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REGULARIZATION OF NON-LINEAR MEASURE DIFFERENTIAL EQUATIONS

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The Cauchy problem for a non-linear system of regularized measure differential equations is studied. It is proved that the solutions of the regularized problems converges to a solution of an integral equation involving measures. In general this equation differs form the formal one corresponding to the original masure differential equation. This theorem generalizes a special result by Kurzweil [2] and a linear result by the author [5]. Sesekin [10] has proved an analogous theorem for a more restricted system in a wider class of regularizations.

1. Introduction.

Let μ and η be signed Borel measures on $\mathbf{R}$, see Royden [9], and let g be a continuous real valued function on $\mathbf{R}$. Let $d \in \mathbf{R}$. Let $\int_{a^+}^{a} = \int_{(a,b]}$. If in a model one describes concentrated phenomena as point phenomena one may be lead to study the Cauchy problem

$$u' + g(u)\mu = \eta,\ u(0) = d. \tag{1.1}$$

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This is formally equivalent to

$$u(x) = d - \int_{0^+}^{x} g(u(s))d\mu(s) + \int_{0^+}^{x} d\eta(s), \;\; x \geq 0, \tag{1.2}$$

and

$$u(x) = d + \int_{x^+}^{0} g(u(s))d\mu(s) - \int_{x^+}^{0} d\eta(s), \;\; x < 0. \tag{1.3}$$

It is then interesting to study the effect of regularization of (1.2) and (1.3). If one regularizes the measures in (1.2) and (1.3) and lets the regularized measures tend to the unregularized ones then solutions of the regularized problems tend to a limit which in general differs from the solution of (1.2)-(1.3) if $u(x)$ exists at all. I call this phenomenon the paradox of measure differential equations. The purpose of this note is to prove the existence of the above limit and to exhibit the modification of (1.2)-(1.3) having this limit as its solution. One should notice that one has continuous dependence of parameters for the limit together with the existence of the limit. This is not explicitly proved but could easily be done.

The linear n-dimensional case has already been treated in Persson [5]. In 1958 J. Kurzweil [2, Theorem 5,1] treated the case when μ is a Dirac measure and η is a function. The result is expressed in terms of solutions of his generalized differential equations. As to the jump in the limit it is the same as we get here. In the mean time there seems to have been no treatment of the regularization problem for measure differential equations. In part this can be justified when the limit of the regularized problems solves the unmodified unregularized problem corresponding to (1.2)-(1.3). In [3] one uses known results for the Sturm-Liouville theory to get analogous results for the vibrating string with point masses by a limiting process of regularized problems. In [6] and [7] the regularization of the wave equation with a measure as a potential is treated. At last in [8] it is proved that if one regularizes linear distribution differential equations the solution of the regularized Cauchy problem tends to the solution of the original one except when the equation is a measure differential equation not having this property according to [5]. In [4] one finds results which have been useful for the proofs in the present paper although the effects of regularization is not included. As is already said in [5] it was Einar Mjølhus at Tromsø who initiated the present author in his search for a solution of the paradox of measure differential equations. See also O. Hájek [1].

Our main theorem, Theorem 2.3, is stated and commented in Section 2. The proof is first described in Section 3. In Section 4 one finds all lemmas and intermediate theorems then used in Section 5 for the proof of Theorem 2.3.

Remark. After the first version of this note was completed we learned that Sesekin [10], [11], has already treated the regularization problem of measure differential equations. See the remark below Theorem 2.3.

2. Preliminaries and main theorem.

We let the space of signed Borel measures on $\mathbf{R}$ be denoted by $\mathcal{P}^0$. Let D denote distribution differentiation. If f is a right continuous real valued function such that $Df \in \mathcal{P}^0$ then f is said to be in $\mathcal{P}^1$. It is clear that $f \in \mathcal{P}^1$ and $\gamma \in \mathcal{P}^0$ implies that $f\gamma \in \mathcal{P}^0$. The same applies if f is Borel measurable and locally bounded.

Let ϕ be real valued continuous such that $\phi(x) > 0$, $|x| < 1$, $\phi(x) = 0$, $|x| \geq 1$, and $\int \phi = 1$. Then let $\varepsilon > 0$ and let $\phi(x, \varepsilon) = \varepsilon^{-1}\phi((x - \varepsilon)/\varepsilon)$. If $\gamma \in \mathcal{P}^0$ we let $\gamma(x, \varepsilon) = \int \phi(x - s, \varepsilon) d\gamma(s)$. We also write $d\gamma(s, \varepsilon) = \gamma(s, \varepsilon)ds$. One notices that

$$\int_0^x d\gamma(s,\varepsilon) \to \int_{0+}^x d\gamma(s),\ x > 0,\ \text{and} \int_0^x d\gamma(s,\varepsilon) \to -\int_{x^+}^0 d\gamma(s),\ x > 0,$$

when $\varepsilon \to 0$. We also notices that for d a constant

$$f(x) = d - \int_{x^+}^{0} d\gamma(s),\ x < 0,\ f(x) = d + \int_{0+}^{x} d\gamma(s),\ x \geq 0,$$

is in $\mathcal{P}^1$ and that $Df = \gamma$.

