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COMPOSITION OPERATORS ON SUMMABLE FUNCTIONS SPACES

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A *composition operator* is a linear operator C_T on a subspace of $\mathbb{K}^X$ by a point trasformation T on a set X (where $\mathbb{K}$ denotes the scalar field) by the formula

$$C_T f(x) := f \circ T(x).$$

We give some necessary and/or sufficient conditions under which the map T induces a continuous composition operator on suitable topological subspaces of summable functions of $\mathbb{K}^X$ as L^p or $W^{1,p}$.

0. Introduction.

Let (X, S, m) be a measure space; We assume X is a standard Borel space, i.e. X is isomorphic to a Borel subset of a complete separable metric space and S is the σ-algebra of Borel subsets of X. The measure m will be finite or σ-finite.

Let $Y \subseteq X$ be a measurable subset of X and let T be a measurable trasformation of Y into X, i.e. T is function $T : Y \rightarrow X$

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such that

$$T^{-1}(E) \in S \quad \text{whenever} \quad E \in S.$$

We define the linear operator C_T on the space of the measurable complex valued functions on X by

(1) $$C_T f(x) : \begin{cases} f(T(x)) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

C_T is said *composition operator.*

Remark 1. We observe that $C_T(f) = 0$ a.e. whenever $f = 0$ a.e. if and only if $mT^{-1} \ll m$ (i.e. the measure mT^{-1} is absolutely continuous with respect to m). ■

We can ask the following question:

Question: «Under what circumstance does C_T induce a bounded operator on $L^p(m)$?».

The complete answer is in the following two theorems:

THEOREM 1. ([1]). *Necessary and sufficient condition for a measurable trasformation T to induce a bounded operator C_T on $L^p(m)$ with $1 \leq p < \infty$, defined by (1), are:*

(i) $mT^{-1} \ll m$;

(ii) the Radon-Nikodym derivative dmT^{-1}/dm of mT^{-1} with respect to m is bounded.

In this case

$$||C_T||_{L^p} = ||dmT^{-1}/dm||_{\infty}^{1/p}.$$ ■

Remark 2. We underline that if $mT^{-1} \ll m$, by using the trasformation of integral formula

$$\int_X |f \circ T|^p dm = \int_X |f|^p (dmT^{-1}/dm) dm$$

([1], pag. 38), one can deduce that the invariance of $L^p(m)$ with respect to C_T implies the boundedness of C_T, i.e.

$$C_T(L^p(m) \subseteq L^p(m)$$

implies that there exists $M > 0$ such that

$$||C_T f||_{L^p} \leq M||f||_{L^p}.$$

From on, thus, we use indifferently

«T induces an operator C_T on $L^p(m)$»

or

«T induces a bounded operator C_T on $L^p(m)$». ∎

THEOREM 2. ([2]). *Are equivalent:*

(a) T induces the bounded operator C_T on $L^\infty(m)$.

(b) $mT^{-1} \ll m$.

(c) T induces the bounded operator C_T on $L^\infty(m)$ and $||C_T||_{L^\infty} \leq 1$ ∎

In the section 2 of the present paper we give some partial answers to the *Question* above with $L^p(m)$ substituted by the Sobolev space $W^{1,p}$, whereas in the section 1 we obtain some little but nice results concerning:

(a) the existence of an element in the point spectrum of C_T;

(b) a necessary and sufficient condition for $||C_T||_{L^p} \leq 1$;

(c) the injectivity of C_T.

1. Composition operators on $L^p(m)$.

Let (X, S, m), Y, T be as in the introduction and $\mathbb{K} = \mathbb{C}$. In this section we assume that T induces the composition operator C_T on $L^p(m)$.

(a) *Existence of eigenvalues.*

The following propositions give two conditions assuring that a complex number $\lambda \in \mathbb{C}$ belongs to point spectrum of C_T on $L^p(m)$ with $1 \leq p < \infty$ and $L^\infty(m)$ respectively.

