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REPRESENTATION THEOREMS WITH VARIABLES IN AN INTRINSIC FORM

SEBASTIANO PENNISI (Catania) (*) (**)

In a Minkowski space V, consideration is given to the tensor-valued isotropic functions of an arbitrary number of tensors of which one is a time-like vector U^α and another is a symmetric tensor $A_{\alpha\beta}$ such that $\hat{A}^{\alpha\beta}$, its space-space part orthogonal to U^α, has distinct eigenvalues.

Representations are given for these functions in terms of an orthonormal basis of eigenvectors U^α_A of $\hat{A}^{\alpha\beta}(A = 0, 1, 2, 3)$. The relationship between the total and the partial derivative of these function with respect to the independent components of U^α_A is also obtained in covariant form.

1. Introduction.

Representation theorems for isotropic functions have been object of much attention due to their utility in the field of physics. In the case of a 3-dimensional euclidean space [2, 6, 8, 9] they characterize the functions obeying the principle of objectivity or material frame

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indifference (as an example of applications, it can be seen the paper [3] by I-Shih Liu and I. Müller).

In the case of a Minkowski space [5, 7] these representation theorems characterize the functions underlying the principle of relativity and papers such us [1, 4] furnish examples of possible applications.

In order to express analitically this concept of isotropic functions let us consider the set w of independent variables constituted by the following tensors of order 0,1,2 respectively: $\lambda_n, V_r^\alpha, T_p^{\alpha\beta}$ for $n = 1, \ldots, N$; $r = 1, \ldots, R$; $p = 1, \ldots, P$.

The applications

$$\phi : w \to R,\ \phi^\alpha : w \to V,\ \phi^{\alpha\beta} : w \to V \otimes V,\ \phi^{\alpha\beta\gamma} : w \to V \otimes V \otimes V$$

are said isotropic functions (of order 0,1,2,3 respectively) iff the conditions

$$\phi(\lambda_n, A_\alpha^{\alpha'}, V_r^\alpha, A_\alpha^{\alpha'} A_\beta^{\beta'} T_p^{\alpha\beta}) = \phi(\lambda_n, V_r^\alpha, T_p^{\alpha\beta})$$

$$\phi^\mu(\lambda_n, A_\alpha^{\alpha'} V_r^\alpha, A_\alpha^{\alpha'} A_\beta^{\beta'} T_p^{\alpha\beta}) = A_{\mu'}^\mu \phi^{\mu'}(\lambda_n, V_r^\alpha, T_p^{\alpha\beta})$$

$$\phi^{\mu\nu}(\lambda_n, A_\alpha^{\alpha'} V_r^\alpha, A_\alpha^{\alpha'} A_\beta^{\beta'} T_p^{\alpha\beta}) = A_{\mu'}^\mu A_{\nu'}^\nu \phi^{\mu'\nu'}(\lambda_n, V_r^\alpha, T_p^{\alpha\beta})$$

$$\phi^{\mu\nu\lambda}(\lambda_n, A_\alpha^{\alpha'} V_r^\alpha, A_\alpha^{\alpha'} A_\beta^{\beta'} T_p^{\alpha\beta}) = A_{\mu'}^\mu A_{\nu'}^\nu A_{\lambda'}^\lambda \phi^{\mu'\nu'\lambda'}(\lambda_n, V_r^\alpha, T_p^{\alpha\beta})$$

are verified for all orthogonal automorphisms $A_\alpha^{\alpha'}$ of V.

In other words, this property means that the following diagrams are commutative

$$\begin{array}{ccc} w & \xrightarrow{A} & w \\ \phi \downarrow & \swarrow \phi ; & \\ R & & \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{A} & w \\ h \downarrow & & \downarrow h \\ Z & \xrightarrow{A} & Z \end{array}$$

where $Z = V$ if $h = \phi^\mu$, $Z = V \otimes V$ if $h = \phi^{\mu\nu}$,

$$Z = V \otimes V \otimes V \quad \text{if} \quad h = \phi^{\mu\nu\lambda},$$

Obviously if V is a 3-dimensional euclidean vector space the summation convention over repeated indices is used from 1 to 3 and the orthogonality of $A_\alpha^{\alpha'}$ means that $A_\alpha^{\alpha'} A_\beta^{\beta'} \delta_{\alpha'\beta'} = \delta_{\alpha\beta}$ (Kronecker symbol) while if V is a Minkowski space the summation convention is used from 0 to 3 and $A_\alpha^{\alpha'}$ is orthogonal iff

$$A_\alpha^{\alpha'} A_\beta^{\beta'} g_{\alpha'\beta'} = g_{\alpha\beta}$$

(Metric tensor).

The representation theorems furnish sets S_i of tensors of order i for $i = 0, 1, 2$ such that:

1 Every scalar-valued isotropic function can be expressed as a function F of the elements in set S_0.

2) Every tensorial isotropic function of order $i = 1, 2$ can be expressed as a linear combination F_i of the elements in set S_i through scalar coefficients.

But the function F (for $i = 0$) or the coefficients of F_i (for $i = 1, 2$) are not in general differentiable functions of their arguments, as stated for example by one of the authors of these papers ([8] p. 915); on the other hand, in the applications they are assumed to be derivable ([1, 3, 4]). Now it is true that derivability may be assumed on physical ground but it is also true that this property depends even on the representation itself; for example let us consider a scalar function f of a symmetric tensor A in a 3-dimensional euclidean space. From the tables in [8] we read that f can be expressed as a function $f = F_1(trA, trA^2, trA^3)$ and then as a function $f = F_2(x, y, z)$ where

$$x = 2trA^3 - 3(trA)(trA^2) + (trA)^3,$$

$$y = 2trA^2 - (trA)^2, \quad z = trA;$$

obviously F_1 is derivable with respect to its arguments iff F_2 is derivable with respect to its owns.