We have to impose some restrictions on the function g in (1.1). We want to have uniqueness in the Cauchy problem and we want to have global solutions. We choose the simplest of such conditions.

DEFINITION 2.1. *Let g be a continuous function on $\mathbf{R} \times \mathbf{R}^n$ with values in $\mathbf{R}^n$. To each $c > 0$ there is an $M > 0$ such that*

$$|g(x,y) - g(x,y')| \leq M|y - y'|, y, y' \in \mathbf{R}^n, |x| \leq c. \tag{2.1}$$

Such a g is said to be in the class $\mathcal{G}$.

If $g \in \mathcal{G}$ and $u \in (\mathcal{P}^1)^n$ then $g(s, u(s))$ is Borel measurable. Let μ be an $n \times n$ matrix with elements in $\mathcal{P}^0$ and let η be in $(\mathcal{P}^0)^n$. Then also $-\mu g(\cdot, u(\cdot)) + \eta$ is in $(\mathcal{P}^0)^n$.

We use l^1 norms on both $\mathbf{R}^n$ and on $n \times n$ matrices regarded as elements in $\mathbf{R}^{n^2}$. We know that

$$\mu = \mu_0 + \sum_{j=1}^{\infty} \mu(\{x_j\})\delta_{x_j}$$

and

$$\eta = \eta_0 + \sum_{j=1}^{\infty} \eta(\{x_j\})\delta_{x_j}$$

Here x_j is a set of real numbers such that $x_j \neq x_{j'}$, $j \neq j'$, and δ_{x_j} is the Dirac measure at $x = x_j$, $\mu_0(\{x\}) = \eta_0(\{x\}) = 0$, $x \in \mathbf{R}$. Further $\sum(|\mu| + |\eta|)(\{x_j\})$ is convergent when summed over those j with x_j in a compact set.

We shall study the problem

$$u(x,\varepsilon) = d - \int_0^x d\mu(s,\varepsilon)g(s, u(x,\varepsilon)) + \int_0^x d\eta(s,\varepsilon), \tag{2.2}$$

DEFINITION 2.2. *Let f be a function. Let $f(x^-) = \lim f(t)$, $t < x$. Let $u \in (\mathcal{P}^1)^n$ be such that its jumps are defined by*

$$dw/ds(s) = -\mu(\{x_j\})g(x_j, w(s)) + \eta(\{x_j\}), \tag{2.3}$$

$$w(0) = u(x_j^-),\ w(1) = u(x_j),\ j = 1, 2, \dots .$$

We define the measure $\lambda(u, \mu, \eta)$ by

$$\lambda(u, \mu, \eta)(A) = (-\mu g(\cdot, u(\cdot)) + \eta)(A), \tag{2.4}$$

$$A \subset \mathbf{R}^n,\ A \text{ Borel set},\ x_j \notin A,\ j = 1, 2, \dots ,$$

and

$$\lambda(u, \mu, \eta)(\{x_j\}) = u(x_j) - u(x_j^-),\ j = 1, 2, \dots . \tag{2.5}$$

THEOREM 2.3. *Let $g \in \mathcal{G}$, let $d \in \mathbf{R}^n$, let μ be an $n \times n$ matrix with elments in $\mathcal{P}^0$ and let $\eta \in (\mathcal{P}^0)^n$. Let λ be given by Definition 2.2. Then the solutions $u(x, \varepsilon)$ of (2.2) tend to a limit $u(x) \in (\mathcal{P}^1)^n$ when $\varepsilon \to 0$ for all $x \in \mathbf{R}$. Further u is the unique solution in $(\mathcal{P}^1)^n$ of*

$$u(x) = d + \int_{0^+}^{x} d\lambda(u, \mu, \eta)(s),\ x \geq 0, \tag{2.6}$$

and

$$u(x) = d - \int_{x^+}^{x} d\lambda(u, \mu, \eta)(s),\ x < 0, \tag{2.7}$$

Remark. The solution is such that λ is well defined. If $g(x, u)(x)) = u(x)$ we are in the linear case of [5]. Let $n = 1$, $a = \mu(\{s\}) \neq 0$ and $b = \eta(\{s\})$. Then we get

$$\lambda(u, \mu, \eta)(\{s\}) = a^{-1}(e^a - 1)(-au(s) + b);$$

just as in [5, Theorem 3.1]. Sesekin treats the non-linear problem for systems in [10]. In [11] he treats a scalar differential equation or rather integral equation. In both cases Sesekin works in a wider class of regularizations. So his systems corresponding to Theorem 2.3 are less general. And even very simple scalar linear equations covered by [5] are excluded. Either one must accept these restrictions on the underlying models or restrict the admissible class of regularizations.