PROPOSITION 1. *We assume that there exists a subset E of Y with $m(E) > 0$ such that if $E_0 := E$, $E_{n+1} := T(E_n)$ and $\tilde{E} := \bigcup_{n \in \mathbb{N}} E_n$, we have:*

(i) $T(C\tilde{E}) \subseteq C\tilde{E}$;

(ii) $m(E_h \cap E_k) = 0$ *whenever* $h \neq k$;

(iii) $\sum_{n \in \mathbb{N}} |\lambda^p|^n m(E_n) < \infty, \lambda \neq 0.$

Then λ belongs to point spectrum of C_T on $L^p(m)$ with $1 \leq p < \infty$.

Proof. If we define $f_\lambda : X \to \mathbb{C}$ by

$$f_\lambda(x) := \begin{cases} 0 & \text{if } x \notin \tilde{E} \\ \lambda^k & \text{if } x \in E_k \end{cases} \tag{2}$$

then we have, (from (ii)), that f_λ is well defined; moreover it is easy to verify, using (i), that $C_T f_\lambda = \lambda f_\lambda$. It remains to show that $f_\lambda \in L^p(m)$. For this, we have

$$\int_X |f_\lambda|^p dm = \int_{\tilde{E}} |f_\lambda|^p dm = \sum_{n \in \mathbb{N}} \int_{E_n} |f_\lambda|^p dm = \sum_{n \in \mathbb{N}} \int_{E_n} |\lambda^n|^p dm =$$
$$\sum_{n \in \mathbb{N}} |\lambda^p|^n m(E_n) < \text{ (from (iii))} < \infty \qquad \blacksquare$$

PROPOSITION 2. *Let T be such that induces composition operator C_T on $L^\infty(m)$. We assume that there exists a subset E of Y with $m(E) > 0$ such that if $E_0 := E$, $E_{n+1} := T(E_n)$ and $\tilde{E} := \bigcup_{n \in \mathbb{N}} E_n$, we have*

(i) $T(C\tilde{E}) \subseteq C\tilde{E}$;

(ii) $m(E_k \cap E_h) = 0$ *whenever* $h \neq k$.

Then the spectrum $\pi(C_T)$ *of* C_T *is the complex unit ball* $B_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Proof. From Theorem 2(c) we have $\pi(C_T) \subseteq B_1$. Since $\pi(C_T) \supseteq \overline{\pi_0(C_T)}$, where $\overline{\pi_0(C_T)}$ denotes the closure of the point spectrum $\pi_0(C_T)$ of C_T, it is enough to show that $\pi_0(C_T) \supseteq \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$. For this, let $0 < |\lambda| < 1$. Then, if we define f_λ as in (2), we have $C_T f_\lambda = \lambda f_\lambda$ and $f_\lambda \in L^\infty(m)$ ■

EXAMPLE 1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^+$, S =Borel σ-algebra, m-Lebesgue measure. If $t \in \mathbb{R}^+$ is fixed, we define $T : Y \to X$ by

$$T(x) := x + t.$$

Then $mT^{-1} \ll m$ and

$$(dmT^{-1}/dm)(x) = \begin{cases} 1 & \text{if } x \geq t \\ 0 & \text{if } x < t. \end{cases}$$

It follows that $||C_T||_{L^p} = 1$ for every $1 \leq p < \infty$ (on the other hand, $||C_T||_{L^\infty} = 1$ also).

The hypotheses (i) and (ii) of Propositions 1 and 2 are verified taking, for instance, $E := [t/2, t]$.

From $m(E_n) = t/2$ for each $n \in \mathbb{N}$, it follows that (iii) of Proposition 1 is satisfied for every λ with $0 < |\lambda| < 1$, in such a way that $\pi_0(C_T) \supseteq \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$. On the other hand, it is straighforward to verify that $\pi_0(C_T) = \{\lambda : |\lambda| < 1\}$ on $L^p(m)$ with $1 \leq p < \infty$ and $\pi_0(C_T) = B_1$ on $L^\infty(m)$, and so $\pi(C_T) = B_1$ on $L^p(m)$ with $1 \leq p \leq \infty$.■

