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SOME REMARKS ON THE HILBERT FUNCTION OF A ZERO-CYCLE IN $\mathbb{P}^2$

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We prove some new results concerning the Hilbert function of a finite set of points in general position in $\mathbb{P}^2$, with assigned multiplicities. In particular, we answer a question raised by E. Davis and A.V. Geramita.

1. Introduction.

Let $P_1, \dots, P_s$ be distinct points in $\mathbb{P}^2 = \mathbb{P}^2(k)$, with $s \geq 3$ and k any algebraically closed field, and let $m_1, \dots, m_s$ be positive integers. Then, if A denotes the ring $k[X_0, X_1, X_2]/I$, where I is the homogeneous ideal $\mathcal{P}_1^{m_1} \cap \mathcal{P}_2^{m_2} \cap \dots \cap \mathcal{P}_s^{m_s}$, $\mathcal{P}_1, \dots, \mathcal{P}_s$ being respectively the prime ideals in $k[X_0, X_1, X_2]$ corresponding to the points $P_1, \dots, P_s$, it is well known (cf., e.g., [3]) that A is a 1-dimensional Cohen-Macaulay ring of multiplicity $\delta(A) = \sum_{i=1}^{s} \binom{m_i + 1}{2}$.

Now, it is a classical problem (see, e.g., [1], [8], [4], [5], [6]) to

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determine the least degree τ of a linear system of plane curves such that the passage through each point P_i with multiplicity at least m_i ($i = 1, \dots, s$) imposes exactly $\delta(A)$ independent linear conditions, or equivalently, the least integer τ for which the Hilbert function, say $\{b_i\}_{i \geq 0}$, of the homogeneous coordinate ring A of the zero-cycle $\sum_{i=1}^{s} m_i P_i$ stabilizes.

This integer is also called the *index of regularity* of A or of the corresponding zero-cycle (see, e.g., [9, Def. 0.2.2]). If we denote by Z the zero-cycle $\sum_{i=1}^{s} m_i P_i$, we write then $\tau = \tau(Z)$ and $b_i = b_i(Z)$ for $i \geq 0$; moreover, we denote by $|\mathcal{O}_{\mathbb{P}^2}(n); m_1P_1, m_2P_2, \dots, m_sP_s|$ or by $|\mathcal{O}_{\mathbb{P}^2}(n); Z|$ the linear system (viewed as a k-vector space) of all plane curves of degree n with multiplicity at least m_i at P_i ($i = 1, \dots, s$).

The aim of this paper is to prove two results concerning a zero-cycle in $\mathbb{P}^2$, given by a finite set of points in *general position* (i.e. such that no three of them are collinear), with certain multiplicities. The first one (Theor. 2.1) provides some new characterizations for a finite set of points lying on an irreducible conic and also gives a positive answer (in a more general setting) to a question raised in [3]. The second statement (Theor. 2.4), which can be thought of as an extension of Theor. 2.1, concerns the relationship between the index of regularity of a zero-cycle of the form $Z = \sum_{i=1}^{s} 2P_i$ and the maximum number of points of $\{P_1, \dots, P_s\}$ actually lying on a conic.

1. Some preliminary observations.

In order to prove our main results, we need two lemmas, which will be proved in this section.

LEMMA 1.1. *Let $P_1, \dots, P_s$ be s points in general position in $\mathbb{P}^2$, with $s \geq 3$. Then* *(i)* $\tau\left(\sum_{i=1}^{s} P_i\right) \leq \left[\frac{s}{2}\right]$;

(ii) $\tau\left(\sum_{i=1}^{s} 2P_i\right) \leq s$;

(iii) if $s \geq 4$, *we get:*

$$\tau\left(\sum_{i=1}^{s} mP_i\right) \leq \tau\left(\sum_{i=1}^{s}(m-2)P_i\right) + s, \text{ for any } m \geq 3;$$

(iv) if $s \geq 4$, *we get:*

$$\tau\left(\sum_{i=1}^{s} mP_i\right) \leq \left[\frac{sm}{2}\right], \text{ for any } m \geq 1;$$

moreover, we have equality, if $P_1, \ldots, P_s$ *lie on a conic.*

Proof. (i) is clear and well known (cf., e.g., [7]).