3. How goes the proof.

We shall prove Theorm 2.3 in several steps. Let

$$\mu_N = \mu - \sum_{j=N+1}^{\infty} \mu(\{x_j\})\delta_{x_j}, \tag{3.1}$$

and

$$\eta_N = \eta - \sum_{j=N+1}^{\infty} \eta(\{x_j\})\delta_{x_j}, \tag{3.2}$$

The solution of (2.2) when μ is replaced by μ_N and η by η_N is called $u_N(x,\varepsilon)$. The corresponding solution of (2.6)-(2.7) is called u_N.

At first one proves that $u(x,\varepsilon)$ and $u_N(x,\varepsilon)$ are equibounded on compact sets for ε, $0<\varepsilon\le 1$, and all N. Then one proves that $u_N(x)$ exists as a limit of $u_N(x,\varepsilon)$ when $\varepsilon\to 0$. In the same step one shows that u_N solves (2.6)-(2.7) with μ replaced by μ_N and η by η_N. The difference $u(x,\varepsilon)-u_N(x,\varepsilon)$ can be made arbitrarily small on compact sets for all big N uniformly in ε, $0<\varepsilon\le 1$. Adding this to the fact that $u_N(x,\varepsilon)\to u_N(x)$, $\varepsilon\to 0$, shows that $u(x,\varepsilon)$ is a pointwise converging Cauchy family when $\varepsilon\to 0$, with a limit $u(x,0)$. Then it is clear that $u_N(x)\to u(x,0)$, $N\to\infty$. One also proves that the sequence u_N is equibounded on compact sets. All this is done in Section 4. Finally one proves the full Theorem 2.3 in Section 5.

4. Intermediate theorems and lemmata.

Let

$$u_0(x,\varepsilon) = d - \int_0^x d\mu(s,\varepsilon)g(s,d) + \int_0^x d\eta(s,\varepsilon), \tag{4.1}$$

and let

$$u_{j+1}(x,\varepsilon) = d - \int_0^x d\mu(s,\varepsilon)g(s,u_j(s;\varepsilon)) + \int_0^x d\eta(s,\varepsilon),\ j = 0,1,2,\dots. \tag{4.2}$$

Let $c > 0$ be fixed. Define $L = \max|g(s,d)|$, $|s| \le c$, and

$$C = |d| + \int_{-c-1}^{c+1} Ld|\mu|(s) + \int_{-c-1}^{c+1} d|\eta|(s). \tag{4.3}$$

We take M from (2.1) and assert that

$$|u_j(x) - u_{j-1}(x)| \le 2^{-j}C\exp\left(\left|2M\int_0^x d|\mu|(s,\varepsilon)\right|\right), \quad |x| \le c,\ j = 0,1,2,\dots. \tag{4.4}$$

It is true for $j = 0$ because of (4.3) if $u_{-1} = 0$. Let it be true for a certain j. Then (4.2) and (4.4) give

$$\begin{aligned} |u_{j+1}(x,\varepsilon) - u_j(x,\varepsilon)| &= \left|\int_0^x (g(u_j(s,\varepsilon) - g(u_{j-1}(s,\varepsilon))d\mu(s,\varepsilon)\right| \\ &\le \left|\int_0^x M|u_j(s,\varepsilon) - u_{j-1}(s,\varepsilon)||\mu|(s,\varepsilon)ds\right| \\ &\le \left|2^{-j}C\int_0^x M|\mu|(s,\varepsilon)\exp\left(2M\left|\int_0^s |\mu|(t,\varepsilon)dt\right|\right)ds\right| \\ &\le 2^{-j-1}C\exp\left(2M\left|\int_0^x |\mu|(s,\varepsilon)ds\right|\right). \end{aligned} \tag{4.5}$$

