GLOBAL CONVERGENCE AND NON EXISTENCE
OF PERIODIC POINTS OF PERIOD 4

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It is given a non trivial example of nonempty subset \( J \) of \( C^0([0,1]^2) \) such that:

Whatever \( F \in J \) be, for the pair \( ([0,1]^2, F) \) the successive approximations method converges globally (i.e. for each \( P \in [0,1]^2 \) the sequence \( (F^n(P))_{n \in \mathbb{N}} \) converges to a fixed point of \( F \)) if and only if \( F \) has no periodic point of period 4.

1. Introduction.

Let \( S \) be a compact metric space and \( C^0(S) \) the set of all continuous functions from \( S \) into itself.

If \( F \in C^0(S) \) it is said that for the pair (\( S, F \)) the successive approximations method (abbr. s.a.m.) converges globally if for each point \( x \in S \) the sequence \( (F^n(x))_{n \in \mathbb{N}} \) converges (to a fixed point of \( F \)).

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A theorem of global convergence relative to the case $S = [0, 1]$ is the following (see [1,2,7,8]):

(1.1). Whatever $F \in C^0([0, 1])$ be, for the pair $([0, 1]), F)$ the s.a.m. converges globally if and only if $F$ has no periodic point of period 2.

A theorem similar to (1.1) is the following (see [3]):

(1.2). Whatever $F \in C^0(S^1)$ ($S^1$ denotes the unit circle) be, for the pair $(S^1, F)$ the s.a.m. converges globally if and only if $F$ has no periodic point of even period.

The theorems (1.1) and (1.2) can be expressed by saying that $[0, 1]$ and $S^1$ are examples of compact metric spaces $S$ for which the following proposition is true.

A) There exists a subset $M$ of $\mathbb{N}$ such that, whatever $F \in C^0(S)$ be, for the pair $(S, F)$ the s.a.m. converges globally if and only if $F$ has no periodic point whose period belongs to $M$.

Another example of compact metric space for which the proposition A) is true is given by the following theorem (see [10]) which generalizes the theorem (1.1):

(1.3). If $X$ is an arcwise connected tree endowed with a finite number, $m$, of end points and $F \in C^0(X)$, then for the pair $(X, F)$ the s.a.m. converges globally if and only if $F$ has no periodic point whose period belongs to $\{2, \ldots, m\}$.

It is not known if the proposition A) is true when $S = [0, 1]^2$.

Then, it is interesting the problem to give meaningful examples of nonempty subsets $J$ of $C^0([0, 1]^2)$ and subsets $M$ of $\mathbb{N}$ for which the following proposition is true:

B) Whatever $F \in J$ be for the pair $([0, 1]^2, F)$ the s.a.m. converges globally if and only if $F$ has no periodic point whose period belongs to $M$.

The theorem (3.2), which will be proved in section 3, gives a solution of above problem$^{(1)}$.

$^{(1)}$ Other solutions of above problem can be obtained from theorem 2.1 of [4], the theorem of section 3 of [5] and theorem (5.3) of [6].
In fact from (3.2) trivially follows that the proposition B) is true if $M$ is equal to $\{4\}$ and $J$ is the set of all functions $F$ from $[0,1]^2$ into itself of kind:

$$F(x,y) = (f(x,y), x), \quad f \in C^0([0,1]^2,[0,1]),$$

such that:

1) $f$ is decreasing with respect to both variables;

2) Set, for each $x \in [0,1]$, $\varphi(x) = f(x, x)$ it results:

$$(\varphi(x) - x)(\varphi^2(x) - x) \geq 0 \quad \forall x \in [0,1];$$

3) There do not exist a periodic point $\xi$ of $\varphi$ of period 2 and a point $P \in [0,1]^2$ such that the point $(\xi, \varphi(\xi))$ is a cluster point of the sequence $(F^n(P))_{n \in \mathbb{N}}$.

In the end, let us observe that theorem (3.2) has been proved using some results, relate to the existence of periodic points of period 4 for a function $F$ of kind (#) satisfying 1), showed in section 2.