But it is obvious that f can be expressed also as a function $f = F_3(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A; now F_3

may be derivable even when F_2 is not; in fact

$$\frac{\partial F_2}{\partial y} = \sum_{i=1}^{3} \frac{\partial F_3}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial y}$$

but $\lambda_i(x, y, z)$ is such that

$$\lambda_1(0, y, 2) = 0; \ \lambda_2(0, y, 2) = 1 - (\sqrt{y})/2; \ \lambda_3(0, y, 2) = 1 + (\sqrt{y})/2$$

and then λ_2 is not derivable with respect to y in the point $(x, y, z) = (0, 0, 2)$.

On the other hand the derivability of F_2 implies that of F_3; in fact

$$\frac{\partial F_3}{\partial \lambda_i} = \frac{\partial F_2}{\partial x}\frac{\partial x}{\partial \lambda_i} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial \lambda_i} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial \lambda_i}$$

and

$$\frac{\partial x}{\partial \lambda_1} = 6\lambda_2\lambda_3; \ \frac{\partial x}{\partial \lambda_2} = 6\lambda_1\lambda_3; \ \frac{\partial x}{\partial \lambda_3} = 6\lambda_1\lambda_2;$$

$$\frac{\partial y}{\partial \lambda_1} = 2(\lambda_1 - \lambda_2 - \lambda_3); \ \frac{\partial y}{\partial \lambda_2} = 2(\lambda_2 - \lambda_1 - \lambda_3); \ \frac{\partial y}{\partial \lambda_3} = 2(\lambda_3 - \lambda_1 - \lambda_2);$$

$$\frac{\partial z}{\partial \lambda_1} = \frac{\partial z}{\partial \lambda_z} = \frac{\partial z}{\partial \lambda_3} = 1.$$

Moreover, in the applications one may be interested to functions that are linear with respect to some of their arguments (see [1, 3, 4] and even this aspect depends on the representation used; in fact in the above mentioned example we can see that F_1 is linear in A iff $F_1 = a tr A + b$ for some constants a, b (and then F_1 is a single variable function), while F_3 is linear in A iff $F_3 = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + b'$ for some constants a_1, a_2, a_3, b' (and then it is a three-variables function).

From these observations it is then evident that with physical motivations one must assume not only derivability of the functions involved and the possibility of expanding some of them up to linear terms, but also the representation in which these properties must be satisfied.

This fact shows the utility to know alternative representations to those alredy known and I think it is interesting the case in which

1) The framework is a Minkowski space V with metric tensor $g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ (The case in which we have a 3-dimensional euclidean space, might be easily considered with the same method),

2) among the independent variables there are a time-like vector U^α, the eigenvalues of

$$\hat{A}^{\alpha\beta} = A^{\alpha\beta} + (U_\gamma U^\gamma)^{-2} U_\lambda U_\mu A^{\lambda\mu} U^\alpha U^\beta - 2(U_\gamma U^\gamma)^{-1} U_\mu A^{\mu(\alpha} U^{\beta)}$$

(i.e. the space-space part orthogonal to U_α of a symmetric tensor A) and an orthonormal basis U_A^α of eigenvectors of $\hat{A}^{\alpha\beta}$ $(A = 0, 1, 2, 3)$ where $U_0^\alpha = (-U_\gamma U^\gamma)^{-1/2} U^\alpha$.

I call this set «intrinsic variable».

When $\hat{A}^{\alpha\beta}$ has distinct eigenvalues then U_A^α are determined *up* to sign; more generally one can assume that U_A^α are a sort of hidden variables with which $\hat{A}^{\alpha\beta}$ assumes the form

$$\hat{A}^{\alpha\beta} = \sum_{A,B=0}^{3} \lambda^{AB} U_A^\alpha U_B^\beta \quad \text{where} \quad ||\lambda^{AB}|| = \mathrm{diag}(0, \lambda_1, \lambda_2, \lambda_3)$$

and $\lambda_1, \lambda_2, \lambda_3$ are three other scalar variables (The eigenvalues of $\hat{A}^{\alpha\beta}$).

Consequently we have

$$A^{\alpha\beta} = \sum_{A,B=0}^{3} \mu^{AB} U_A^\alpha U_B^\beta$$

where

$$||\mu^{AB}|| = \left\| \begin{matrix} \mu^0 & \mu^1 & \mu^2 & \mu^3 \\ \mu^1 & \lambda_1 & 0 & 0 \\ \mu^2 & 0 & \lambda_2 & 0 \\ \mu^3 & 0 & 0 & \lambda_3 \end{matrix} \right\|$$

and μ_0, μ_1, μ_2, μ_3 are four other scalar variables, i.e. $\mu^A = U_\alpha^0 U_\beta^A A^{\alpha\beta}$ for $A = 0, 1, 2, 3$, where *I* have defined $U_\alpha^A = \sum_{B=0}^{3} g^{AB} U_{B\alpha}$.

With these variables among the others, I list in Section 2 complete representations for scalar-valued, vector-valued and tensor-valued isotropic functions of order 2 and 3; it can also be proved that they are irreducible in the sense that no proper subset of them will suffice to furnish a complete representation.

In section 3 it is shown how the results previously known in the literature may be derived from the present ones.

Lastly, in section 4 the problem is considered of expressing in covariant form the total derivatives of a function with respect to the independent components between U_A^α, in terms of the partial derivatives.