(ii) The statement is certainly true for $s = 3$ and $s = 4$, the corresponding Hilbert functions being respectively:

$$1\ 3\ 6\ 9 \ldots,\ 1\ 3\ 6\ 10\ 12 \ldots.$$

So, let $s > 4$. To prove (ii), it is enough to show that

$$b_s\left(\sum_{i=1}^{s} 2P_i\right) \geq 3s \text{ i.e. } \dim_k |\mathcal{O}_{\mathbb{P}^2}(s);\ 2P_1, \ldots, 2P_s| \leq \binom{s-1}{2}.$$

Now, if we impose to the above linear system the passage through $s-3$ new points on the line P_1P_2, $s-3$ new points on the line $P_2P_3, \ldots$, one more point on the line $P_{s-2}P_{s-1}$, i.e. to contain, as fixed components, the $s-2$ lines $P_1P_2, P_2P_3, \ldots, P_{s-2}P_{s-1}$, we get:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(s);\ 2P_1, \ldots, 2P_s| \leq [(s-3)+(s-3)+(s-4)+\ldots+2+1]+1 = \binom{s-1}{2},$$

which proves (ii).

(iii) It is enough to show that if

$$b_i\left(\sum_{i=1}^{s}(m-2)P_i\right)=\delta\left(\sum_{i=1}^{s}(m-2)P_i\right),$$

then

$$b_{i+s}\left(\sum_{i=1}^{s}mP_i\right)=\delta\left(\sum_{i=1}^{s}mP_i\right).$$

Now, if we impose to the linear system $|\mathcal{O}_{\mathbb{P}^2}(i+s);\ mP_1,\dots,mP_s|$ to contain, as fixed components, the s lines $P_1P_2, P_2P_3,\dots,P_sP_1$ (by imposing, as before, the passage through a suitable number of new points on each of our s lines), we get:

$$\dim_k|\mathcal{O}_{\mathbb{P}^2}(i+s);mP_1,\dots,mP_s|\le[(i+s)-2m+1]+\ldots+[(i+1)-(2m-2)+1]+$$
$$+\left[\binom{i+2}{2}-s\binom{m-1}{2}\right]=\binom{i+s+2}{2}-s\binom{m+1}{2}.$$

It follows that $b_{i+s}\left(\sum_{i=1}^{s}mP_i\right)\ge s\binom{m+1}{2}$, which proves (iii).

(iv) We first prove that $\tau\left(\sum_{i=1}^{s}mP_i\right)\le\left[\frac{sm}{2}\right]$, for any $m\ge1$.

In fact, if m is *even*, say $m=2t$ $(t\ge2)$, we get from statements (ii), (iii) above:

$$\tau\left(\sum_{i=1}^{s}mP_i\right)\le(t-1)s+s=\left[\frac{sm}{2}\right].$$

If m is *odd*, say $m=2t+1$ $(t\ge2)$, we get from statements (i), (iii) above:

$$\tau\left(\sum_{i=1}^{s}mP_i\right)\le ts+\left[\frac{s}{2}\right]=\left[\frac{sm}{2}\right],$$

which shows our claim.

Now we prove that, if $P_1, \ldots, P_s$ lie on a conic, we get:

$$\tau\left(\sum_{i=1}^{s} mP_i\right) = \left[\frac{sm}{2}\right].$$

In fact, if we put $u = \left[\frac{sm}{2}\right]$, it is enough to show (in view of the inequality just proved before) that $b_{u-1}\left(\sum_{i=1}^{s} mP_i\right) < s\binom{m+1}{2}$.

