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GLOBAL CONVERGENCE AND NON EXISTENCE OF PERIODIC POINTS OF PERIOD 4

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It is given a non trivial example of nonempty subset $\mathcal{J}$ of $C^0([0,1]^2)$ such that:

Whatever $F \in \mathcal{J}$ be, for the pair $([0,1]^2, F)$ the successive approximations method converges globally (i.e. for each $P \in [0,1]^2$ the sequence $(F^n(P))_{n\in\mathbb{N}}$ converges to a fixed point of F) if and only if F has no periodic point of period 4.

1. Introduction.

Let S be a compact metric space and $C^0(S)$ the set of all continuous functions from S into itself.

If $F \in C^0(S)$ it is said that *for the pair* (S, F) *the successive approximations method* (abbr. *s.a.m.*) *converges globally if for each point* $x \in S$ *the sequence* $(F^n(x))_{n\in\mathbb{N}}$ *converges* (to a fixed point of F).

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A theorem of global convergence relative to the case $S = [0,1]$ is the following (see [1,2,7,8]):

(1.1). *Whatever* $F \in C^0([0,1])$ *be, for the pair* $([0,1]), F)$ *the s.a.m. converges globally if and only if* F *has no periodic point of period 2.*

A theorem similar to (1.1) is the following (see [3]):

(1.2). *Whatever* $F \in C^0(S^1)$ *(*S^1 *denotes the unit circle) be, for the pair* (S^1, F) *the s.a.m. converges globally if and only if* F *has no periodic point of even period.*

The theorems (1.1) and (1.2) can be espressed by saying that $[0,1]$ and S^1 are examples of compact metric spaces S for which the following proposition is true.

A) *There exists a subset* M *of* $\mathbb{N}$ *such that, whatever* $F \in C^0(S)$ *be, for the pair* (S, F) *the s.a.m. converges globally if and only if* F *has no periodic point whose period belongs to* M.

Another example of compact metric space for which the proposition A) is true is given by the following theorem (see [10]) which generalizes the theorem (1.1):

(1.3). *If* X *is an arcwise connected tree endowed with a finite number,* m*, of end points and* $F \in C^0(X)$*, then for the pair* (X, F) *the s.a.m. converges globally if and only if* F *has no periodic point whose period belongs to* $\{2, \ldots, m\}$.

It is not known if the proposition A) is true when $S = [0,1]^2$.

Then, it is interesting the problem to give meaningful examples of nonempty subsets $\mathcal{J}$ of $C^0([0,1]^2)$ and subsets M of $\mathbb{N}$ for which the following proposition is true:

B) *Whatever* $F \in \mathcal{J}$ *be for the pair* $([0,1]^2, F)$ *the s.a.m. converges globally if and only if* F *has no periodic point whose period belongs to* M.

The theorem (3.2), which will be proved in section 3, gives a solution of above problem(1).

(1) Other solutions of above problem can be obtained from theorem 2.1 of [4], the theorem of section 3 of [5] and theorem (5.3) of [6].

In fact from (3.2) trivially follows that the proposition B) is true if M is equal to $\{4\}$ and $\mathcal{J}$ is the set of all functions F from $[0,1]^2$ into itself of kind:

$$(\#) \qquad F(x,y) = (f(x,y), x), \quad f \in C^0([0,1]^2, [0,1]),$$

such that:

1) f is decreasing with respect to both variables;

2) Set, for eqach $x \in [0,1]$, $\varphi(x) = f(x,x)$ it results:

$$(\varphi(x) - x)(\varphi^2(x) - x) \geq 0 \quad \forall x \in [0,1];$$

3) There do not exist a periodic point ξ of φ of period 2 and a point $P \in [0,1]^2$ such that the point $(\xi, \varphi(\xi))$ is a cluster point of the sequence $(F^n(P))_{n \in \mathbb{N}}$.

In the end, let us observe that theorem (3.2) has been proved using some results, relate to the existence of periodic points of period 4 for a function F of kind (#) satisfying 1), showed in section 2.

2. Existence of periodic points of F of period 4.

In what follows, f represents a continuous function of $[0,1]^2$ into $[0,1]$ decreasing with respect to both variables, F the function of $[0,1]^2$ into itself defined as follows:

$$F(x,y) = (f(x,y), x),$$

and φ the function of $[0,1]$ into itself defined as follows:

$$\varphi(x) = f(x,x).$$

Let us set for each point (x_1, x_0) of $[0,1]^2$ and for each positive integer n:

$$x_{n+1} = f(x_n, x_{n-1}).$$

In this section let us denote a periodic point of φ of period two by y and setting $z = \varphi(y)$, without lose of generality, we assume that y is less than z.