2. Complete and irreducible representations with the variables U_A^α.

Let us consider the set S of independent variables constituted by

1) an orthonormal basis of vectors U_A^α for $A = 0, 1, 2, 3$ of which U_0^α is time-like.
2) an arbitrary number of other 4-vectors V_r^α (for $r = 1, \ldots, R$),
3) an arbitrary number of second order symmetric tensors $A_p^{\alpha\beta}$ (for $p = 1, \ldots, P$),
4) an arbitrary number of second order skew-symmetric tensors $W_q^{\alpha\beta}$ (for $q = 1, \ldots, Q$),
5) an arbitrary number of scalar variables λ_n (for $n = 1, \ldots, N$).

Then the following theorems hold

THEOREM 1. *Every scalar-valued isotropic function ϕ of the variables in set S can be expressed as a single-valued function of the* *scalars in the following set T_0:*

$$V_r^{\alpha} U_{A\alpha} \qquad \text{for } r = 1, \dots, R;\ A = 0, 1, 2, 3$$

$$A_p^{\alpha\beta} U_{A\alpha} U_{B\beta} \qquad \text{for } p = 1, \dots, P;\ A, B = 0, 1, 2, 3,\ A \leq B$$

$$W_q^{\alpha\beta} U_{A\alpha} U_{B\beta} \qquad \text{for } q = 1, \dots, Q;\ A, B = 0, 1, 2, 3,\ A < B$$

$$\lambda_n \qquad \text{for } n = 1, \dots, N''.$$

THEOREM 2. *Every vector-valued isotropic function ϕ^α of the variables in set S can be expressed as a linear combination through scalar coefficients of the vectors U_A^α, for $A = 0, 1, 2, 3$ (set T_1).*

THEOREM 3. *Every second order symmetric tensor-valued isotropic function $\phi^{\alpha\beta}$ of the variables in set S can be expressed as a linear combination through scalar coefficients of the tensors $U_A^{(\alpha} U_B^{\beta)} = (U_A^\alpha U_B^\beta + U_A^\beta U_B^\alpha)/2$ for $A, B = 0, 1, 2, 3,\ A \leq B$ (set T_{2s}).*

THEOREM 4. *Every second order skew-symmetric tensor-valued isotropic function $W^{\alpha\beta}$ of the variables in set S can be expressed as linear combination through scalar coefficients of the tensors $U_A^{[\alpha} U_B^{\beta]} = (U_A^\alpha U_B^\beta - U_A^\beta U_B^\alpha)/2$ for $A, B = 0, 1, 2, 3,\ A < B$ (set T_{2a}).*

THEOREM 5. *Every third order tensor-valued isotropic function $A^{\alpha\beta\gamma}$ of the variables in set S can be expressed as linear combination through scalar coefficients of the tensors $U_A^\alpha U_B^\beta U_C^\gamma$ for $A, B, C = 0, 1, 2, 3$ (set T_3).*

The proofs of these theorems are trivial and then I omit them.

The sets $T_0, T_1, T_{2s}, T_{2a}, T_3$ are called complete representations and it is not difficult to see that such representations are irreducible in the sense expressed in section 1.

In the next section these representations are related to those already known in the literature.

3. Relationship with the previously known results.

The most complete paper on representation theorems for isotropic functions in a 4-dimensional pseudoeuclidean vector space (with signature $-+++$) is ref. [7]. This fact does not mean that the present paper is a particular case of that one because I have now more restrictive hypotesis and thus the results are also different. Moreover it can be shown that the results in that paper, may be obtained by using the method of the present one; I show now how this purpose may be realized.

The independent variables in [7] are a time-like four-vector V_0^α, an arbitrary number of other 4-vectors V_r^α and of second order tensors (of which some, $A_p^{\alpha\beta}$, are symmetric and the others $W_q^{\alpha\beta}$ skew-symmetric; $r = 1, \dots, R$; $p = 1, \dots, P$; $q = 1, \dots, Q$).

There are some cases to be considered; each of them will give a set of generators S_i for isotropic functions of order i, for $i = 0, 1, 2, 3$. The union of all this sets of the same order will give a representation for these functions that is equivalent to the corresponding one in ref. [7]. (If R_i and $R_{i'}$ are two representations for isotropic functions of order i here R_i is said equivalent to $R_{i'}$ iff each element of R_i can be expressed as function of those of R_i', if $i = 0$, or as a linear combination through scalar coefficients of those of R_i', if $i > 0$).

The first case to be considered is

Case 1. Among the independent variables there are three 4-vectors $V_1^\alpha, V_2^\alpha, V_3^\alpha$ that are linearly independent (l.i.) with V_0^α.

In this case we may consider

$$U_A^\alpha = \sum_{B=0}^{A} \lambda_{AB} V_B^\alpha \qquad \text{for } A = 0, 1, 2, 3 \tag{3.1}$$

and choose λ_{AB} such that U_A^α constitute an orthonormal basis of vectors; obviously λ_{AB} depends only on $V_A^\alpha \, V_B^\beta \, g_{\alpha\beta}$ that must then be included in the representation for scalar-valued isotropic functions. By substituting the relation (3.1) in the representations found in sect.

2 of this paper we obtain the first of the above mentioned sets of generators for isotropic functions.