Assume m *even*, say $m = 2t (t \geq 1)$, hence $2u = sm$. It follows that $\dim_k |\mathcal{O}_{\mathbb{P}^2}(u-1); mP_1, \ldots, mP_s| = \dim_k |\mathcal{O}_{\mathbb{P}^2}(u-3); (m-1)P_1, \ldots, (m-1)P_s|$, since $2(u-1) = sm - 2 < sm$. On the other hand, we have:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(u-3); (m-1)P_1, \ldots, (m-1)P_s| \geq \binom{u-1}{2} - s\binom{m}{2}.$$

Hence:

$$b_{u-1}\left(\sum_{i=1}^{s} mP_i\right) \leq \binom{u+1}{2} - \left[\binom{u-1}{2} - s\binom{m}{2}\right] = s\binom{m+1}{2} - 1,$$

which shows our claim.

If m is *odd*, the proof essentially runs as above, which completes the proof of statement (iv) and also of the lemma.

LEMMA 1.2. *Let $X := \{P_1, \ldots, P_s\}$ be a set of $s \geq 6$ points in general position in $\mathbb{P}^2$ such that:*

(i) $P_1, \ldots, P_h$ lie on a conic, say C, with $h \leq s - \beta$, for some integer $\beta \geq 1$;

(ii) $P_{h+1}, \ldots, P_s \notin C$;

(iii) $\tau(2P_{h+1} + \ldots + 2P_s) \leq s - \beta - 4$.

Then we get: $\tau\left(\sum_{i=1}^{s} 2P_i\right) \leq s - \beta$. Moreover, if $h = s - \beta$, then we have: $\tau\left(\sum_{i=1}^{s} 2P_i\right) = s - \beta$.

Proof. In fact, by imposing to the forms of degree $s-\beta$ through $Z=\sum_{i=1}^{s} 2P_i$ to contain, as a fixed component, the conic C counted twice, we get:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(s-\beta); Z| \leq (2s-2\beta+1-2h)+(2s-2\beta-3-h)+$$

$$+\dim_k |\mathcal{O}_{\mathbb{P}^2}(s-\beta-4);\ 2P_{h+1},\ldots,2P_s| = \binom{s-\beta-2}{2}+s-4\beta-2.$$

Hence $b_{s-\beta}(Z) \geq \binom{s-\beta+2}{2}-\binom{s-\beta-2}{2}-s+4\beta+2=3s$, i.e. $\tau(Z)\leq s-\beta$.

The last assertion is clear, since in any case, in view of Lemma 1.1 (iv), we have: $\tau(2P_1+\ldots+2P_h)=h$.

2. The main results.

In this section we will prove the two theorems announced in the Introduction.

THEOREM 2.1. *Let $P_1,\ldots,P_s$ be points in general position in $\mathbb{P}^2$, with $s\geq 6$, and let $\left[\frac{sm}{2}\right]$ denote the integral part of the rational number $\frac{sm}{2}$.*

Then the following are equivalent:

(a) the points $P_1,\ldots,P_s$ lie on a conic;

(b) there exists some integer $m\geq 2$ such that

$$\tau\left(\sum_{i=1}^{s} mP_i\right)=\left[\frac{sm}{2}\right];$$

(c) for every $m\geq 2$, we get: $\tau\left(\sum_{i=1}^{s} mP_i\right)=\left[\frac{sm}{2}\right]$.

Proof. (a)⇒(c) follows directly from Lemma 1.1 (iv), and (c)⇒(b) is clear. It remains to prove that (b)⇒(a).

Now, in view of Lemma 1.1, it is enough to show that, for $s \geq 6$:

(I) $\tau\left(\sum_{i=1}^{s} 2P_i\right) = s \Rightarrow P_1, \dots, P_s$ lie on a conic, and

(II) $\tau\left(\sum_{i=1}^{s} 3P_i\right) = s + \left[\frac{s}{2}\right] \Rightarrow P_1, \dots, P_s$ lie on a conic.

We will prove, in both cases, that the points $P_1, \dots, P_s$ lie on the conic $C = (P_1, \dots, P_5)$.

