly $\{B_n\}$ is also a positive solution of the equation

$$B_{n+1} - B_n + \sum_{i=0}^{N}(p_n - q_{n+l-k})Q^i B_{n-k-il} = 0$$

which contradicts the hypothesis and completes the proof.

Remark 2. From the proof of the theorem it is easy to see that under the conditions of Theorem 3 if every solution of the difference equation with constant coefficients

$$B_{n+1} - B_n + \sum_{i=0}^{N}\alpha Q^i B_{n-k-il} = 0$$

oscillates then every solution of Eq. (1) also oscillates.

Now we consider the delay difference equation with constant coefficients

$$y_{n+1} - y_n + py_{n-k} - qy_{n-l} = 0 \tag{43}$$

where

$$p, q \in \mathbb{R}^+ \quad \text{and} \quad k, l \in \mathbb{N}. \tag{44}$$

If $q = 0$ or $k = l$, Eq. (43) reduces to the equation

$$y_{n+1} - y_n + (p - q)y_{n-k} = 0 \tag{45}$$

for which it is known, see [2], that

$$p - q > \frac{k^k}{(k+1)^{k+1}} \quad \text{for } k \geq 1$$

and

$$p \geq 1 \quad \text{for } k = 0$$

is a necessary and sufficient condition for the oscillation of all solutions.

The following result is a corollary of Theorem 3.

COROLLARY 2. *Assume that*

$$p > q \geq 0, \quad k > l \geq 0, \quad q(k-l) \leq 1 \tag{46}$$

and that there exists a nonnegative integer IN *such that every solution of the delay difference equation*

$$B_{n+1} - B_n + \sum_{i=0}^{N} (p-q)q^i(k-l)^i B_{n-k-il} = 0 \tag{47}$$

oscillates. Then every solution of Eq. (43) also oscillates.

The next corollary gives an explicit sufficient condition for the oscillation of all solution of Eq. (43).

COROLLARY 1. *Assume that (46) holds and that either*

$$\sum_{i=0}^{\infty} (p-q)q^i(k-l)^i \frac{(k+il+1)^{k+il+1}}{(k+il)^{k+il}} > 1. \tag{48}$$

or

$$\lim_{m \to \infty} \left[\prod_{i=0}^{m} (p-q)q^i(k-l)^i \right]^{\frac{1}{m+1}} \frac{(K+1)^{K+1}}{K^K} > 1 \tag{49}$$

where

$$K = k + \frac{1}{2}ml.$$

Then every solution of Eq. (43) oscillates.

Proof.. If (48) holds, then there exists a nonnegative integer N such that

$$\sum_{i=0}^{N}(p-q)^i(k-l)^i\frac{(k+il+1)^{k+il+1}}{(k+il)^{k+il}}>1.$$

Also if (49) holds, then there exists a nonnegative integer N such that

$$\left[\prod_{i=0}^{N}(p-q)q^i(k-l)^i\right]^{\frac{1}{N+1}}\frac{(K+1)^{K+1}}{K^K}>1$$

where

$$K=k+\frac{1}{2}ml.$$

Hence by Lemma 1 every solution of Eq. (47) oscillates. Therefore by Corollary 1 every solution of Eq. (43) also oscillates.

Remark 3. In [4] it was shown that if (46) holds and

$$p-q>\frac{k^k}{(k+1)^{k+1}} \tag{50}$$

then every solution Eq. (43) oscillates. Clearly condition (48) is a substantial improvement of (50).

Example 1. The delay difference equation

(51)

$$y_{n+1}-y_n+2\left(\sin^2\frac{n\pi}{6}\right)y_{n-5}-\frac{1}{3}\left(\cos^2\frac{n\pi}{6}\right)y_{n-2}=0,\quad n=0,1,2,\ldots$$

satisfies the conditions (12)-(15) and (24). Therefore by Theorem 2 every solution of Eq. (51) oscillates.

Example 2. The delay difference equation

$$y_{n+1}-y_n+\left(\frac{31}{54}+\frac{2}{n+1}\right)y_{n-2}-\left(\frac{1}{2}+\frac{1}{n+2}\right)y_{n-1}=0 \tag{52}$$

satisfies the hypotheses of Theorem 3. Therefore every solution of Eq. (52) oscillates.

REFERENCES

[1] Erbe L. H., Zhang B. G., *Oscillation of discrete analogues of delay equations,* Proceedings of the International Conference on Theory and Applications of Differential Equations, March 21-25, 1988, Ohio Uneversity, to be published by Marcel Dekker, Inc.

[2] Ladas G., *Oscillations of equations with piecewise constant mixed arguments,* Proceedings of the International conference on Theory and Applications of Differential Equations, March 21-25, 1988, Ohio Uneversity, to be published by Marcel Dekker, Inc.

[3] Ladas G., *Oscillations of difference equations with positive and negative coefficients,* Proceedings of Geoffrey J. Butler Memorial Conference, University of Alberta, Edmonton, June 20-25, 1988.

[4] Ladas G., *Explicit conditions for the oscillation of difference equations,* J. Math. Anal. Appl. **153** (1990), 276-287.

[5] Ladas G., Philos Ch. G., Sficas Y. G., *Sharp conditions for the oscillation for delay difference equations,* Journal of Applied Mathematics **2** (1989), 101-112.

[6] Chuanxi Q., Ladas G., *Oscillation in differential equations with positive and negative coefficients,* Canad. Math. Bull. **33** (1990), 442-450.

Department of Mathematics
The University of Rhode Island
Kingston, R.I 02881-0816, U.S.A.