EXAMPLE 2. Let $X = \mathbb{R} = Y$, S=Borel σ-algebra, m-Lebesgue measure. If $t \in \mathbb{R}^+$ is fixed, we define $T : X \to X$ by

$$T(x) := \begin{cases} x + t & \text{if } x > 0 \\ x - t & \text{if } x \leq 0. \end{cases}$$

Then $||C_T||_{L^p} = 1$ for $1 \leq p \leq \infty$ and one can use the Propositions 1 and 2 to see that $\pi_0(C_T) = \{\lambda : |\lambda| < 1\}$ on $L^p(m), 1 \leq p < \infty$, $\pi_0(C_T) = B_1$ on $L^\infty(m)$ and $\pi(C_T) = B_1$ for $1 \leq p \leq \infty$. ∎

(b) C_T *non-expansive.*

We recall that a map $f : N_1 \to N_2$, N_1, N_2 normed spaces, is said *non-expansive* if $||f(x_1) - f(x_2)|| \leq ||x_1 - x_2||$, $\forall x_1, x_2 \in N_1$. Of course a composition operator C_T is non-expansive iff $||C_T|| \leq 1$ and Theorem 2 affirm that any composition operatôr in $L^\infty(m)$ is non-expansive. The following proposition characterizes the maps T for which C_T is non-expansive on $L^p(m)$, $1 \leq p < \infty$. First, we need a definition.

With suggestive words, we can say that the map $T : Y \to X$ is *m-expansive* if $m(E) \leq m(T(E))$ for each $E \subseteq Y$, $E \in S$.

PROPOSITION 3. *Let* $1 \leq p < \infty$. *Are equivalent:*

(i) $||C_T||_{L^p} \leq 1$;

(ii) C_T *is non-expansive on* $L^p(m)$;

(iii) T *is m-expansive*

(iv) $mT^{-1} \leq m$, *i.e.* $m(T^{-1}(E)) \leq m(E)$ *for each* $E \in S$.

Proof. We prove only (i) iff (iv).

Let $||C_T||_{L^p} \leq 1$ and $E \subset X$, $E \in S$. If $m(E) = \infty$, we are done. If $m(E) < \infty$, then

$$mT^{-1}(E) = \int_X \chi_{T^{-1}(E)} dm = \int_X |C_T \chi_E|^p dm$$

$$= ||C_T \chi_E||^p_{L^p} \leq ||C_T||^p_{L^p} ||\chi_E||^p_{L^p} \leq ||\chi_E||^p_{L^p} = m(E).$$

Let now $mT^{-1} \leq m$. We must to prove that

$$(*) \qquad ||C_T f||_{L^p} \leq ||f||_{L^p} \text{ for each } f \in L^p(m).$$

If $f = \chi_E$, $m(E) < \infty$, then

$$||C_T \chi_E||^p_{L^p} = \int_X |C_T \chi_E|^p dm = \int_X \chi_{T^{-1}(E)} dm = mT^{-1}(E) \leq m(E) = ||\chi_E||^p_{L^p}.$$

At this point, the general case follows by standard arguments.■

(c) C_T *injective.*

We say that $T : Y \to X$ is essentially surjective if $m(X \backslash T(Y)) = 0$.

PROPOSITION 4. *C_T is injective iff T is essentially surjective.*

Proof. Sufficience. If $X = T(Y) \cup A$, where $m(A) = 0$, then $\ker C_T = \{f \in L^p(m) : f_{/T(Y)} = 0\} = \{0\}$.

Necessity. Let

$X_0 := \{x \in X : (dmT^{-1}/dm)(x) = 0\}, L^p(X_0) := \{f \in L^p(m) : f(x) = 0$ a.e. on $X \backslash X_0\}$ and $A(f) := \{x \in X : f(x) \neq 0\}$ for each $f \in L^p(m)$.

Step 1. $L^p(X_0) = \{f \in L^p(m) : (dmT^{-1}/dm)_{/A(f)} = 0 \text{ a.e.}\}$.

Indeed,

$$L^p(X_0) = \{f \in L^p(m) : f(x) = 0 \text{ a.e. on } X \backslash X_0\} =$$

$$= \{f \in L^p(m) : A(f) \subseteq X_0 \text{ a.e.}\} =$$

$$= \{f \in L^p(m) : (dmT^{-1}/dm)_{/A(f)} = 0 \text{ a.e.}\}.$$

Step 2. $L^p(X_0) = \ker C_T$.