It follows that $u_j(x,\varepsilon)$ converges towards a solution $u(x,\varepsilon)$ of (2.2) for $|x| \le c$ and that

$$|u(x,\varepsilon)| \le 2C\exp\left(2M\int_{-c-1}^{c+1} d|\mu|(s)\right),\ |x| \le c,\ 0 < \varepsilon \le 1. \tag{4.6}$$

We realize that the same bound applies to $u_N(x,\varepsilon)$ for all N and $0<\varepsilon\le 1$. We get

LEMMA 4.1. *The families* $u(x,\varepsilon)$ *and* $u_N(x,\varepsilon)$ *are equibonded on compact sets.*

We also get

LEMMA 4.2. *Let* $c>0$. *To each* $\varepsilon'>0$ *there is an* N' *such that*

$$|u(x,\varepsilon)-u_N(x,\varepsilon)|<\varepsilon', |x|\le c,\ 0<\varepsilon\le 1,\ N\ge N'. \tag{4.7}$$

Proof. Let $v_N(x,\varepsilon)=u(x,\varepsilon)-u_N(x,\varepsilon)$. Then

$$\begin{aligned} v_N(x,\varepsilon) &= \int_o^x d\mu(s,\varepsilon)(g(s,u(s,\varepsilon))-g(s,u_N(s,\varepsilon))) \\ &+\int_0^x (d\mu(x,\varepsilon)-d_{\mu_N}(s,\varepsilon))g(s,u_N(s,\varepsilon)). \end{aligned} \tag{4.8}$$

Because of Lemma 4.1 there is a $C>0$ such that

$$|g(s,u_N(s,\varepsilon))|\le C,\ |s|\le c,\ 0<\varepsilon\le 1,\ \text{all } N.$$

It follows that

$$\begin{aligned} &\int_0^x |g(s,u_N(s,\varepsilon))|d|\mu(s,\varepsilon)-\mu_N(s,\varepsilon))|\le \\ &\le C\int_{-c-1}^{c+1} d|\mu-\mu_N|(s)=C(N). \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) gives

$$|v_N(x,\varepsilon)|\le \left|\int_0^x M|v_N(s,\varepsilon)|d|\mu|(s,\varepsilon)\right|+C(N). \tag{4.10}$$

Since $C(N)\to 0$ when $N\to\infty$ the same applies to $|v_N(x,\varepsilon)|$ uniformly in $|x|\le c$ because of (4.8) and Grönwall's lemma giving

$$|v_N(x,\varepsilon)|\le C(N)\exp\left(\left|\int_0^x Md|\mu|(s,\varepsilon)\right|\right)\le C(N)\exp\left(\int_{-c-1}^{c+1} Md|\mu|(s)\right).$$

The lemma is proved.

THEOREM 4.3. *Replace μ by μ_N and η by η_N in (2.3)-(2.5). Then to each N there is a unique solution u_N of the modified equations (2.6)-(2.7). Further the solution $u_N(x,\varepsilon)$ of the modified equation (2.2) converges pointwise towards $u_N(x)$.*

Proof. In order to simplify the notation we assume that $\mu = \mu_N$ and $\eta = \eta_N$. Then we let $c > 0$ be fixed. We can now choose numbers a_j, $j = 0, \pm 1, \pm 2, \ldots$, such that $a_{j-1} < a_j$ and $|a_j| \to \infty$, $|j| \to \infty$ fulfilling

$$(|\mu| + |\eta|)(\{x\}) = 0, \ x \neq a_j. \tag{4.11}$$

Let $I_j = (a_{j-1}, a_j)$ and let $c > 0$ be fixed. In addition we assume that we have chosen the numbers a_j such that $a_0 = 0$ and such that

$$|\mu|(I_j) < (2M)^{-1}, \ I_j \subset \{x; |x| \leq c\}, \tag{4.12}$$

further modifying the a_j such that $a_j = c$ and $a_{j'} = -c$ for some j and j'. Let

$$u_0(x) = d + \int_{0^+}^{x} d\lambda(d, \mu, \eta)(s), x \geq 0, \tag{4.13}$$

$$u_0(x) = d - \int_{x^+}^{0} d\lambda(d, \mu, \eta)(s), \ x < 0, \tag{4.14}$$

and

$$u_{j+1}(x) = d + \int_{0^+}^{x} d\lambda(u_j, \mu, \eta)(s), x \geq 0, \tag{4.15}$$

$$u_{j+1}(x) = d - \int_{x^+}^{0} d\lambda(u_j, \mu, \eta)(s), \ x < 0. \tag{4.16}$$

We let $a_0 = 0$. We start by looking at the interval $0 \leq x < a_1$. We notice that λ is well defined in that interval. Let

$$C = |d| + \int_0^{a_1} |g(s,d)| d|\mu|(s) + \int_0^{a_1} d|\eta|(s), \tag{4.17}$$

We assert that with $u_{-1} = 0$

$$|u_j(x) - u_{j-1}(x)| \leq C2^{-j}, \; 0 \leq x < a_1, \; j = 0, 1, 2, \ldots, \tag{4.18}$$

It follows from (4.13) and (4.17) that (4.18) is true for $j = 0$. Let it be true for a certain j.