Set $I = [y, z]$, it is useful bearing in mind that the square I^2 is positively invariant for F, i.e; $F(I^2) \subseteq I^2$, (see (4.1) of [9]), and that, consequently, $f(I^2) \subseteq I$.

In what follows it is convenient to denote the following sets:

$$\{(t, y)\}_{t\in I},\ \{(z, t)\}_{t\in I},\ \{(t, z)\}_{t\in I},\ \{(y, t)\}_{t\in I},$$

respectively, by the simbols S_1, S_2, S_3, S_4.

That being stated, we consider the following property referred to a generic point (x_1, x_0) of S_1:

$$(w) \qquad F(x_1, x_0) \in S_2,\ F^3(x_1, x_0) \in S_4,\ pr_1F^4(x_1, x_0) \leq x_1.$$

(2.1). *If there exists a point* (x_1, x_0) *of* S_1 *satisfying the property* (w) *then there exists in* S_1 *a periodic point of* F *of period 4 satisfying the property* (w).

Proof. Since $f(I^2) \subseteq I$, it is obvious that:

$$(1) \qquad x_n \in I,\ \forall n \in \mathbb{N}_0.$$

Moreover, whatever $n \in \mathbb{N}_0$ be, the following propositions are equivalent:

$$(2) \quad x_{4n} = x_{4n+4} = y \leq x_{4n+5} \leq x_{4n+1} \leq x_{4n+2} = x_{4n+6} = z,\ x_{4n+3} \leq x_{4n+7}.$$

$$(3) \qquad x_{4n} = x_{4n+4} = y \leq x_{4n+5} \leq x_{4n+1} \leq x_{4n+2} = z.$$

Indeed, to deduce (2) from (3) it suffices bearing in mind that f is decreasing with respect to both variables and that (1) is satisfied.

That being stated, let us prove by induction that (3) is true for each $n \in N_0$.

Well, since (x_1, x_0) satisfies (w) and S_4 is transformed by F into S_1, it results:

$$x_0 = x_4 = y \leq x_5 \leq x_1 \leq x_2 = z,$$

thus (3) is true for $n = 0$.

Assuming (3) is true for $n = m$, it will be shown to be true for $n = m + 1$.

By being (3) and (2) equivalent it follows that:

$$(4) \qquad \begin{aligned} &x_{4m} = x_{4m+4} = y \leq x_{4m+5} \leq x_{4m+1} \leq x_{4m+2} = x_{4m+6} = z, \\ &x_{4m+3} \leq x_{4m+7}. \end{aligned}$$

Then it results:

$$x_{4m+8} = f(x_{4m+7}, x_{4m+6}) \leq f(x_{4m+3}, x_{4m+2}) = x_{4m+4} = y,$$

hence being, from (1), $y \leq x_{4m+8}$ we have:

$$(5) \qquad x_{4m+4} = x_{4m+8} = y.$$

Consequently, the inequality $x_{4m+3} \leq x_{4m+7}$ implies:

$$(6) \qquad x_{4m+9} = f(x_{4m+8}, x_{4m+7}) \leq f(x_{4m+4}, x_{4m+3}) = x_{4m+5}.$$

From (5), (1), (6), (4) it follows that (3) is true for $n = m + 1$.

So we have proved that (3) is true for each $n \in N_0$.

Thus, being (2) and (3) equivalent, it follows that:

$$x_{4n} = y, \ \forall n \in \mathbb{N}_0; \quad (x_{4n+1})_{n \in \mathbb{N}_0} \text{ is decreasing;}$$

$$x_{4n+2} = z, \ \forall n \in \mathbb{N}_0; \quad (x_{4n+3})_{n \in \mathbb{N}_0} \text{ is increasing.}$$

Set:

$$c = \lim_n x_{4n+1}, \quad d = \lim_n x_{4n+3},$$

it can be easily proved that the following equalities are true:

$$c = f(y, d), \ d = f(z, c), \ y = f(d, z), \ z = f(c, y),$$

and, moreover, that the points (y,d), (z,c), (d,z), (c,y) are pairwise distinct and each of the above points is a periodic point of F of period 4 satisfying the property (w).