Let $f \in L^p(X_0)$. Then $\int_X |f|^p (dmT^{-1}/dm) dm = 0$ from step 1, and so

$$\int_X |f \circ T|^p dm = \int_X |f|^p \circ T dm = \int_X |f|^p dmT^{-1} =$$

$$= \int_X |f|^p (dmT^{-1}/dm) dm = 0,$$

concluding that $C_T = f \circ T = 0$ a.e., i.e. $f \in \ker C_T$.

Viceversa, let $f \in \ker C_T$. Then $f \circ T = 0$ a.e., so that $0 = |f \circ T|^p = |f|^p \circ T$, hence $0 = \int_X |f|^p \circ T dm = \int_X |f|^p (dmT^{-1}/dm) dm$, from which $(dmT^{-1}/dm)_{/A(f)} = 0$ a.e. concluding that $f \in L^p(X_0)$ (step 1).

Step 3. T essentially surjective.

If C_T is injective, then $L^p(X_0) = \{0\}$ (step 2), i.e. $f = 0$ a.e. if $(dmT^{-1}/dm)_{/A(f)} = 0$ a.e.. If follows that $dmT^{-1}/dm \neq 0$ a.e., i.e. $m(X_0) = 0$.

At this point it is sufficient to prove that $X \backslash X_0 = T(Y)$. Indeed, $X \backslash X_0 = A(dmT^{-1} \backslash dm)$ is contained in $T(Y)$, since $E \subseteq X \backslash T(Y)$ implies $0 = mT^{-1}(E) = \int_E (dmT^{-1}/dm)dm$ and so $(dmT^{-1}/dm)_{/E} = 0$.■

2. Composition operators on $W^{1,p}(I)$.

In this last section we label:

– $I =]a,b[$ bounded interval of $\mathbf{R}$.

– m Lebesgue measure on I.

– $T : I \to I$ Borel trasformation such that $mT^{-1} \ll m$ (by Remark 1, §0, this last condition assures that $C_T f$ depends only on the equivalence class of f).

– $W^{1,p}(I) := \{f \in L^p(I) : \exists \tilde{f} \in L^p(I)$ such that $\int_I fg' dm = -\int_I \tilde{f} g dm$ for each C^∞-function g with support in $I\}$.

For $f \in W^{1,p}(I)$ we write $\tilde{f} = f'$. The Sobolev space $W^{1,p}$ endowed with the norm $||f||_{W^{1,p}} := ||f||_{L^p} + ||f'||_{L^p}$ is a Banach space for $1 \leq p \leq \infty$.

– $W_0^{1,p}(I) := \{f \in W^{1,p}(I) : f(a) = f(b) = 0\}$.

Remark 1. It is well known (for instance see [3] Theorem VIII. 2) that if $f \in W^{1,p}(I)$, then there exists a unique function $\bar{f} \in C[a,b]$ such that $f = \bar{f}$ a.e. on I; In this section with «$f \in W^{1,p}(I)$» we understand that f belongs to $C[a,b]$; this clarifies the above definition of $W_0^{1,p}(I)$. ■

Our purpose here is to give a kind of answer to question: «Under what circumstances does T induce bounded composition operator on

$W^{1,p}(I)$?».

The obtained results are very partial, but they are the first (that we know!) in the literature on composition operators.

We begin observing that if T induces bounded C_T on $L^p(I)$, then T not necessarily induces bounded C_T on $W^{1,p}(I)$ and viceversa.

Indeed, if $I =]-1,1[$ and $T : I \to I$ is defined by

$$T(x) := \begin{cases} -x & \text{if } x \leq 0 \\ x-1 & \text{if } x > 0, \end{cases}$$

then $mT^{-1} \ll m$ and $dmT^{-1}/dm = 1$, so that T induces bounded C_T on $L^p(I)$ for $1 \leq p \leq \infty$, whereas T does not induce C_T on $W^{1,p}(I)$: in fact, as will see, if T induces C_T on $W^{1,p}(I)$, then $T| \in W^{1,p}(I)$, while this is not the case here (see Remark 1).