We use (4.15) and (4.15) with j replaced by $j-1$, (2.2), (2.7), (4.12) and (4.18). We get

$$|u_{j+1}(x) - u_j(x)| \leq \int_{0^+}^{x} M|u_j(s) - u_{j-1}(s)| d|\mu|(s) \leq C2^{-j-1}.$$

That means that $u_j \to u$ solving (4.13) for $0 < x < a_1$. If $(|\mu| + |\eta|)(\{a_1\}) = 0$ then we solve (4.13) in an interval $a_1 < x < a_2$ with d replaced by $u(a_1)$ which then of course exists since we have uniform convergence of continuous functions in $0 < x \leq a_1$. At last we might arrive at an a_k with $(|\mu| + |\eta|)(\{a_k\}) \neq 0$. We let it be a_1 here. Then we define $u(a_1) = w(1)$ with w from (2.3) when $x_j = a_1$ and $w(0) = u(a_1^-)$. We get the right jump at $x = a_1$. We continue by solving (4.13) in (a_1, a_2) with d replaced by $u(a_1)$ and 0 by a_1 etc. In this way we solve (2.6) in $0 \leq x \leq c$.

In the other direction we start by letting $u(0^-) = w(0)$ where w is taken from (2.3) with $x_j = 0$ and $w(1) = d$. Then we solve (2.7) in $a_{-1} \leq x < 0$ just as we did with (2.6) in the positive direction. We let $u(a_{-1}^-) = w(0)$, where w is taken from (2.3) with $x_j = a_{-1}$ and $w(1) = u(a_{-1})$. Then we solve in the interval $[a_{-2}, a_{-1})$ etc. We get a solution of (2.6)-(2.7) in $|x| \leq c$. It also follows from the construction that it must be unique. Since c is arbitrary the solution exists and is unique in all $\mathbf{R}$.

We prove that $u(x,\varepsilon) \to u(x)$ in $0 < x < a_1$. We see that

$$u(x,\varepsilon) - u(x) = \int_{0^+}^{x} d\mu(s,\varepsilon)(g(s,u(s,\varepsilon)) - g(s,u(s))) + \int_{0^+}^{x} d(\mu(s) - \mu(s,\varepsilon))g(s,u(s)). \tag{4.19}$$

Let $v(x,\varepsilon) = u(x,\varepsilon) - u(x)$. Since u is bounded on bounded sets we know that for some constant C

$$|g(x,u(x))| \leq C,\ 0 \leq x \leq a_1.$$

For fixed $\delta > 0$ let

$$C(\varepsilon) = \int_{0^+}^{a_1-\delta} d(|\mu(\cdot) - \mu(\cdot,\varepsilon)|(s). \tag{4.20}$$

It follows from (4.19) and (4.20) that

$$|v(x,\varepsilon)| \leq CC(\varepsilon) + M\int_{0^+}^{x} |v(s,\varepsilon)|d|\mu|(s,\varepsilon),\ 0 \leq x \leq a_1 - \delta. \tag{4.21}$$

Grönwalls lemma then says that $u(x,\varepsilon) \to u(x)$ uniformly on compact sets of $[0,a_1)$ since $C(\varepsilon) \to 0$, $\varepsilon \to 0$. It is also clear that the limit $u(x)$ in $0 < x < a_1$ is independent of the point masses at $x = a_1$. For technical reasons we assume that there is a $\delta > 0$ such that

$$(|\mu| + |\eta|)(\{x; 0 < |x - a_1| < \delta\} = 0. \tag{4.22}$$

This means that

$$\begin{aligned} u(x,\varepsilon) =& u(a_1 - \delta,\varepsilon) - \\ & \int_{a_1-\delta}^{x} \mu(\{a_1\})(g(s,u(s,\varepsilon)) - \eta(\{a_1\}))\phi(s - a_1,\varepsilon)ds, \\ & a_1 - \delta \leq x \leq a_1,\ 0 < \varepsilon < \delta/2. \end{aligned} \tag{4.23}$$