In what follows, $f$ represents a continuous function of $[0,1]^2$ into $[0,1]$ decreasing with respect to both variables, $F$ the function of $[0,1]^2$ into itself defined as follows:

$$F(x,y) = (f(x,y), x),$$

and $\varphi$ the function of $[0,1]$ into itself defined as follows:

$$\varphi(x) = f(x, x).$$

Let us set for each point $(x_1, x_0)$ of $[0,1]^2$ and for each positive integer $n$:

$$x_{n+1} = f(x_n, x_{n-1}).$$
In this section let us denote a periodic point of \( \varphi \) of period two by \( y \) and setting \( z = \varphi(y) \), without lose of generality, we assume that \( y \) is less than \( z \).

Set \( I = [y, z] \), it is useful bearing in mind that the square \( I^2 \) is positively invariant for \( F \), i.e; \( F(I^2) \subset I^2 \), (see (4.1) of [9]), and that, consequently, \( f(I^2) \subset I \).

In what follows it is convenient to denote the following sets:

\[
\{(t, y)\}_{t \in I}, \{(z, t)\}_{t \in I}, \{(t, z)\}_{t \in I}, \{(y, t)\}_{t \in I},
\]

respectively, by the symbols \( S_1, S_2, S_3, S_4 \).

That being stated, we consider the following property referred to a generic point \( (x_1, x_0) \) of \( S_1 \):

\[(w) \quad F(x_1, x_0) \in S_2, \; F^3(x_1, x_0) \in S_4, \; pr_1 F^4(x_1, x_0) \leq x_1.\]

(2.1). *If there exists a point \( (x_1, x_0) \) of \( S_1 \) satisfying the property \( (w) \) then there exists in \( S_1 \) a periodic point of \( F \) of period 4 satisfying the property \( (w) \).*

**Proof.** Since \( f(I^2) \subset I \), it is obvious that:

\[(1) \quad x_n \in I, \; \forall n \in \mathbb{N}_0.\]

Moreover, whatever \( n \in \mathbb{N}_0 \) be, the following propositions are equivalent:

\[(2) \quad x_{4n} = x_{4n+4} = y \leq x_{4n+5} \leq x_{4n+1} \leq x_{4n+2} = x_{4n+6} = z, \; x_{4n+3} \leq x_{4n+7}.\]

\[(3) \quad x_{4n} = x_{4n+4} = y \leq x_{4n+5} \leq x_{4n+1} \leq x_{4n+2} = z.\]

Indeed, to deduce (2) from (3) it suffices bearing in mind that \( f \) is decreasing with respect to both variables and that (1) is satisfied.

That being stated, let us prove by induction that (3) is true for each \( n \in \mathbb{N}_0 \).
Well, since \((x_1, x_0)\) satisfies \((w)\) and \(S_4\) is transformed by \(F\) into \(S_1\), it results:
\[
x_0 = x_4 = y \leq x_5 \leq x_1 \leq x_2 = z,
\]
thus (3) is true for \(n = 0\).

Assuming (3) is true for \(n = m\), it will be shown to be true for \(n = m + 1\).

By being (3) and (2) equivalent it follows that:
\[
x_{4m} = x_{4m+4} = y \leq x_{4m+5} \leq x_{4m+1} \leq x_{4m+2} = x_{4m+6} = z,
\]
(4)
\[
x_{4m+3} \leq x_{4m+7}.
\]

Then it results:
\[
x_{4m+8} = f(x_{4m+7}, x_{4m+6}) \leq f(x_{4m+3}, x_{4m+2}) = x_{4m+4} = y,
\]
hence being, from (1), \(y \leq x_{4m+8}\) we have:
(5)
\[
x_{4m+4} = x_{4m+8} = y.
\]

Consequently, the inequality \(x_{4m+3} \leq x_{4m+7}\) implies:
(6)
\[
x_{4m+9} = f(x_{4m+8}, x_{4m+7}) \leq f(x_{4m+4}, x_{4m+3}) = x_{4m+5}.
\]

From (5), (1), (6), (4) it follows that (3) is true for \(n = m + 1\).