Viceversa, let $I =]0,1[$ and $T : I \to I$ defined by $T(x) := x^2$. Then T does not induce C_T on $L^1(I)$. Neverthless T induces bounded C_T on $W^{1,1}(I)$ as one can see using Proposition 5 below.

Remark 2. Let $1 \leq p \leq \infty$. Assume that $C_T(W^{1,p}(I)) \subseteq W^{1,p}(I)$. Then T belongs to $W^{1,p}(I)$. This result is peculiar of boundedness of I: if I is unbounded and T is the identity fucntion on I, then T induces the identity operator on $W^{1,p}(I)$, but $T \notin W^{1,p}(I)$. ■

PROPOSITION 1. *Let* $1 \leq p_0 < \infty$. *We suppose that* T *induces bounded* C_T *on* $L^{p_0}(I)$. *Moreover we assume that* $T \in W^{1,\infty}(I)$. *Then* T *induces bounded* C_T *on* $W^{1,p}(I)$ *for any* $1 \leq p < \infty$.

Proof. We observe first of all, that from hypothesis that T induces C_T on $L^{p_0}(I)$, it follows that T induces bounded C_T on $L^p(I)$ for any $1 \leq p < \infty$ (Theorem 1, §0). Now, let $f \in W^{1,p}(I)$. Then from $W^{1,p}(I) \subseteq L^p(I)$ and $T \in W^{1,\infty}(I) \subseteq W^{1,p}(I)$, we have that the functions $f \circ T$ and $f' \circ T$ both belong to $L^p(I)$. Moreover, also $(f' \circ T)T'$ belongs to $L^p(I)$, since $\int_I |(f' \circ T)T'|^p dm \leq ||T'||^p_{L^\infty} ||f' \circ T||^p_{L^p} < \infty$. Finally, from the fact that $f \circ T \in W^{1,p}(I)$ for each $f \in W^{1,p}(I)$ and $(f \circ T)' = (f' \circ T)T'$ ([3], Theorem VIII. 6 and Corollary VIII.10), the claim follows. ■

For $p=\infty$, we obtain the stronger result:

PROPOSITION 2. *T induces bounded C_T on $W^{1,\infty}(I)$ if and only if $T \in W^{1,\infty}(I)$.*

Proof. Let $f \in W^{1,\infty}(I)$. Then $f \circ T$ belongs to $L^\infty(I)$. In order to prove that $f \circ T \in W^{1,\infty}(I)$, it is sufficient to show that there exists a constant $c > 0$ such that

$$|(f \circ T)(x) - (f \circ T)(y)| \leq c|x-y| \quad \text{a.e. on} \quad I,$$

and this is true since f and T both belong to $W^{1,\infty}(I)$.

At this point the proof proceeds as the proof of Proposition 1 using Theorem 2, §0. ■

Remark 3. In the hypotheses of Proposition 1 one has

$$||C_T||_{W^{1,p}} \leq ||C_T||_{L^p} \max(1, ||T'||_{L^\infty}).$$

Such inequality remains true also for $p=\infty$.

The following proposition shows that the assumption of Proposition 1 is also necessary in order T to induce bounded C_T on $W^{1,p}(I)$ if T induces *surjective* C_T on $L^p(I)$. ■

PROPOSITION 3. *Let $1 \leq p \leq \infty$. We suppose that T induces bounded surjective C_T on $L^p(I)$. Then T induces bounded C_T on $W^{1,p}(I)$ iff $\in W^{1,\infty}(I)$.*

Proof. By Proposition 2, it is sufficient to consider the case $1 \leq p < \infty$.

Sufficience. It is proved in Proposition 1.

Necessity. We assume that T induces bounded C_T on $W^{1,p}(I)$. Then by Remark 2 we have $T \in W^{1,p}(I)$. It remains to show that $T' \in L^\infty$. Now, $(f \circ T)' = (f' \circ T)T'$ belongs to $L^p(I)$ for every $f \in W^{1,p}(I)$; on the other hand, from Theorem VIII.2 of [3], we have that for every $h \in L^p(I)$ there exists $f \in W^{1,p}(I)$ such that $f' = h$. It follows that for every $h \in L^p(I)$ we have that $(h \circ T)T'$ belongs to $L^p(I)$ also.