Since $u(x,\varepsilon)$ is equibounded on compact sets and g is continuous we commit an error tending to zero when $\varepsilon \to 0$ if we instead solve

$$(4.24) \qquad \begin{aligned} u(x,\varepsilon) =& u(a_1-\delta,\varepsilon) - \\ & \int_{a_1-\delta}^{x} (\mu(\{a_1\})g(a_1,u(s,\varepsilon)) - \eta(\{a_1\}))\phi(s-a_1,\varepsilon)ds. \end{aligned}$$

We change variable. Let

$$t = \int_{a_1-2\varepsilon}^{x} \phi(s-a_1,\varepsilon)ds.$$

Let $w(t) = u(x,\varepsilon)$. We get

$$w(t) = w(0) - \int_0^t (\mu(\{a_1\})g(a_1,w(s)) - \eta(\{a_1\}))ds$$

which is equivalent to (2.3) with $x_j = a_1$.

We see that

$$w(0) = u(a_1-\delta,\varepsilon) \to u(a_1-\delta) = u(a_1^-), \varepsilon \to 0.$$

So

$$w(1) = u(a_1,\varepsilon) \to u(a_1,0) = u(a_1), \ \varepsilon \to 0.$$

We get the right jump at $x = a_1$.

We now remove the restriction (4.22). We define new signed measures μ_j and η_j by

$$(4.25) \qquad \begin{aligned} &\mu_j(A) = \mu(A), \ \eta_j(A) = \eta(A), \text{ if } A \cap \{x; |x-a_1| < j^{-1}\} = \emptyset, \\ &\mu_j(A) = 0, \ \eta(A) = 0, \text{ if } A \subset \{x; |x-a_1| < j^{-1}\}. \end{aligned}$$

We replace μ by μ_j and η by η_j in (2.7)-(2.9) getting the solutions $u_j(x,\varepsilon)$ and $u_j(x)$ in $0 \le x \le a_1$. For a fixed x, $0 < x < a_1$, there is a j such that for all small ε $u(x,\varepsilon) = u_j(x,\varepsilon) \to u_j(x) = u(x)$, $\varepsilon \to 0$. One also realizes that $u(x,\varepsilon) - u_j(x,\varepsilon) \to 0$, $j \to \infty$, equiuniformly on compact sets for $0 < \varepsilon \leq 1$. We just use Grönwall's lemma for that just as above.

We then get

$$u_j(a_1, \varepsilon) \to u_j(a_1), \varepsilon \to 0. \tag{4.26}$$

We want to prove that

$$u(a_1, \varepsilon) \to w(1), \varepsilon \to 0, \ when \ w(0) = u(a_1^-), \tag{4.27}$$

with w from (2.3) and $x_j = a_1$. One gets from (4.26)

$$\begin{aligned} |u(a_1, \varepsilon) - w(1)| &\leq |u(a_1, \varepsilon) - u_j(a_1, \varepsilon)| + \\ &|u_j(a_1, \varepsilon) - w_j(1)| + |w_j(1) - w(1)|. \end{aligned} \tag{4.28}$$

Here w_j is the solution w of (2.5) when μ is replaced by μ_j and η by η_j and when we choose $w_j(0) = u_j(a_1^-)$. It is obvious that $u_j(a_1^-) \to u(a_1^-)$, $j \to \infty$. Then one sees from (4.28) that one can prove (4.27) by first choosing a big j and then letting $\varepsilon \to 0$.

If we start with $u(a_1, \varepsilon) = u(a_1)$ in $a_1 \leq x \leq a_2$ then we commit an error tending to zero uniformly in $a_1 \leq x \leq a_2$ when $\varepsilon \to 0$. We get that the modified $u(x, \varepsilon)$ and thus the original $u(x, \varepsilon)$ tends to $u(x)$ in $a_1 \leq x < a_2$. Then we prove that we get the right jump in the limit at $x = a_2$. Continuing in this way we see that $u(x, 0)$ exists and is equal to $u(x)$ in $x \geq 0$.

We repeat the procedure in the other direction. We start by assuming that for some $\delta > 0$,

$$(|\mu| + |\eta|)(\{x; 0 < |x| < \delta\}) = 0. \tag{4.29}$$

It is then clear that

$$u(x, \varepsilon) \to w(0), \ \varepsilon \to 0, \ -\delta < x \leq 0,$$

if w is taken from (2.5) with $x_j = 0$ and $w(1) = d$. We notice that $u(x) = w(0)$ too. We replace 0 by $-\delta$ and d by $w(0)$ in (2.2) and 2.(7).