This completes the proof.

(2.2). *If the condition:*

$$(*) \qquad (\varphi(x)-x)(\varphi^2(x)-x) \geq 0, \quad \forall x \in I,$$

is satisfied, then the following statements are true:

a) $((x_1,x_0) \in S_1 - \{(z,y)\}, F(x_1,x_0) \in \partial I^2) \Rightarrow F(x_1,x_0) \in S_2$;

b) $((x_1,x_0) \in S_3 - \{(y,z)\}, F(x_1,x_0) \in \partial I^2) \Rightarrow F(x_1,x_0) \in S_4$.

Proof. We shall prove a); b) can be proved in the similar way.

Let (x_1,x_0) be a point of $S_1 - \{(z,y)\}$, i.e.:

$$x_0 = y \leq x_1 < z.$$

If $F(x_1,x_0) \notin S_2$, i.e. $z \neq x_2$, we have $x_1 \neq y$ (otherwise, it would result $x_2 = f(x_1,x_0) = f(y,y) = \varphi(y) = z$).

Then it follows that:

$$(7) \qquad y < x_1 < z,$$

and consequently, being $(x_2,x_1) = F(x_1,x_0) \in \partial I^2$ and $x_2 \neq z$, it results:

$$x_2 = y < x_1.$$

Since, on the other hand:

$$y \leq \varphi(x_1) = f(x_1,x_1) \leq f(x_1,x_0) = x_2,$$

we have:

$$y = \varphi(x_1) < x_1.$$

Thus, bearing in mind $(*)$, it follows:

$$z = \varphi(y) = \varphi^2(x_1) \leq x_1.$$

But this is in contrast with (7).

This completes the proof.

(2.3). *If the condition* $(*)$ *of (2.2) is satisfied and there exists a nonempty closed set* C*, positively invariant for* F*, such that:*

(8). $$C \subseteq \partial I^2 \quad \text{and} \quad C \cap \{(y,z),(z,y)\} = \phi,$$

then there exists a periodic point of F *of period 4 satisfying* (w).

Proof. Set:

$$S_5 = S_1, \; C_i = C \cap S_i, \quad \forall i \in \{1,\dots,5\},$$

we observe that for each $i \in \{1,\dots,4\}$ it results:

(9) $$F(C_i) \subseteq C_{i+1}.$$

In fact (9) is trivial for $i \in \{2,4\}$ and for $i \in \{1,3\}$ it follows easily from (2.2).

Since $C \neq \phi$, from (9) it results:

$$C_i \neq \phi, \quad \forall i \in \{1,\dots,4\}.$$

Set:

$$x_1 = \max pr_1 C_1, \; x_0 = y,$$

according to (9) it results:

$$F(x_1,x_0) \in S_2, \; F^3(x_1,x_0) \in S_4, \; pr_1 F^4(x_1,x_0) \leq x_1,$$

and consequently, from (2.1), the assertion follows.

3. Global convergence.

Such being the meaning of the symbols f, F and φ given in section 2, let us recall (see (5.1) of [9]) that, if φ satisfies the following condition:

$(*)$ $$(\varphi(x)-x)(\varphi^2(x)-x) \geq 0 \qquad \forall x \in [0,1],$$

the following statement:

iii) *φ has no periodic point ξ of period 2 such that the boundary (in R^2) of the square:*

$$[min\{\xi, \varphi(\xi)\}, max\{\xi, \varphi(\xi)\}]^2$$

contains a closed subset of $[0, 1]^2$ *invariant for F,*

is equivalent to global convergence of the *s.a.m.* for the pair $([0, 1]^2, F)$.

Let us now consider the following condition:

I) *φ has no periodic point ξ of period 2 such that the boundary (in R^2) of the square:*

$$[min\{\xi, \varphi(\xi)\}, max\{\xi, \varphi(\xi)\}]^2$$

contains a set $\Omega(P)$ for some $P \in [0, 1]^2$, where $\Omega(P)$ is the set of all cluster points of the sequence $(F^n(P))_{n\in\mathbb{N}}$.

Since (see C) of [11]) $\Omega(P)$ is a closed set invariant for F, I) is weaker than iii).

Well, if φ satisfies $(*)$ also I) implies the global convergence.