From surjectivity of C_T on $L^p(I)$ one has that gT' belongs to $L^p(I)$ for every $g \in L^p(I)$ and this is sufficient to conclude $T' \in L^\infty(I)$ (see [4], Theorem 20.15 for the case $p = 1$ and its suitable modifications for $p > 1$). ■

In the previous Propositions 1 and 3 we have supposed that T induces C_T on $L^p(I)$. The next two Propositions use not such hypothesis.

First, we give a remark.

Remark 4. Let $T : I \to I$ be absolutely continuous and strictly monotone. Then $f \circ T$ is absolutely continuous for each f absolutely continuous (routine calculation). ■

PROPOSITION 4. *We assume that:*

(i) $T \in W^{1,\infty}(I)$;

(ii) T *strictly monotone.*

Then T *induces bounded* C_T *on* $W^{1,p}(I)$ *for every* $1 < p \leq \infty$.

Proof. Let T strictly increasing (analogously we proceed if T strictly decreasing). We can assume $1 < p < \infty$, since for $p = \infty$ the claim is contained in Proposition 2.

First, we observe that (i) and (ii) imply

(2) $$(T'(x))^p \leq T'(x)||T'||_{L^\infty}^{p-1}.$$

As in the proof of Proposition 1 we have $C_T(W^{1,p}(I)) \subseteq W^{1,p}(I)$.

For the boundedness, we use the fact that for each $t_0 \in I$ fixed, we have that

$$||f||_{t_0} := |f(t_0)| + ||f'||_{L^p}$$

is a norm on $W^{1,p}(I)$ equivalent to usual norm $|| \, ||_{W^{1,p}}$ and the equivalence is independent from t_0, i.e. there exist two positive constants H, K such that for every $t_0 \in I$ one has

$$H|| \, ||_{t_0} \leq || \, ||_{W^{1,p}} \leq K|| \, ||_{t_0}$$

(this is a routine calculation). Hence

$$||C_T f||_{W^{1,p}} = ||f \circ T||_{W^{1,p}} \leq K(|f(T(t_0))| + ||(f' \circ T)T'||_{L^p}) \leq$$

$$\leq K(|f(T(t_0))| + ||T'||_{L^\infty}^{(p-1)/p} ||f'||_{L^p}).$$

So, if $||T'||_{L^\infty} > 1$, it results

$$||C_T f||_{W^{1,p}} \leq K||T'||_{L^\infty}^{(p-1)/p} (|f(T(t_0))| + ||f'||_{L^p}) \leq$$

$$\leq (K/H)||T'||_{L^\infty}^{(p-1)/p} ||f||_{W^{1,p}},$$

whereas, if $||T'||_{L^\infty} \leq 1$, then

$$||C_T f||_{W^{1,p}} \leq (K/H)||f||_{W^{1,p}}$$

in such a way that $||C_T||_{W^{1,p}} \leq (K/H)\max(1, ||T'||_{L^\infty}^{(p-1)/p})$ ■

For $p = 1$ the hypotheses of Proposition 4 can be weakened.

PROPOSITION 5. *We assume* $T \in W^{1,1}(I)$ *and* T *strictly monotone. Then* T *induces bounded* C_T *on* $W^{1,1}(I)$.

Proof. The fact that $f \circ T \in W^{1,1}(I)$ whenever $f \in W^{1,1}(I)$ is the content of Remark 4. We show that C_T is bounded, following the broad outlines of proof Proposition 4:

$$||C_T f||_{W^{1,1}} \leq K(|f(T(t_0))| + \int_I |(f' \circ T)T'| dm) = \text{(Theorem 3 of [6])} =$$

$$= K(|f(T(t_0))| + \int_{T(I)} |f'| dm) \leq K(|f(T(t_0))| + ||f'||_{L^1}) \leq$$

$$\leq (K/H)||f||_{W^{1,1}}$$ ■

The following is the main result of this section.