We also let $v(x,\varepsilon)=u(x,\varepsilon)-u(x)$. We further use (2.1) and get

$$|v(x,\varepsilon)|\leq\int_x^{-\delta}M|v(s,\varepsilon)d|\mu|(s,\varepsilon)+\int_x^{-\delta}|g(s,u(s))|d|\mu(\cdot)-\mu(\cdot,\varepsilon)|(s)$$
$$+\int_x^{-\delta}d|\eta(\cdot,\varepsilon)-\eta(\cdot)|(s).$$

We notice that $g(x,u(x))$ is bounded on bounded sets. So the above inequality gives

$$|v(x,\varepsilon)|\leq\int_x^{-\delta}M|v(s,\varepsilon)|d|\mu|(s,\varepsilon)+C(\varepsilon)$$

with $C(\varepsilon)\to 0$, $\varepsilon\to 0$ uniformly in $a_{-1}\leq x\leq-\delta$. Grönwall's lemma then says that $v(x,\varepsilon)\to 0$, $\varepsilon\to 0$ uniformly on $a_{-1}\leq x\leq-\delta$. We now remove the restriction (4.29) and repeat the construction in (4.25) with a_1 replaced by 0. We call the corresponding solutions $u_j(x,\varepsilon)$ and $u_j(x)$. We have just proved that for every fixed j $u_j(x,\varepsilon)\to u_j(x)$, $\varepsilon\to 0$. We also have to prove that $v_j(x)=u_j(x)-u(x)\to 0$, $j\to\infty$ in $a_{-1}\leq x\leq 0$. We notice that $v_j(0)=0$ and that v_j is continuous in this intervall. If follows that for fixed j and all small ε

$$\int_x^0 M|v_j(s)|d|\mu|(s)\leq 2\int_x^0 M|v_j(s)|d|\mu|(s,\varepsilon). \tag{4.30}$$

From (2.7) and (2.7) with μ and η replaced by μ_j and η_j we get

$$|v_j(x)|\leq\int_x^0 M|v_j(s)|d|\mu|(s)+C(j), \tag{4.31}$$

where $C(j)\to 0$, $j\to\infty$, uniformly in $a_{-1}\leq x\leq 0$. We combine (4.30) and (4.31) and use Grönwall's lemma. We get

$$|v_j(x)|\leq C(j)\exp(2M\int_x^0|\mu|(s,\varepsilon)ds). \tag{4.32}$$

Then one realizes that we can remove the regularization in (4.32) and even the number 2 since any number greater than one will do. We have proved that $v_j\to 0$, $j\to\infty$, uniformly in $a_{-1}\leq x\leq 0$.

It is then obvious from

$$|u(x,\varepsilon)-u(x)| \leq |u(x,\varepsilon)-u_j(x,\varepsilon)| + |u_j(x,\varepsilon)-u_j(x)| + |u_j(x)-u(x)|$$

that $u(x,\varepsilon) \to u(x)$, $\varepsilon \to 0$ in $a_{-1} \leq x < 0$. We now replace $u(a_{-1},\varepsilon)$ by $u(a_{-1})$. By this we commit an error in $u(x,\varepsilon)$ tending uniformly to zero in $a_{-2} \leq x \leq a_{-1}$. Just as above we get that the modified $u(x,\varepsilon) \to u(x)$, $\varepsilon \to 0$, in the same interval proving that this is also true for the original $u(x,\varepsilon)$. In this way we prove that the limit exists and is equal to $u(x)$, $x < 0$. That completes the proof of Theorem 4.3.

THEOREM 4.4. *The solution $u(x,\varepsilon)$ of (2.2) converges pointwise to a limit $u(x,0)$ when $\varepsilon \to 0$.*

Proof. Let $u_N(x)$ and $u_N(x,\varepsilon)$ be defined as in Theorem 4.3. We shall prove that $u(x,\varepsilon)$ for fixed x is a Cauchy family in ε. Let $\varepsilon > 0$ and $\varepsilon' > 0$. Then

$$|u(x,\varepsilon)-u(x,\varepsilon')| \leq |u(x,\varepsilon)-u_N(x,\varepsilon)| + |u_N(x,\varepsilon)-u_N(x)|$$
$$+|u_N(x)-u_N(x,\varepsilon')| + |u_N(x,\varepsilon')-u(x,\varepsilon')|.$$

It follows from this, Lemma 4.2 and from Theorem 4.3 that $u(x,\varepsilon)$ for fixed x is a Cauchy family in ε. The limit when $\varepsilon \to 0$ is called $u(x,0)$. The theorem is proved.