In the other cases the following set I of vector-valued isotropic functions is used:

$$\mathbf{V}_0, \mathbf{V}_r, A_{p_3}\mathbf{V}_0, W_{q_3}\mathbf{V}_0, W_{q_3}\mathbf{V}_0, A_{p_1}\mathbf{V}_r, A_{p_1}A_{p_3}\mathbf{V}_0, A_{p_1}W_{q_3}\mathbf{V}_0,$$

$$A_{p_1}A_{p_1}\mathbf{V}_r, A_{p_1}A_{p_1}A_{p_3}\mathbf{V}_0, A_{p_1}A_{p_1}A_{q_3}\mathbf{V}_0, W_{q_1}\mathbf{V}_r, W_{q_1}A_{p_3}\mathbf{V}_0,$$

$$W_{q_1}W_{q_3}\mathbf{V}_0, W_{q_1}W_{q_1}\mathbf{V}_r, W_{q_1}W_{q_1}A_{p_3}\mathbf{V}_0, W_{q_1}W_{q_1}W_{q_3}\mathbf{V}_0,$$

$$(A_{p_1}A_{p_2} - A_{p_2}A_{p_1})\mathbf{V}_r, (A_{p_1}A_{p_2} - A_{p_2}A_{p_1})A_{p_3}\mathbf{V}_0, (A_{p_1}A_{p_2} - A_{p_2}A_{p_1})W_{q_3}\mathbf{V}_0,$$

$$(W_{q_1}W_{q_2} - W_{q_2}W_{q_1})\mathbf{V}_r, (W_{q_1}W_{q_2} - W_{q_2}W_{q_1})A_{p_3}\mathbf{V}_0, (W_{q_1}W_{q_2} - W_{q_2}W_{q_1})W_{q_3}\mathbf{V}_0,$$

$$(A_{p_1}W_{q_1} - W_{q_1}A_{p_1})\mathbf{V}_r, (A_{p_1}W_{q_1} - W_{q_1}A_{p_1})A_{p_3}\mathbf{V}_0, (A_{p_1}W_{q_1} - W_{q_1}A_{p_1})W_{q_3}\mathbf{V}_0,$$

for $r = 1, \ldots, R$; $p_1, p_2, p_3 = 1, \ldots, P$ and $p_1 < p_2$; $q_1, q_2, q_3 = 1, \ldots, Q$ and $q_1 < q_2$.

Case 1.1. There are not three 4-vectors $V_1^\alpha, V_2^\alpha, V_3^\beta$ l.i., with V_0^α, but there are three elements $Y_1^\alpha, Y_2^\alpha, Y_3^\alpha$ of set I that are l.i., with V_0^α.

The method of case (1) can be used here, but with Y_a^α instead of V_a^α for $a = 1, 2, 3$.

Case 2. We have that:

a) There are not three elements of set I that are l.i., with V_0^α;

b) There are two-4-vectors V_1^α, V_2^α that are l.i., with V_0^α.

As in case (1) we may consider U_A^α defined by (3.1) but for $A = 0, 1, 2$ and λ_{AB} such that U_A^α constitutes an orthonormal basis of V generated by V_0^α, V_1^α, V_2^α. Moreover we may complete U_0^α, U_1^α, U_2^α to an orthonormal basis of V with another 4-vector U_3^α (or its opposite $-U_3^\alpha$). Let Σ be the reference frame in which the coordinate axis are directed along U_A^α; from condition a) we have that

$$V_r^\alpha U_{3\alpha} = 0; A_p^{'\alpha\beta} U_{3\alpha} = 0; W_q^{\alpha\beta} u_{3\alpha} = 0 \tag{3.2}$$

where

$$A'^{\alpha\beta}_p = A^{\alpha\beta}_p + g^{\alpha\beta} A^{\lambda\mu}_p (-g_{\lambda\mu} - U_{0\lambda}U_{0\mu} + U_{1\lambda}U_{1\mu} + U_{2\lambda}U_{2\mu}) \tag{3.3}$$

that substitutes $A'^{\alpha\beta}_p$ and the scalars $g_{\lambda\mu}A^{\lambda\mu}_p$, $U_{A\lambda}U_{B\mu}A^{\lambda\mu}_p$ (for $A = 0, 1, 2$) to $A^{\alpha\beta}_p$.

We find then the representation for scalar-valued isotropic functions constituted by $g_{\lambda\mu}A^{\lambda\mu}_p$ and that of theorem 1, sect. 2, but only for $A, B = 0, 1, 2$ (as it can be seen from relation (3.2)).

From theorem 2 of sect. 2 we have that a vector-valued isotropic function ϕ^α may be expressed as

$$\phi^\alpha = \sum_{A=0}^{3} \lambda_A U^\alpha_A \quad \text{where} \quad \lambda_3 = \phi^3 \quad \text{in} \quad \Sigma; \tag{3.4}$$

now if we change the sign of the 3-axis of Σ, ϕ^3 must change sign (because ϕ^α is an isotropic function), while it must remain the same as before (because it is a function of variables that remain the same with this inversion of the 3-axis) and then $\phi^3 = 0$, i.e. $\lambda_3 = 0$ in (3.4). Then $\{U^\alpha_0, U^\alpha_1, U^\alpha_2\}$ (or equivalently $\{V^\alpha_0, V^\alpha_1, V^\alpha_2\}$ constitutes the requested set for vector-valued isotropic function.

The same method may be used for the isotropic functions $\phi^{\alpha\beta} = \phi^{(\alpha\beta)}$, $W^{\alpha\beta} = W^{[\alpha\beta]}$, $A^{\alpha\beta\gamma}$ and shows that in Σ we have $\phi^{\alpha 3} = W^{\alpha 3} = A^{\alpha\beta 3} = A^{\alpha 3\beta} = A^{3\alpha\beta} = A^{333} = 0$; for $\alpha, \beta = 0, 1, 2$.