THEOREM 6. *Let* $1 < p \leq \infty$. *We assume:*

(i) T *induces bounded* C_T *on* $W^{1,p}(I)$;

(ii) $T' > 0$.

Then $T \in W^{1,\infty}(I)$.

Proof. We can assume $1 < p < \infty$ since for $p = \infty$ the claim is contained in Proposition 2.

To prove $T \in W^{1,\infty}(I)$ it is enough to show $T' \in L^\infty$, because we know already that $T \in L^\infty$ (see [3], Theorem VIII.7).

Step 1. There exists a constant $c > 0$ such that

$$||(f \circ T)'||_{L^p} \leq c||f'||_{L^p}. \tag{3}$$

First of all,

$$\begin{aligned} ||(f \circ T)'||_{L^p} &\leq ||C_T f||_{W^{1,p}} \leq ||C_T||_{W^{1,p}}||f||_{W^{1,p}} \leq \\ &\leq ||C_T||_{W^{1,p}} K||f||_{t_0} = ||C_T||_{W^{1,p}} K(|f(T_0)| + ||f'||_{L^p}) \end{aligned} \tag{4}$$

where t_0 is a fixed point in I.

We begin proving that (3) holds if $\int_I f dm = 0$.

Indeed, let $f \in W^{1,p}(I)$ be a continuous function such that $\int_I f dm = 0$. Then for the Media Theorem there exists $t_0 \in I$ such that $f(t_0) = 0$ and so (3) follows by (4). Let now $f \in W^{1,p}(I)$ be a continuous function. Then there exists a constant A and a continuous function g such that $\int_I g dm = 0$ and $f = g + A$ ($A = \int_I f dm$ and $g = f - \int_I f dm$). Then

$$f' = g' \quad \text{and} \quad (f \circ T)' = (g \circ T)'$$

and so (3) follows by (4).

Step 2. $T' \in L^\infty(I)$.

We write (3) as

$$\int_I |(f' \circ T)T'|^p dm \leq c^p \int_I |f'|^p dm. \tag{5}$$

Now, for every $g \in L^p(I)$ there exists $f \in W^{1,p}(I)$ such that $f' = g$ ([3], Lemma VIII.2); from this and (5) we obtain that for every $g \in L^p(I)$,

$$\int_I |(g \circ T)T'|^p dm \leq c^p \int_I |g|^p dm. \tag{6}$$

For any measurable set $A \subset T(I)$ the characteristic function χ_A belongs to $L^p(I)$ so that (6) gives

$$\int_I |(\chi_A \circ T)T'|^p dm \leq c^p \int_I |\chi_A|^p dm,$$

i.e.

$$\int_{T^{-1}(A)} |T'|^p dm \leq c^p m(A). \tag{7}$$

Now, from the hypothesis $T' > 0$ and from the fact that $T(x) - T(y) = \int_y^x T' dm$, $x, y \in I$ ([3], theorem VIII.2), we have that T is strictly increasing, so we write

$$T'^p = T'^{p-1} T' = (T' \circ T^{-1} \circ T)^{p-1} T'.$$

We note that T absolutely continuous and strictly increasing implies that T^{-1} is absolutely continuous also and

$$(T^{-1})' = 1/(T' \circ T^{-1}),$$

so that from [6], Corollary 6, it follows that if we define $f := \chi_{T^{-1}(A)} |T'|^p$, then $(f \circ T^{-1}) 1/(T' \circ T^{-1})$ is a summable function and the change of variables formula holds:

$$\int_{T(I)} (f \circ T^{-1}) 1/(T' \circ T^{-1}) dm = \int_I f dm.$$

Then

$$\int_A T'^{p-1} \circ T^{-1} dm = \int_A (T'^{p-1} \circ T^{-1}) T' \circ T^{-1} 1/(T' \circ T^{-1}) dm =$$

$$\int_A (T'^p \circ T^{-1}) 1/(T' \circ T^{-1}) dm =$$

$$= \int_{T(I)} (f \circ T^{-1}) 1/(T' \circ T^{-1}) dm = \int_I f dm =$$

$$= \int_{T^{-1}(A)} T'^p dm \leq c^p m(A)$$

i.e.