LEMMA 4.5. *Let u_N be defined as in Theorem 4.3 and let $u(x,0)$ be defined as in Theorem 4.4. Then $u_N(x) \to u(x,0)$, $N \to \infty$, pointwise.*

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned} |u_N(x)-u(x,0)| &\leq |u_N(x)-u_N(x,\varepsilon)| + \\ &\quad |u_N(x,\varepsilon)-u(x,\varepsilon)| + |u(x,\varepsilon)-u(x,0)|. \end{aligned} \tag{4.33}$$

Let $\delta > 0$. Then we know from Lemma 4.2 that there is a N' such that

$$|u_N(x,\varepsilon)-u(x,\varepsilon)| < \delta/2,\ N \geq N',\ 0 < \varepsilon \leq 1. \tag{4.34}$$

From Theorem 4.4 we get an ε' such that

$$|u(x,\varepsilon) - u(x,0)| < \delta/2,\ 0 < \varepsilon \leq \varepsilon'. \tag{4.35}$$

From Theorem 4.3 we know that for fixed N,

$$u_N(x,\varepsilon) \to u_N(x),\ \varepsilon \to 0. \tag{4.36}$$

We first choose N' such that (4.34) is true. Then for each fixed $N \geq N'$ we let $\varepsilon \to 0$ in (4.33). Then (4.35) and (4.36) show that $|u_N(x) - u(x,0)| < \delta,\ N \geq N'$. The lemma is proved.

LEMMA 4.6. *Let u_N be defined as in Theorem 4.3. Then the sequence u_N is equibounded on compact sets.*

Proof. Lemma 4.1 and Theorem 4.3 prove the lemma.

5. Proof of Theorem 2.3.

Let $u(x) = u(x,0)$ with $u(x,0)$ from Theorem 4.4. Let u_N be defined as in Theorem 4.3. We known that $u_N(x) \to u(x)$ and that the sequence u_N is equibounded on compact sets, Lemma 4.5 and Lemma 4.6. We shall prove that

$$d + \int_{0^+}^{x} d\lambda(u_N, \mu_N, \eta_N)(s) \to d + \int_{0^+}^{x} d\lambda(u, \mu, \eta)(s),\ x > 0,\ N \to \infty. \tag{5.1}$$

Let

$$A(x) = \{s; 0 < s \leq x,\ (|\mu| + |\eta|)(\{s\}) = 0\}. \tag{5.2}$$

Let

$$\begin{aligned} &\lambda_j(u,\mu,\eta)(B) = 0,\ B \text{ Borel set},\ x_j \notin B, \\ &\lambda_j(u,\mu,\eta)(\{x_j\}) = \lambda(u,\mu,\eta)(\{x_j\}). \end{aligned} \tag{5.3}$$

We shall use the following assertion which we prove later on

$$\sum |\lambda_j|(u,\mu,\eta)(\{x_j\}) < \infty, \textit{ summed over } j \textit{ with } |x_j| \le c. \tag{5.4}$$

We see that

$$\int_{A(x)} d(-\mu g(\cdot, u_N(\cdot)) + \eta)(s) \to \int_{A(x)} d(-\mu(g(\cdot, u(\cdot)) + \eta)(s),\ N \to \infty. \tag{5.5}$$

One realizes that

$$\lambda_j(u_N, \mu_N, \eta_N)(\{x_j\}) = \lambda_j(u_N, \mu, \eta)(\{x_j\}),\ j \le N, \tag{5.6}$$

From (5.3), (5.6), Definition 2.2 and Lemma 4.5 we get

$$\begin{gathered}\lambda_j(u_N, \mu_N, \eta_N)(\{x_j\}) \to \lambda_j(u, \mu, \eta)(\{x_j\}),\ N \to \infty\ j \\ \textit{with } |x_j| \le c.\end{gathered} \tag{5.7}$$

Now (5.4)-(5.7) together with the existence of a constant C such that

$$|\lambda_j|(u_N, \mu, \eta)(\{x_j\}) \le C(|\mu| + |\eta|)(\{x_j\}),\ \text{all } N, \tag{5.8}$$

and

$$|\lambda_j(u, \mu, \eta)(\{x_j\})| \le C(|\mu| + |\eta|)(\{x_j\}), \tag{5.9}$$

shows (5.1) is true. It remains to prove (5.8) and (5.9).

We see that

$$\lambda_j(\{x_j\}) = u(x_j) - u(x_j^-) = -\int_0^1 (\mu(\{x_j\})g(x_j, w(s)) + \eta(\{x_j\}))ds \tag{5.10}$$

From the equiboundedness of w, $|x_j| \le c$ and the continuity of g it follows from (5.10) that for all j with $|x_j| \le c$, there is a constant C such that (5.8) and (5.9) are true. The proof is completed.

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