«LE MATEMATICHE»
Vol. XLIV (1989) - Fasc. II, pp. 345-350

AN EXACT VALUE FOR THE PATH-CHROMATIC INDEX OF A COMPLETE GRAPH

FRANK HARARY (Las Cruces) - ŹSOLT TUZA (Budapest) (*) (**)

We prove that the edge of K_n cannot be partitioned into less than $(n-1)/t$ P_{t+2}-free subgraphs. We show that this inequality is sharp and characterize the edge partitions which attain it. In the process, we point out a surprising connection between combinatorial designs and the conditional chromatic index.

Coloring problems are among the most frequently, studied questions in graph theory. They are often easy to state and difficult to solve. The most general treatment of the subject was proposed in [5], and independently in several other articles. Let $G = (V, E)$ be a graph with node set V and edge set E, and let $\mathcal{P}$ be a property of graphs. The

(*) Entrato in Redazione il 2 ottobre 1990.
(**) Research supported in part by grant N00014-90J1860 from the U.S. Office of Naval Research.

conditional chromatic number with respect to this property, written $\chi(G:\mathcal{P})$, is the smallest number of colors in a node coloring $f: V \to N$ (the natural numbers) such that the subgraph induced by the node set $f^{-1}(i)$ has property $\mathcal{P}$ for every color i. Similarly, the *conditional chromatic index*, $\chi'(G:\mathcal{P})$, is the smallest number of colors in an edge coloring $f': E \to N$ such that the subgraph formed by each edge set $f^{-1}(i)$ has property $\mathcal{P}$. The study of $\chi(G:\mathcal{P})$ and $\chi'(G:\mathcal{P})$ for various properties $\mathcal{P}$ was discussed in [3, 6, 8], and other papers.

In an interesting particular case, $\mathcal{P}$ is defined in terms of forbidden subgraphs. For a fixed forbidden graph F, $\chi(G:-F)$ and $\chi'(G:-F)$ denote the conditional chromatic number and index, respectively, where the property is that no monochromatic subgraph isomorphic to F occurs.

We now consider $F = P_t$ the path on t nodes, following the notation and terminology of [4]. Note that $\chi(G) = \chi(G:-P_2)$ and $\chi'(G):\chi'(G:-P_3)$. The study of $\chi(G:-P_t)$ was initiated in [1]. Here we investigate $\chi'(G:-P_t)$, proving a general lower bound and pointing out a surprising connection between combinatorial designs and the conditional chromatic index.

Recall that a *Steiner System* $S(2,t,n)$ is a collection $\mathcal{B}$ of t-subsets B (called *blocks*) of an n-set X, such that any two distinct elements $x, x' \in X$ are contained in precisely one block $B \in \mathcal{B}$. In a *resolvable system*, there is a partition $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_q$ into «parallel classes» with $B \cap B' = \emptyset$ for $B, B' \in \mathcal{B}_i$ and $\cup_{B\in\mathcal{B}_i} B = X$ for each i $(1 \le i \le q)$.

THEOREM 1. *For every grapgh G with n nodes and m edges, and every positive integer $t \le n-2$,*

$$\chi'(G:-P_{t+2}) \ge 2m/nt.$$

In particular,

$$\chi'(K_n:-P_{t+2}) \ge (n-1)/t$$

with equality if and only if there exists a resolvable Steiner System $S(2, t+1, n)$.

Proof. Let us begin with the upper bound. Assume that $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_q$ is a resolution of $S(2, t+1, n)$. Identify its point set X with the node set V of K_n. Then each pair of nodes $u, v \in V$ is contained in just one block $B \in \mathcal{B}$, and B belongs to a unique parallel class. Hence, we can define an edge coloring $f': E \to \{1, \ldots, q\}$ by the rule $f'(uv) = i$ if and only if edge $uv \in B$ for some $B \in \mathcal{B}_i$.

Since each parallel class $\mathcal{B}_i$ consists of pairwise disjoint blocks, the monochromatic connected subgraphs in f' have precisely $t+1$ nodes each. Thus, f' is P_{t+2}-free, and this fact implies that $\chi'(K_n : -P_{t+2}) \leq q$. To determine the exact value of q, note that each $\mathcal{B}_i$ consists of $n/(t+1)$ blocks (since those blocks are mutually disjoint), and each block covers exactly $t(t+1)/2$ pairs of nodes. Thus, the number of edges of color i is equal to $nt(t+1)/(2t+2) = nt/2$. If the $n(n-1)/2$ edges of K_n are partitioned into q color classes, then $n(n-1)/2 = qnt/2$ must hold, implying $\chi'(K_n : -P_{t+2}) \leq q = (n-1)/t$.

We deduce the lower bound from a more general inequality as follows. Let F be an arbitrary fixed graph and n a natural number. The *Turán number*, $T(n, F)$, has been defined as the largest integer h for which there is a graph with n nodes and h edges that does not contain a subgraph isomorphic to F. To complete the proof we require the following inequality.

LEMMA 1. *For any two graphs F and G,*

$$\chi'(G : -F) \geq m/T(n, F).$$

Proof. By the definition of $T(n, F)$, every F-free subgraph of G has at most $T(n, F)$ edges. Thus, an edge-coloring without any monochromatic subgraph isomorphic to F must use at least $m/T(n, F)$ colors.