$$\int_A T'^{p-1} \circ T^{-1} dm \leq c^p m(A). \tag{8}$$

At this point we recall that if $\bar{f}$ is a summable function such that $\int_A \bar{f} dm \leq L m(A)$ for each measurable set $A \subseteq T(I)$, then $\bar{f} \leq L$ a.e. on $T(I)$. From (8) we obtain thus

$$|T' \circ T^{-1}|^{p-1} \leq c^p \quad \text{a.e. on} \quad T(I),$$

from which

$$T' \circ T^{-1} \leq c^{p/(p-1)} \quad \text{a.e. on} \quad T(I),$$

i.e. there exists a Borel set $N \subseteq T(I)$ such that $m(N) = 0$ and

$$T' \circ T^{-1}(y) \leq c^{p/(p-1)} \quad \text{if} \quad y \notin N,$$

i.e.

$$T'(x) \leq c^{p/(p-1)} \quad \text{if} \quad x \in I \backslash T^{-1}(N).$$

On the other hand $m(T^{-1}(N)) = 0$ since $m(N) = 0$ and T^{-1} is absolutely continuous; so finally we conclude $T' \in L^\infty(I)$ ■

Remark 5. The claim of previous theorem is not true if $p = 1$. Indeed, if $I =]0, 1[$, then the transformation $T(x) = x^{\frac{1}{2}}$ induces bounded composition operator on $W^{1,1}(I)$ because T is absolutely continuous and strictly increasing (Proposition 5), neverthless $T' \notin L^{\infty}(I)$. Moreover for each $1 < p < \infty$, if $f(x) := x^{2(p-1)/p}$, then $f \in W^{1,p}(I)$, but $(f \circ T)(x) = x^{(p-1)/p} \notin W^{1,p}(I)$, so that T induces not C_T on $W^{1,p}(I)$ (of course this follows from Theorem 6 also for the case $p = \infty$). ■

We give at last a necessary and sufficient condition under which $W_0^{1,p}(I)$ is invariant for C_T.

PROPOSITION 7. *Let T be to induce bounded C_T on $W^{1,p}(I)$, $1 \leq p < \infty$. Then*

$$C_T(W_0^{1,p}(I)) \subseteq W_0^{I,p}(I) \quad \textit{iff} \quad T(\{a,b\}) \subseteq \{a,b\}.$$

Proof. Necessity. For absurd, let $T(a) \in I$ (analogously we proceed if $T(b) \in I$). Then there exists $\varepsilon > 0$ such that $J :=]T(a) - \varepsilon, T(a) + \varepsilon[\subseteq I$. We define the function $u : I \to \mathbb{R}$ by

$$U(x) := \begin{cases} e^{-1/(\varepsilon^2 - (x - T(a))^2) + 1/\varepsilon^2} & \text{if } x \in J \\ 0 & \text{if } x \notin J. \end{cases}$$

Then u is a C^1-function with support in I, so that $u \in W_0^{I,p}(I)$, while $u(T(a)) = 1$, in such a way that $C_T u \notin W_0^{1,p}(I)$.

Sufficience. Trivial. ■

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REFERENCES

[1] Nordgren E.A., *Composition operators on Hilbert spaces,* Lect. Notes Math.**693** (1977) 37-62.

[2] Ridge W.C., *SPectrum of a composition operator,* Proc. Amer. Math. Soc. **37** (1973), 121-127.

[3] Brezis H., *Analyse fonctionnelle,* Masson ED. Paris. 1983.

[4] Hewitt E., Stromberg K., *Real and abstract analysis,* Springer-Verlag 1969.

[5] CH.J De la Valle'e Poussin., *Sur l'intègral de Lebesgue,* Trans. Amer. Math. Soc., **16**(1915) 435-501.

[6] Serrin J., Varberg D.E., *A general chain rule for derivatives and the change of variables formula for the Lebesgue integral,* Amer. Math. Montly, **76** (1969) 514-520.

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