So we have proved that (3) is true for each \(n \in N_0\).

Thus, being (2) and (3) equivalent, it follows that:
\[
x_{4n} = y, \ \forall n \in N_0; \quad (x_{4n+1})_{n \in N_0} \text{ is decreasing};
\]
\[
x_{4n+2} = z, \ \forall n \in N_0; \quad (x_{4n+3})_{n \in N_0} \text{ is increasing}.
\]

Set:
\[
c = \lim_{n \to \infty} x_{4n+1}, \quad d = \lim_{n \to \infty} x_{4n+3},
\]
it can be easily proved that the following equalities are true:
\[
c = f(y, d), \quad d = f(z, c), \quad y = f(d, z), \quad z = f(c, y),
\]
and, moreover, that the points \((y, d), (z, c), (d, z), (c, y)\) are pairwise distinct and each of the above points is a periodic point of \(F\) of period 4 satisfying the property \((w)\).

This completes the proof.

(2.2). If the condition:

\[ (\varphi(x) - x)(\varphi^2(x) - x) \geq 0, \quad \forall x \in I, \]

is satisfied, then the following statements are true:

a) \(((x_1, x_0) \in S_1 - \{(z, y)\}, F(x_1, x_0) \in \partial I^2) \Rightarrow F(x_1, x_0) \in S_2;\)

b) \(((x_1, x_0) \in S_3 - \{(y, z)\}, F(x_1, x_0) \in \partial I^2) \Rightarrow F(x_1, x_0) \in S_4.\)

Proof. We shall prove a); b) can be proved in the similar way.

Let \((x_1, x_0)\) be a point of \(S_1 - \{(z, y)\},\) i.e.:

\[ x_0 = y \leq x_1 < z. \]

If \(F(x_1, x_0) \notin S_2,\) i.e. \(z \neq x_2,\) we have \(x_1 \neq y\) (otherwise, it would result \(x_2 = f(x_1, x_0) = f(y, y) = \varphi(y) = z).\)

Then it follows that:

\[ y < x_1 < z, \]

and consequently, being \((x_2, x_1) = F(x_1, x_0) \in \partial I^2\) and \(x_2 \neq z,\) it results:

\[ x_2 = y < x_1. \]

Since, on the other hand:

\[ y \leq \varphi(x_1) = f(x_1, x_1) \leq f(x_1, x_0) = x_2, \]

we have:

\[ y = \varphi(x_1) < x_1. \]

Thus, bearing in mind \((*)\), it follows:

\[ z = \varphi(y) = \varphi^2(x_1) \leq x_1. \]
But this is in contrast with (7).

This completes the proof.

(2.3) If the condition (*) of (2.2) is satisfied and there exists a nonempty closed set $C$, positively invariant for $F$, such that:

$$C \subseteq \partial I^2 \quad \text{and} \quad C \cap \{(y, z), (z, y)\} = \emptyset,$$

then there exists a periodic point of $F$ of period 4 satisfying (w).

Proof. Set:

$$S_5 = S_1, \quad C_i = C \cap S_i, \quad \forall i \in \{1, \ldots, 5\},$$

we observe that for each $i \in \{1, \ldots, 4\}$ it results:

$$F(C_i) \subseteq C_{i+1}.$$

In fact (9) is trivial for $i \in \{2, 4\}$ and for $i \in \{1, 3\}$ it follows easily from (2.2).

Since $C \neq \emptyset$, from (9) it results:

$$C_i \neq \emptyset, \quad \forall i \in \{1, \ldots, 4\}.$$

Set:

$$x_1 = \max pr_1 C_1, \quad x_0 = y,$$

according to (9) it results:

$$F(x_1, x_0) \in S_2, \quad F^3(x_1, x_0) \in S_4, \quad pr_1 F^4(x_1, x_0) \leq x_1,$$

and consequently, from (2.1), the assertion follows.