Indeed, we have that:

(3.1). *If φ satisfies $(*)$, for the pair $([0, 1]^2, F)$ the s.a.m. converges globally if and only if the condition I) is satisfied.*

Proof. First of all let us observe that theorem (2.1) of [9] is still true if the condition 3) is replaced by the following statement:

3') *It does not exist a set of $C - \{\{a\}\}$ whose boundary contains a set $\Omega(x)$, for some $x \in S$.*

It can be verified changing, in obvious way, the proof of (2.1) of [9] where 3) is applied.

Consequently, also (2.2) of [9] is still true if 3) is replaced by 3'); thus (5.1) of [9] is true if iii) is replaced by I), and this shows the assertion.

Utilizing the propositions (2.3) and (3.1) let us prove that:

(3.2). *If* φ *satisfies* $(*)$ *then for the pair* $([0,1]^2, F)$ *the s.a.m. converges globally if and only if the following statement is satisfied:*

II) The function F *has no periodic point of period 4 and there do not exist a periodic point* ξ *of* φ *of period 2 and a point* P *of* $[0,1]^2$ *such that:*

$$(\xi, \varphi(\xi)) \in \Omega(P),$$

where $\Omega(P)$ *is the set of cluster points of the sequence* $(F^n(P))_{n\in\mathbb{N}}$.

Proof. Let us suppose that II) is true. If there is not global convergence of the *s.a.m.* for the pair $([0,1]^2, F)$, according to (3.1), $P \in [0,1]^2$ and $a \in [0,1]$ exist, where $a = \varphi(b) < \varphi(a) = b$, such that:

$$\Omega(P) \subseteq \partial[a,b]^2. \tag{1}$$

First of all, let us observe that, according to proposition C) of [11], $F(\Omega(P)) = \Omega(P)$ and moreover, from II) it results $(a,b) \notin \Omega(P)$ and $(b,a) \notin \Omega(P)$.

Consequently, from (1) and proposition (2.3) there exists a periodic point of F of period 4, in contrast with II).

Conversely, if for the pair $([0,1]^2, F)$ the *s.a.m.* converges globally, then periodic points of F do not exist and moreover, for each point $P \in [0,1]^2$, it results:

$$\Omega(P) = \{(\alpha, \alpha)\},$$

(α, α) being the unique fixed point of F, thus the condition II) is verified.

The theorem is so proved.

REFERENCES

[1] Bashurov V.V., Ogibin V.N., *Conditions for the convergence of iterative processes on the real axis,* U.S.S.R. Comp. Math. and Math. Phys. **6,5**, (1966), 178-184.

[2] Chu S.C., Moyer R.D., *On continuous functions, commuting functions and fixed points,* Fund. Math. **59**, (1966), 91-95.

[3] Di Lena G., *Global Convergence of the Method of Successive Approximations on* S^1, J. Math. Anal. and Applic., Vol. 106, n. 1; February 15, (1985), 196-201.

[4] Di Lena G., *Sulla convergenza del procedimento iterativo a due passi e metodo della secante,* preprint.

[5] Di Lena G., Messano B., Zitarosa A., *On the iterative process* $x_{n+1} = f(x_n, x_{n-1})$, Rend. del Semin. Mat. dell'Università di Padova Vol. 80 (1988), 139-150.

[6] Di Lena G., Messano B., Zitarosa A., *On the generalized successive approximations method,* Calcolo Vol. 25, July-September 1988, n. 3, 249-267.

[7] Di Lena G., Peluso R.I., *A characterization of global convergence for fixed point iteration in* R^1, Pubbl. I.A.C. Serie III, n. 133, 3-11 (1978).

[8] Di Lena G., Peluso R.I., *Sulla convergenza del metodo delle approssimazioni successive in* R^1, Calcolo XVIII, (1980), 313-319.

[9] Messano B., *On the successive approximations method and on the iterative process* $x_{n+1} = f(x_n, x_{n-1})$, Rendiconti di Matematica, Serie VII, Vol. 7, Fasc. 3-4, luglio-dicembre 1987, 281-297.

[10] Messano B., *Continuous function of an arcwise connected tree into itself: periodic points, global convergence, plus-global convergence,* to appear in Ricerche di Matematica.

[11] Messano B., Zitarosa A., *Sul metodo delle approssimazioni successive: convergenza globale e plus-convergenza globale,* Rend. del Circ. Matem. di Palermo, Serie II, Tomo XXXVII (1988), 246-260.

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