To continue the proof Theorem 1, we recall the following important result of Erdös and Gallai [2].

LEMMA 2. *For $n > t \geq 1$, the Turán number of a path is bounded* *by* $T(n, P_{t+2}) \leq nt/2$. *Moreover, equality holds if and only if* $n = k(t+1)$ *for some integer* $k \geq 1$, *and in this case the unique extremal graph is* kK_{t+1}, *the node-disjoint union of* k *copies of* K_{t+1}.

Putting $F = P_{t+2}$ and substituting $nt/2$ into Lemma 1, as the upper bound for $T(n, F)$, we obtain the desired inequality $\chi'(G{:}\, P_{t+2}) \geq 2m/nt$. When $G = K_n$, we have $m = n(n-1)/2$ implying $\chi'(K_n{:} -P_{t+2}) \geq (n-1)/t$.

If $\chi'(K_n{:} -P_{t+2}) = (n-1)/t$, then equality holds throughout the above computation. In particular, $T(n, P_{t+2}) = nt/2$, and in the P_{t+2}-free edge-coloring f' of K_n each color class is isomorphic to $n/(t+1)K_{t+1}$. Let a set B of nodes be a block if and only if it induces a monochromatic component in the edge-coloring f' of K_n. By Lemma 2, each bloch has size $t+1$ and each pair of nodes is contained in precisely one block. Hence, a Steiner System $S(2, t+1, n)$ is obtained. Moreover, the set $\mathcal{B}_i$ of blocks belonging to edges of color i consists of $n/(t+1)$ pairwise disjoint blocks for each i, $1 \leq i \leq q = (n-1)/t$. Thus, $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_q$ is a resolution of $S(2, t+1, n)$.

We conclude with three classic examples of combinatorial structures, showing that Theorem 1 is sharp for infinitely many values of n and t.

Example 1. $n = 6k+3 \geq 9$, $t = 2$.

For every such n there exists a resolvable Steiner Triple System $\mathcal{B}$ of order n. The number of its blocks is $n(n-1)/6 = (2k+1)(3k+1)$, and the size of each parallel class is $n/3 = 2k+1$. Hence, $\mathcal{B}$ provides a P_4-free edge coloring of K_n with $(n-1)/2 = 3k+1$ colors.

Example 2. $n = q^2$, $t = q-1$, q a prime power.

For every such n there exists a finite affine plane $AG(2, q)$ of order q on q^2 points. The number of its lines (viewed as q-element blocks) is $q(q+1)$, and each parallel class consists of q lines. Hence, $AG(q)$ provides a P_{q+1}-free edge coloring of K_n with $(n-1)/(q-1) = q+1$ colors.

As generalization, we obtain

Example 3. $n = q^d$, $t = q - 1$, q a prime power, $d \geq 2$.

For every such n and d there exists a d-dimensional finite affine space $AG(d, q)$ of order q on q^d points. The number of its lines is $q(q+1)$, and each parallel class consists of $(n^2 - n)/(q^2 - q)$ lines. Hence, a P_{q+1}-free edge coloring of K_n with $(n-1)/(q-1) = q^{d-1} + q^{d-2} + \cdots + 1$ colors is obtained.

Fig. 1 displays the isomorphic factorization [7] of three disjoint triangles into K_9, written $3K_3|K_9$. This edge partition corresponds to the (unique) resolvable Steiner System $S(2,3,9)$, isomorphic to the affine plane $AG(2,3)$.

Fig.1 - Monochromatic edge classes in the unique P_4-free 4-edge-coloring of K_9.

REFERENCES

[1] Chartrand G., Geller D., Hedemiemi S., *A generalization of the chromatic number,* Proc. Cambridge Philos. Soc. **64** (1968) 265-271.

[2] Erdös P., Gallai T., *On maximal paths and circuits of graphs,* Acta Mat. Acad. Sci. Hungar. **10** (1959) 337-356.

[3] Faudree R. J., Gyárfás A., Schelp R. H., Tuza Zs., *The strong chromatic index of graphs,* Ars Combinatoria **29B** (1990) 205-211.

[4] Harary F., *Graph Theory,* Addison-wesley, Reading (1969).

[5] Harary F., *Conditional colorability in graphs,* in Graphs and Applications (Harary F., Maybee J. S., Eds), Proc. Boulder,. Colorado 1982, Wiley New York (1985) 127-136.

[6] Harary F., Jones K. F., *Conditional colorability II: Bipartite variations,* Congressus Numerantium **50** (1985) 205-218.

[7] Harary F., Robinson R., Warmald N., *Isomorphic factorization of graphs*

I: Complete graphs, Trans. Amer. Math. Soc. **242** (1978) 243-260.

[8] Truszczński M., Tuza Zs., *Rado's Selection Principe: Applications to binary relations, graph and hypergraph colorings and partially ordered sets,* Discrete Math. (to appear).

Frank Harary
Department of Computer Science
New Mexico State University
Las Cruces, NM 88003, USA

Zsolt Tuza
Computer and Automation Institute
Hungarian Academy of Sciences
H-1502 Budapest, Hungary