We first prove (I). Suppose, by absurd, that $P_1, \dots, P_s$ don't lie on any conic and denote by h the maximum number of the P_i's lying on a conic, hence $h \leq s - 1$. Then we distinguish two cases:

(i) $s \geq 8$, in which case it follows from Lemma 1.2 (see also Lemma 1.1 (ii)) that $\tau\left(\sum_{i=1}^{s} 2P_i\right) \leq s - 1$: contradiction.

(ii) $6 \leq s \leq 7$, in which case one checks directly that $\sum_{i=1}^{s}(2P_i) \leq s - 1$ again, and so we are done.

We turn to the proof of (II). We distinguish two cases:

(j) s *even*, say $s = 2u (u \geq 3)$. By hypothesis, we have:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(3u-1);\ 3P_1, \dots, 3P_s| \geq \binom{3u+1}{2} - (12u - 1).$$

Now, if we impose to the linear system above to contain, as fixed components, the u lines $P_1P_2, \dots, P_{s-1}P_s$, we get:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(2u-1);\ 2P_1, \dots, 2P_s| \geq \binom{3u+1}{2} - (12u-1) -$$

$$-[((3u-1)-5) + ((3u-2)-5) + \dots + (2u-5)] = \binom{2u+1}{2} - (6u-1).$$

Hence, in view of statement (I) proved before, the points $P_1, \dots, P_s$ all lie on a conic, which proves statement (II) for s even.

(jj) s *odd* ($s \geq 7$). For $s = 7$, the proof runs as follows. By hypothesis, we have $b_9\left(\sum_{i=1}^{7} 3P_i\right) \leq 41$ i.e. $\dim_k |\mathcal{O}_{\mathbb{P}^2}(9); 3P_1, \ldots, 3P_7| \geq 14$. Suppose, by absurd, that $P_1, \ldots, P_7$ don't lie on a conic. Then, if $P_6 \in C = (P_1 \ldots P_5)$ and $P_7 \notin C$, we get: $\dim_k |\mathcal{O}_{\mathbb{P}^2}(8); 3P_1, \ldots, 3P_5, 2P_6, 2P_7| = \dim_k |\mathcal{O}_{\mathbb{P}^2}(6); 2P_1, \ldots, 2P_5, P_6, 2P_7| \geq 10$, hence $\dim_k |\mathcal{O}_{\mathbb{P}^2}(5);\ 2P_1, \ldots, 2P_5, 2P_7| \geq 4$: contradiction (in view of (I)); otherwise $P_6, P_7 \notin C$, in which case we get: $\dim_k |\mathcal{O}_{\mathbb{P}^2}(8); 3P_1, \ldots, 3P_5, 2P_6, 2P_7| \geq 10$, hence $\dim_k |\mathcal{O}_{\mathbb{P}^2}(6); 2P_1, \ldots, 2P_7| \geq 8$: contradiction (in view of (I) again).

Assume $s > 7$, say $s = 2u + 1$ ($u \geq 4$). Then, by hypothesis, we have: $\dim_k |\mathcal{O}_{\mathbb{P}^2}(3u); 3P_1, \ldots, 3P_s| \geq \binom{3u+2}{2} - (6s - 1)$.

Now, if we impose to the linear system above to contain, as fixed components, the $(u - 3)$ lines $P_8P_9, \ldots, P_{s-1}P_s$, each of which counted three times, we get:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(9); 3P_1, \ldots, 3P_7| \geq \binom{3u+2}{2} - (6s-1) -$$

$$-\{[(3u-5)+((3u-1)-3)+((3u-2)-1)]+\ldots+[(12-5)+(11-3)+(10-1)]\} = 14.$$

It follows, in view of the case $s = 7$ just discussed above, that P_6, P_7 lie on the conic $C = (P_1 \ldots P_5)$. Hence, by renumbering our s points, we get that the points $P_1, \ldots, P_s$ all lie on the conic C, which proves statement (II) also for s odd.

This completes the proof of the implication (b)$\Rightarrow$(a) in our theorem, and so we are done.