Consequently, from theorem 3,4,5 of sect. 2 we have that

$\phi^{\alpha\beta}$ is a linear combination of $U^{(\alpha}_A U^{\beta)}_B$ (for $A, B = 0, 1, 2$; $A \leq B$) and

$$U^\alpha_3 U^\beta_3;$$

$W^{\alpha\beta}$ is a linear combination of $U^{[\alpha}_A U^{\beta]}_B$ (for $A, B = 0, 1, 2$; $A < B$), $A^{\alpha\beta\gamma}$ is a linear combination of $U^\alpha_A U^\beta_B U^\gamma_C$ and $U^\alpha_A U^\beta_3 U^\gamma_3$,

$$U^\alpha_3 U^\beta_A U^\gamma_3, U^\alpha_3 U^\beta_3 U^\gamma_A \qquad \text{(for } A, B, C = 0, 1, 2).$$

Substituting to $U^\alpha_3 U^\beta_3$ the identity

$$U^\alpha_3 U^\beta_3 = g^{\alpha\beta} + U^\alpha_0 U^\beta_0 - U^\alpha_1 U^\beta_1 - U^\alpha_2 U^\beta_2$$

and to U_A^α the relation (3.1) for $A = 0, 1, 2$, we find the requested sets of generators.

Case 2.1. We have that:

a) there are not three elements of set I that are l.i. with V_0^α;

b) there are not two 4-vectors V_1^α, V_2^β that are l.i., with V_0^α;

c) there are two elements Y_1^α, Y_2^α of set I that are l.i., with V_0^α.

The results of case (2) can be used with Y_a^α instead of V_a^α (for $a = 1, 2$) thus obtaining another of the requested sets.

The remaining cases to be considered may be treated in a similar way and I omit them in order to be brief.

I want now to come back to the hypotesis of the present paper, (i.e. that among the independent variables there is an orthonormal basis U_A^α of 4-vectors, for $A = 0, 1, 2, 3$) and show how the total derivatives with respect to the independent components of U_A^α may be expressed in terms of the partial derivatives in covariant form. This problem is treated in the next section.

4. The total derivatives with respect to U_A^α.

For the orthonormality condition

$$U_A^\alpha U_{\beta\alpha} = g_{AB}, \tag{4.1}$$

it is clear that the 16 variables U_A^α are not all independent, but we may choose six of them x_m (for $m = 1, \ldots, 6$) as independent and call y_n (for $n = 1, \ldots, 10$) the remaining ones, obtaining $y_n = y_n(x_m)$ from the condition (4.1). A possible choice is indicated in the following theorem.

THEOREM 1. *There is a permutation of U_1^α, U_2^α, U_3^α such that U_0^i, U_1^a, U_2^3 (for $i = 1, 2, 3$; $a = 2, 3$) can be assumed as independent variables.*

Proof. From (4.1) with $A = B = 0$ we see that we can take

$$x_1 = U_0^1, x_2 = U_0^2, x_3 = U_0^3$$

and from

$$(U_0^0)^2 = 1 + \sum_{i=1}^{3} (U_0^i)^2 > 0$$

we can take $y_1 = U_0^0$.

Moreover there is $A \in \{0, 1, 2, 3\}$ such that

$$\begin{vmatrix} -U_0^0 & U_0^1 \\ -U_A^0 & U_A^1 \end{vmatrix} \neq 0 \tag{4.2}$$

otherwise we would have $U_A^0 = \lambda_A U_0^0$; $U_A^1 = \lambda_A U_0^1$ for some λ_A and then if μ_A is a non null solution of

$$\sum_{A=0}^{3} \mu_A \lambda_A = 0; \; \sum_{A=0}^{3} \mu_A U_A^2 = 0; \; \sum_{A=0}^{3} \mu_A U_A^3 = 0;$$

we would obtain $\sum_{A=0}^{3} \mu_A U_A^\alpha = 0$ against the linear independence of the 4-vectors U_A^α.

The property (4.2) is not verified for $A = 0$ and we may suppose that it is true for $A = 1$ because in the other eventualities we come back to this case with a permutation of the 4-vectors U_1^α U_2^α, U_3^α.

From (4.2) and (4.1) for $A = 1$; $B = 0, 1$ we see that we can take

$$x_4 = U_1^2, \; x_5 = U_1^3, \; y_2 = U_1^0, \; y_3 = U_1^1.$$

Similarly there is $A \in \{2, 3\}$ such that

$$\begin{vmatrix} -U_0^0 & U_0^1 & U_0^2 \\ -U_1^0 & U_1^1 & U_1^2 \\ -U_A^0 & U_A^1 & U_A^2 \end{vmatrix} \neq 0 \tag{4.3}$$

otherwise we would have $U_A^0 = \lambda_A U_0^0 + \mu_A U_1^0$;

$$U_A^1 = \lambda_A U_0^1 + \mu_A U_1^1; U_A^2 = \lambda_A U_0^2 + \mu_A U_1^2$$

for some λ_A, μ_A and then if ν_A is a non null solution of

$$\sum_{A=0}^{3} \nu_A \lambda_A = 0; \sum_{A=0}^{3} \nu_A \mu_A = 0; \sum_{A=0}^{3} \nu_A U_A^3 = 0$$

we would obtain $\sum_{A=0}^{3} \nu_A U_A^\alpha = 0$ against the linear independence of the 4-vectors U_A^α.

The property (4.3) obviously is not verified for $A = 0, 1$ and we may suppose that it is true for $A = 2$ because in the other cas ewe may come back to the present one with a permutation of U_2^α, U_3^α.