Such being the meaning of the symbols $f$, $F$ and $\varphi$ given in section 2, let us recall (see (5.1) of [9]) that, if $\varphi$ satisfies the following condition:

$$(\ast) \quad (\varphi(x) - x)(\varphi^2(x) - x) \geq 0 \quad \forall x \in [0, 1],$$
the following statement:

iii) \( \varphi \) has no periodic point \( \xi \) of period 2 such that the boundary (in \( R^2 \)) of the square:

\[ [\min\{\xi, \varphi(\xi)\}, \max\{\xi, \varphi(\xi)\}]^2 \]

contains a closed subset of \([0, 1]^2\) invariant for \( F \),

is equivalent to global convergence of the s.a.m. for the pair \(([0, 1]^2, F)\).

Let us now consider the following condition:

I) \( \varphi \) has no periodic point \( \xi \) of period 2 such that the boundary (in \( R^2 \)) of the square:

\[ [\min\{\xi, \varphi(\xi)\}, \max\{\xi, \varphi(\xi)\}]^2 \]

contains a set \( \Omega(P) \) for some \( P \in [0, 1]^2 \), where \( \Omega(P) \) is the set of all cluster points of the sequence \( (F^n(P))_{n \in \mathbb{N}} \).

Since (see C) of [11]) \( \Omega(P) \) is a closed set invariant for \( F \), I) is weaker than iii).

Well, if \( \varphi \) satisfies (\#) also I) implies the global convergence.

Indeed, we have that:

(3.1). If \( \varphi \) satisfies (\#), for the pair \(([0, 1]^2, F)\) the s.a.m. converges globally if and only if the condition I) is satisfied.

Proof. First of all let us observe that theorem (2.1) of [9] is still true if the condition 3) is replaced by the following statement:

3') It does not exist a set of \( C - \{\{s\}\} \) whose boundary contains a set \( \Omega(x) \), for some \( x \in S \).

It can be verified changing, in obvious way, the proof of (2.1) of [9] where 3) is applied.

Consequently, also (2.2) of [9] is still true if 3) is replaced by 3'); thus (5.1) of [9] is true if iii) is replaced by I), and this shows the assertion.

Utilizing the propositions (2.3) and (3.1) let us prove that:
(3.2). If $\varphi$ satisfies (*) then for the pair $([0,1]^2,F)$ the s.a.m. converges globally if and only if the following statement is satisfied:

II) The function $F$ has no periodic point of period 4 and there do not exist a periodic point $\xi$ of $\varphi$ of period 2 and a point $P$ of $[0,1]^2$ such that:

$$ (\xi, \varphi(\xi)) \in \Omega(P), $$

where $\Omega(P)$ is the set of cluster points of the sequence $(F^n(P))_{n\in\mathbb{N}}$.

Proof. Let us suppose that II) is true. If there is not global convergence of the s.a.m. for the pair $([0,1]^2,F)$, according to (3.1), $P \in [0,1]^2$ and $a \in [0,1]$ exist, where $a = \varphi(b) < \varphi(a) = b$, such that:

\begin{equation}
\Omega(P) \subseteq \partial[a,b]^2.
\end{equation}

First of all, let us observe that, according to proposition C) of [11], $F(\Omega(P)) = \Omega(P)$ and moreover, from II) it results $(a,b) \notin \Omega(P)$ and $(b,a) \notin \Omega(P)$.

Consequently, from (1) and proposition (2.3) there exists a periodic point of $F$ of period 4, in contrast with II).

Conversely, if for the pair $([0,1]^2,F)$ the s.a.m. converges globally, then periodic points of $F$ do not exist and moreover, for each point $P \in [0,1]^2$, it results:

$$ \Omega(P) = \{(\alpha, \alpha)\}, $$

$(\alpha, \alpha)$ being the unique fixed point of $F$, thus the condition II) is verified.

The theorem is so proved.

REFERENCES


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