Remark 2.2. It is worth observing that the implication (b)$\Rightarrow$(a) in Theorem 2.1 is not true, in general, for $m = 1$, as is shown, for example, by any set of nine points in general position which are the complete intersection of two plane cubics. Finally we note that the implication (a)$\Rightarrow$(c) is already shown in [3] (see also [2]).

Before stating our next theorem, we need a lemma.

LEMMA 2.3. *Let* $X = \{P_1, \ldots, P_s\}$ *be a set of* $s > 12$ *points in* *general position in* $\mathbb{P}^2$ *and let* Z *denote the zero-cycle* $\sum_{i=1}^{s} 2P_i$. *Then:*

(1) if $s = 2h$ $(h \geq 7)$ *and* X *admits a partition, say* $X = Y_1 \cup Y_2$, *with* $Y_i \subset C_i$, C_i *being a conic* $(i = 1, 2)$ *and moreover* $|Y_i| \leq h+2$, *we get:* $\tau(Z) \leq h+2$;

(2) if $s = 2h(h \geq 7)$ *and there exist two conics* C_1, C_2 *such that* $P_1, \ldots, P_h \in C_1, P_{h+1}, \ldots, P_{s-1} \in C_2, P_s \notin C_1 \cup C_2$, *we get:* $\tau(Z) \leq h+3$;

(3) if $s = 2h+1$ $(h \geq 6)$ *and there exist two conics* C_1, C_2 *satisfying one of the following conditions:*

(i) $P_1, \ldots, P_{h+1} \in C_1, P_{h+2}, \ldots, P_s \in C_2$, *or*

(ii) $P_1, \ldots, P_h \in C_1, P_{h+1}, \ldots, P_{s-1} \in C_2, P_s \notin C_1 \cup C_2$,

we get: $\tau(Z) \leq h+3$.

Proof. (1) First we note that, under our hypotheses on X, we immediately obtain (see, e.g. [7, Lemma 2.1)]: $b_4(X) \geq 12$, hence $b_{h-2}(X) \geq 2(h-2)+4$, i.e. $\tau(X) \leq h-2$.

Now, assume for example that $P_1, \ldots, P_h \in C_1$, $P_{h+1}, \ldots, P_s \in C_2$. Then, by imposing to the forms of degree $h+2$ through Z to contain, as fixed components, both C_1 and C_2, we get:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(h+2); Z| \leq 5 + 1 + \dim_k |\mathcal{O}_{\mathbb{P}^2}(h-2); X| = 6 + \binom{h}{2} - 2h,$$

hence $b_{h+2}(Z) \geq \binom{h+4}{2} - 6 - \binom{h}{2} + 2h = 3s$, which proves (1) in this case. A similar argument proves statement (1) in the other two cases left out.

(2) We note that, under our hypotheses on X, we get (arguing as in (1) above): $\tau(X \backslash \{P_i\}) \leq h-2$, for $i = 1, \ldots, s-1$.

Now, by imposing to the forms of degree $h+3$ through Z to contain, as fixed components, the conics C_1, C_2 and the line $P_{s-1}P_s$, we obtain:

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(h+3); Z| \leq 7 + 5 + (h-3) + \dim_k |\mathcal{O}_{\mathbb{P}^2}(h-2); X \backslash \{P_{s-1}\}| =$$

$$= 10 + \binom{h}{2} - h, \text{ hence } b_{h+3}(Z) \geq \binom{h+5}{2} - 10 - \binom{h}{2} + h = 3s,$$

which proves (2).

Finally the proofs of statements (3)(i), (3)(ii) essentially run as the proofs of (1), (2) respectively, developed above.