From (4.3) and (4.1) for $A = 2$; $B = 0, 1, 2$ we see that we can take

$$x_6 = U_2^3, \ y_4 = U_2^0, \ y_5 = U_2^1, \ y_6 = U_2^2.$$

Moreover we can see (with the same method) that

$$\text{(4.4)} \qquad \begin{vmatrix} -U_0^0 & U_0^1 & U_0^2 & U_0^3 \\ -U_1^0 & U_1^1 & U_1^2 & U_1^3 \\ -U_2^0 & U_2^1 & U_2^2 & U_1^3 \\ -U_3^0 & U_3^1 & U_3^2 & U_3^3 \end{vmatrix} \neq 0$$

From (4.4) and (4.1) for $A = 3$; $B = 0, 1, 2, 3$ we have that

$$y_7 = U_3^0, \ y_8 = U_3^1, \ y_9 = U_3^2, \ y_{10} = U_3^3.$$

It will be useful to know the derivatives of these y_n with respect to the x_m; to this end let us now consider the relation (4.1) for $AB = 00$, 01, 11, 02, 12, 22, 03, 13, 23, 33 and take their derivatives with respect to U_0^i, U_1^a, U_2^3 (for $i = 1, 2, 3$; $a = 2, 3$); we obtain respectively

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{cA'} \frac{\partial}{\partial U_0^i} U_{A'}^{\gamma'} = \sum_{A=0}^{3} p^{CA} U_{Ai} \tag{4.5}$$

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{cA'} \frac{\partial}{\partial U_1^a} U_{A'}^{\gamma'} = \sum_{A=0}^{3} q^{CA} U_{Aa} \tag{4.6}$$

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{cA'} \frac{\partial}{\partial U_2^3} U_{A'}^{\gamma'} = \sum_{A=0}^{3} r^{CA} U_{A3} \tag{4.7}$$

where

$$\mathbf{A}_0^{c0} \equiv (-U_0^0, -U_1^0, 0, -U_2^0, 0, 0, -U_3^0, 0, 0, 0)^T$$

(In the sense that $\mathbf{A}_0^{C0}$ is the c-th component of the second member),

$$\mathbf{A}_0^{c1} \equiv (0, -U_0^0, -U_1^0, 0, -U_2^0, 0, 0, -U_3^0, 0, 0)^T$$

$$\mathbf{A}_1^{c1} \equiv (0, U_0^1, U_1^1, 0, U_2^1, 0, 0, U_3^1, 0, 0)^T$$

$$\mathbf{A}_0^{c2} \equiv (0, 0, 0, -U_0^0, -U_1^0, -U_2^0, 0, 0, -U_3^0, 0)^T$$

$$\mathbf{A}_1^{c2} \equiv (0, 0, 0, U_0^1, U_1^1, U_2^1, 0, 0, U_3^1, 0)^T$$

$$\mathbf{A}_2^{c2} \equiv (0, 0, 0, U_0^2, U_1^2, U_2^2, 0, 0, U_3^2, 0)^T$$

$$\mathbf{A}_0^{c3} \equiv (0, 0, 0, 0, 0, 0, -U_0^0, -U_1^0, -U_2^0, -U_3^0)^T$$

$$\mathbf{A}_1^{c3} \equiv (0, 0, 0, 0, 0, 0, U_0^1, U_1^1, U_2^1, U_3^1)^T$$

$$\mathbf{A}_2^{c3} \equiv (0, 0, 0, 0, 0, 0, U_0^2, U_1^2, U_2^2, U_3^2)^T$$

$$\mathbf{A}_3^{c3} \equiv (0, 0, 0, 0, 0, 0, U_0^3, U_1^3, U_2^3, U_3^3)^T$$

$$p^{CA} \equiv (-\delta_0^A, -\delta_1^A, 0, -\delta_2^A, 0, 0, -\delta_3^A, 0, 0, 0)^T$$

$$q^{CA} \equiv (0, -\delta_0^A, -\delta_1^A, 0, -\delta_2^A, 0, 0, -\delta_3^A, 0, 0)^T$$

$$r^{CA} \equiv (0,0,0,-\delta_0^A,-\delta_1^A,-\delta_2^A,0,0,-\delta_3^A,0)^T$$

Obviously the determinant of $\mathbf{A}_{\gamma'}^{CA'}$ is the product of those in (4.2), (4.3), (4.4) times $(-U_0^0)$ and then $\mathbf{A}_{\gamma'}^{CA'}$ is invertible so that from (4.5), (4.6), (4.7) we can obtain

(4.8)
$$\frac{\partial U_{A'}^{\gamma'}}{\partial U_0^i} = \sum_{A=0}^{3} p_{A'}^{\gamma' A} U_{Ai}; \quad \frac{\partial U_{A'}^{\gamma'}}{\partial U_1^a} = \sum_{A=0}^{3} q_{A'}^{\gamma' A} U_{Aa};$$
$$\frac{\partial U_{A'}^{\gamma'}}{\partial U_2^3} = \sum_{A=0}^{3} r_{A'}^{\gamma' A} U_{A3};$$

where $p_{A'}^{\gamma' A}$, $q_{A'}^{\gamma' A}$, $r_{A'}^{\gamma' A}$ are the solutions of

(4.9)
$$\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} p_{A'}^{\gamma' A} = p^{CA}$$

(4.10)
$$\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} q_{A'}^{\gamma' A} = q^{CA}$$

(4.11)
$$\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} r_{A'}^{\gamma' A} = r^{CA}$$

Lastly let now f be a tensor-valued isotropic function of some order ν (I omit the eventual indices for the sake of brevity), depending among the other variables also on the orthonormal 4-vectors U_A^α (for $A = 0,1,2,3$).

From $f = f(x_m, y_n)$ we may define the composite function

(4.12)
$$g(x_m) = f(x_m, y_n(x_m))$$

and the function

(4.13)
$$F(x_m, y_n) = g(x_m) \quad \text{for which} \quad \frac{\partial F}{\partial y_n} = 0.$$

Then the following theorem holds

THEOREM 2. *There are 16 functions f^{AB} (for $A, B = 0, 1, 2, 3$) satisfying the condition $f^{AB} = f^{BA}$ and such that*

$$\frac{\partial F}{\partial U_A^\mu} = \frac{\partial f}{\partial U_A^\mu} + \sum_{B=0}^{3} f^{AB} U_{B\mu}. \tag{4.14}$$

Proof. We may consider the case in which the variables x_m are those of theorem 1 because the permutation that appears in it does not effect the present proposition (it results only in a permutation of the rows and columns of the matrix of components f^{AB}).

From the derivation property of composite functions and relations (4.8) we have

$$\begin{aligned}\frac{\partial F}{\partial U_0^i} &= \frac{\partial f}{\partial U_0^i} + \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} \frac{\partial U_{A'}^{\gamma'}}{\partial U_o^i} = \\ &= \frac{\partial f}{\partial U_o^i} + \sum_{B=0}^{3}\left(\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} p_{A'}^{\gamma' B}\right) U_{Bi}\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial U_1^a} &= \frac{\partial f}{\partial U_1^a} + \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} \frac{\partial U_{A'}^{\gamma'}}{\partial U_1^a} = \\ &= \frac{\partial f}{\partial U_1^a} + \sum_{B=0}^{3}\left(\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} q_{A'}^{\gamma' B}\right) U_{Ba}\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial U_2^3} &= \frac{\partial f}{\partial U_2^3} + \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} \frac{\partial U_{A'}^{\gamma'}}{\partial U_2^3} = \\ &= \frac{\partial f}{\partial U_2^3} + \sum_{B=0}^{3}\left(\sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} r_{A'}^{\gamma' B}\right) U_{B3}.\end{aligned}$$

If we define

$$f^{0B} = \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} p_{A'}^{\gamma' B}$$

$$f^{1B} = \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} q_{A'}^{\gamma' B}$$

(4.15) $$f^{2B} = \sum_{A'=0}^{3}\sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} r_{A'}^{\gamma' B}$$

$$f^{3B} = \begin{cases} f^{B3} & \text{for } B = 0,1,2 \\ -\sum_{\gamma'=0}^{3} \frac{\partial f}{\partial U_3^{\gamma'}} U_3^{\gamma'} & \text{for } B = 3 \end{cases}$$

the above relations prove the proposition (4.14) for $A\mu$ = 01, 02, 03, 12, 13, 23. Then it is true if it is verified also for $A\mu$ = 00, 10, 11, 20, 21, 22, 30, 31, 32, 33, i.e.

(4.16)
$$\begin{aligned} 0 &= \frac{\partial f}{\partial U_0^0} + \sum_{B=0}^{3} f^{0B} U_{B0} = \frac{\partial f}{\partial U_1^0} + \sum_{B=0}^{3} f^{1B} U_{B0} = \\ &= \frac{\partial f}{\partial U_1^1} + \sum_{B=0}^{3} f^{1B} U_{B1} = \frac{\partial f}{\partial U_2^0} + \sum_{B=0}^{3} f^{2B} U_{B0} = \\ &= \frac{\partial f}{\partial U_2^1} + \sum_{B=0}^{3} f^{2B} U_{B1} = \frac{\partial f}{\partial U_2^2} + \sum_{B=0}^{3} f^{2B} U_{B2} = \\ &= \frac{\partial f}{\partial U_3^0} + \sum_{B=0}^{3} f^{3B} U_{B0} = \frac{\partial f}{\partial U_3^1} + \sum_{B=0}^{3} f^{3B} U_{B1} = \\ &= \frac{\partial f}{\partial U_3^2} + \sum_{B=0}^{3} f^{3B} U_{B2} = \frac{\partial f}{\partial U_3^3} + \sum_{B=0}^{3} f^{3B} U_{B3}. \end{aligned}$$

Moreover the simmetry condition $f^{AB} = f^{BA}$ holds iff

(4.17) $$f^{01} = f^{10};\ f^{02} = f^{20};\ f^{12} = f^{21}.$$

Now we can see that $\sum_{A=0}^{3} p^{CA} U_{A0} = -\mathbf{A}_0^{c0}$ so that from (4.9) we obtain

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} \left(\sum_{A=0}^{3} p_{A'}^{\gamma' A} U_{A0} \right) = -\mathbf{A}_0^{C0}$$

whose solution is

$$\sum_{A=0}^{3} p_{A'}^{\gamma' A} U_{A0} = -\delta_{A'}^{0} \delta_0^{\gamma'} .$$

(By substitution it becomes evident that it is a solution; moreover the invertibility of $\mathbf{A}_{\gamma'}^{CA'}$ assures that it is the only solution).

This relation and $(4.15)_1$ prove $(4.16)_1$.

Similarly we see that

$$\sum_{A=0}^{3} q^{CA} U_{A0} = -\mathbf{A}_0^{c1}; \ \sum_{A=0}^{3} q^{CA} U_{A1} = -\mathbf{A}_1^{c1}$$

so that from (4.10) we may obtain two sistems whose solutions are

$$\sum_{A=0}^{3} q_{A'}^{\gamma' A} U_{A0} = -\delta_{A'}^{1} \delta_0^{\gamma'}; \ \sum_{A=0}^{3} q_{A'}^{\gamma' A} U_{A1} = -\delta_{A'}^{1} \delta_1^{\gamma'}$$

that by use of $(4.15)_2$ give the proof of $(4.16)_2$, $(4.16)_3$ respectively.

In the same way we have $\sum_{A=0}^{3} r^{CA} U_{A\mu} = -\mathbf{A}_\mu^{c2}$ for $\mu = 0, 1, 2$; that gives (jointly with (4.11)),

$$\sum_{A=0}^{3} r_{A'}^{\gamma' A} U_{A\mu} = -\delta_{A'}^{2} \delta_\mu^{\gamma'} .$$

This relation with the aid of $(4.15)_3$ proves $(4.16)_{4,5,6}$.