THEOREM 2.4. *Let* $X = \{P_1, \ldots, P_s\}$ *be a set of* $s \geq 6$ *points in general position in* $\mathbb{P}^2$*, not lying on any conic, and denote by* $h = h(X)$ *the maximum number of points of* X *lying on a conic, and by* Z *the zero-cycle* $\sum_{i=1}^{s} 2P_i$*. Then:*

(1) $\tau(Z) = s - 1$*, if and only if,* $h(X) = s - 1$*;*

(2) if $s > 9$*, we get:* $\tau(Z) = s - 2$*, if and only if,* $h(X) = s - 2$*;*

(3) if $s > 2(\alpha + 3)$*, with* $\alpha \geq 3$*, the following are equivalent:*

(i) $\tau(Z) = s - \alpha$*, and moreover* $h(X) \geq \alpha + 3$*;*

(ii) $h(X) = s - \alpha$*.*

Proof. First we note that, in view of Theor. 2.1, we get:

$$(*) \qquad \tau(Z) \geq h(X).$$

Also, we will assume that $P_1, \ldots, P_h$ (with $h = h(X)$) lie on a conic, say C, and we will denote by W the zero-cycle $\sum_{i=h+1}^{s} 2P_i$.

(1) $\Rightarrow$ Suppose, by absurd, that $h(X) \neq s - 1$. Hence, in view of (*), we get: $h(X) \leq s - 2$. We distinguish two cases:

(i) $s > 10$, in which case it follows from Lemma 1.2 (see also Lemma 1.1 (ii)) that $\tau(Z) \leq s - 2$: contradiction.

(ii) $7 \leq s \leq 10$, in which case one checks directly that $\tau(Z) \leq s - 2$ again, and so we are done (the case $s = 6$ being clear).

The reverse implication follows from Lemma 1.2.

(2) $\Rightarrow$ Suppose not; hence, in view of (*), $h(X) \leq s - 3$. We distinguish two cases:

(i) $s > 12$, in which case we get:

$$(**) \qquad \tau(W) \leq s - 7.$$

In fact, this is clear if $h \geq 7$ (in view of Lemma 1.1 (ii)); if $h = 5, 6$, (**) follows from Lemma 1.1 (ii) and Theor. 2.1 (see also statement (1) above). Therefore, it follows from Lemma 1.2 that $\tau(Z) \leq s - 3$: contradiction.

(ii) $10 \leq s \leq 12$. In this case one checks directly that $\tau(Z) \leq s - 3$ again, and so we are done.

The reverse implication follows from Lemma 1.2.

(3) $\Rightarrow$ Suppose not; hence, in view of (*), $h(X) \leq s - \alpha - 1$ ($\alpha \geq 3$). Now, if we show that:

$$(***) \qquad \tau(W) \leq s - \alpha - 5,$$

we get immediately from Lemma 1.2 that $\tau(Z) \leq s - \alpha - 1$, which is a contradiction. We distinguish four cases:

(i) $h \geq \alpha + 5$, or $h = \alpha + 4$ and $s > 2h$, or $h = \alpha + 3$ and $s \geq 2h + 2$, in which hypotheses (***) is true. In fact, this is clear, if $h \geq \alpha + 5$ (in view of Lemma 1.1 (ii)); in the other two subcases this follows from Lemma 1.1 (ii) and Theor. 2.1 (see also statements (1), (2) above).

(ii) $h = \alpha + 4$ and $s = 2h$ ($h \geq 7$). In this case, if the points $P_{h+1}, \ldots, P_s$ lie on a conic, we get from Lemma 2.3 (1): $\tau(Z) \leq s - \alpha - 2$, which is a contradiction; otherwise, we get (***) and we are done.

(iii) $h = \alpha + 4$ and $s = 2h - 1$. In this case, if the points $P_{h+1}, \ldots, P_s$ lie on a conic, we get from Lemma 2.3 (3): $\tau(Z) \leq s - \alpha - 1$, which is a contradiction; otherwise, we get (***) and we are done.

(iv) $h = \alpha + 3$ and $s = 2h + 1$. In this case, if h of the $s - h$ points $P_{h+1}, \ldots, P_s$ also lie on a conic, we get from Lemma 2.3 (3): $\tau(Z) \leq s - \alpha - 1$, which is a contradiction; otherwise, we get (***) again and we are done.

Finally, the reverse implication of statement (3) also follows from Lemma 1.2, which completes the proof of the theorem.

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