Lastly we notice that

$$\mathbf{A}_\mu^{c3} \equiv (0, 0, 0, 0, 0, 0, U_{0\mu}, U_{1\mu}, U_{2\mu}, U_{3\mu})^T$$

for $\mu = 0, 1, 2, 3$; from which and (4.1) we have

$$\sum_{\gamma'=0}^{3} \mathbf{A}_{\gamma'}^{C3} U_3^{\gamma'} \equiv (0,0,0,0,0,0,0,0,0,1)^T;$$

then it becomes evident that

$$p^{C3} U_{0\mu} + q^{C3} U_{1\mu} + r^{C3} U_{2\mu} - \sum_{\gamma'=0}^{3} \mathbf{A}_{\gamma'}^{C3} U_{3\mu} U_3^{\gamma'} = -\mathbf{A}_{\mu}^{C3};$$

But the first member is also equal to

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} (p_{A'}^{\gamma'3} U_{0\mu} + q_{A'}^{\gamma'3} U_{1\mu} + r_{A'}^{\gamma'3} U_{2\mu} - \delta_{A'}^{3} U_3^{\gamma'} U_{3\mu})$$

as it can be seen from (4.9), (4.10) and (4.11); as consequence we have that

$$p_{A'}^{\gamma'3} U_{0\mu} + q_{A'}^{\gamma'3} U_{1\mu} + r_{A'}^{\gamma'3} U_{2\mu} - \delta_{A'}^{3} U_3^{\gamma'} U_{3\mu} = -\delta_{A'}^{3} \delta^{\gamma'}{}_{\mu}$$

from which and (4.15) we have

$$\sum_{B=0}^{3} f^{3B} U_{B\mu} = f^{03} U_{0\mu} + f^{13} U_{1\mu} + f^{23} U_{2\mu} - \sum_{\gamma'=0}^{3} \frac{\partial f}{\partial U_3^{\gamma'}} U_3^{\gamma'} U_{3\mu} =$$

$$= \sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \frac{\partial f}{\partial U_{A'}^{\gamma'}} (p_{A'}^{\gamma'3} U_{0\mu} + q_{A'}^{\gamma'3} U_{1\mu} + r_{A'}^{\gamma'3} U_{2\mu} - \delta_{A'}^{3} U_3^{\gamma'} U_{3\mu}) =$$

$$= -\frac{\partial f}{\partial U_3^{\mu}}$$

thus proving $(4.16)_{7,8,9,10}$.

It remains now to prove only the conditions (4.17); to this end it is sufficient to prove that

$$p_{A'}^{\gamma'1} = q_{A'}^{\gamma'0}; \; p_{A'}^{\gamma'2} = r_{A'}^{\gamma'0}; \; q_{A'}^{\gamma'2} = r_{A'}^{\gamma'1} \tag{4.18}$$

as it can be seen from (4.15).

Now we have that $p^{C1} - q^{C0} = 0$; $p^{C2} - r^{C0} = 0$; $q^{C2} - r^{C1} = 0$ and then (4.9), (4.10), (4.11) give

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} (p_{A'}^{\gamma' 1} - q_{A'}^{\gamma' 0}) = 0$$

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} (p_{A'}^{\gamma' 2} - r_{A'}^{\gamma' 0}) = 0$$

$$\sum_{A'=0}^{3} \sum_{\gamma'=0}^{A'} \mathbf{A}_{\gamma'}^{CA'} (q_{A'}^{\gamma' 2} - r_{A'}^{\gamma' 1}) = 0$$

that prove (4.18).

I conclude now with some remarks:

In a given physical problem one may be interested only in the function $F(x_m)$ and then the dependence of f on the variables $U_{A'}^{\gamma'}$ is arbitrary; in this case the functions f^{AB} are also arbitrary (because from (4.14) we can find

$$\frac{\partial f}{\partial U_{A'}^{\gamma'}} = -\sum_{B=0}^{3} f^{A'B} U_{B\gamma'}$$

that can be inverted because, for $A' = 3$ it becomes a linear system in the unknowns f^{3B} whose matrix of the coefficients has (4.4) as determinant; likewise for $A' = 2$ the unknowns are f^{20}, f^{21}, f^{22} and the matrix of the coefficients has (4.3) as determinant; in the same way for $A' = 1$ we find f^{10}, f^{11} and for $A' = 0$ we find f^{00}).

Then if F is an unknown function, also the f^{AB} are independent unknown functions to be determined; restrictions on their generalities may obviously derive from the requirement that f is isotropic. In some problems (see [8] and [9] as examples) there is a finite set of functions f_q, λ_q (for $q = 1, \ldots, Q$) of which at least f_1 (the entropy-entropy flux vector) is unknown and $\lambda_1 = 1$; one of the requirements that must be satisfied is

(4.19)
$$\sum_{q=1}^{Q} \lambda_q \frac{\partial F_q}{\partial x_m} = 0$$

where x_m may indicate the independent components of U_A^α; from (4.14) we have now that it is equivalent to assume the existence of f_q^{AB} such that

$$\sum_{q=1}^{Q} \lambda_q \frac{\partial f_q}{\partial U_A^\mu} + \sum_{B=0}^{3} f^{AB} U_{B\mu} = 0 \tag{4.20}$$

holds for $f^{AB} = \sum_{q=1}^{Q} \lambda_q f_q^{AB}$;

Then (4.20) imposes restrictions on f_q, λ_q, f^{AB} and it is not necessary to find the functions f_q^{AB} because their contribution is all contained in f^{AB}.

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Università di Catania
Dipartimento di Matematica
Viale A. Doria 6
95125 Catania