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ON THE HILBERT SERIES OF VERTEX COVER ALGEBRAS OF COHEN-MACAULAY BIPARTITE GRAPHS

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We study the Hilbert series of the vertex cover algebra $A(G)$, where G is a Cohen-Macaulay bipartite graph.

Introduction

Let $G = (V, E)$ be a simple (i.e., finite, undirected, loopless and without multiple edges) graph with the vertex set $V = [n]$ and the edge set $E = E(G)$. A *vertex cover* of G is a subset $C \subset V$ such that $C \cap \{i, j\} \neq \emptyset$, for any edge $\{i, j\} \in E(G)$. A vertex cover C of G is called *minimal* if no proper subset $C' \subset C$ is a vertex cover of G. A graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of R generated by those quadratic monomials $x_i x_j$ with $\{i, j\} \in E(G)$. It is said that a graph G is *Cohen-Macaulay* (over K) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every Cohen-Macaulay graph is unmixed.

A vertex cover $C \subset [n]$ can be represented as a $(0,1)-$vector c that satisfies the restriction $c_i + c_j \geq 1$, for every $\{i, j\} \in E(G)$. For each $k \in \mathbb{N}$, a *vertex cover* of G of *order* k, or simply a $k-$*vertex cover* of G, is a vector $c \in \mathbb{N}^n$ such that $c_i + c_j \geq k$, for every $\{i, j\} \in E(G)$. The *vertex cover algebra* $A(G)$ is

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defined as the subalgebra of the one variable polynomial ring $R[t]$ generated by all monomials $x_1^{c_1}\dots x_n^{c_n}t^k$, where $c=(c_1,...,c_n)\in\mathbb{N}^n$ is a k-vertex cover of G. This algebra was introduced and studied in [5]. Let $\mathfrak{m}$ be the maximal graded ideal of R. The graded K-algebra $\bar{A}(G)=A(G)/\mathfrak{m}A(G)$ is called the *basic cover algebra* and it was introduced and studied in [4, Section 3].

In Section 1, we study the Hilbert series of vertex cover algebras of Cohen-Macaulay bipartite graphs. The main result of the paper is given in Theorem 1.5, which shows that one may reduce the computation of the Hilbert series of the vertex cover algebra $A(G)$ to the computation of the Hilbert series of the basic cover algebra of all Cohen-Macaulay bipartite subgraphs of G. The theorem has several consequences. For instance, we prove that the h-vector of $A(G)$ is unimodal.

In Section 2, we give sharp bounds for the components of the h-vector and for the multiplicity of $A(G)$. We show that both chains and antichains are uniquely determined up to a poset isomorphism by the Hilbert series of their corresponding vertex cover algebras.

1. The Hilbert series of the vertex cover algebra of a Cohen-Macaulay bipartite graph

Let $S=K[x_1,...,x_n,y_1,...,y_n]$ and let $P_n=\{p_1,p_2,\dots,p_n\}$ be a poset with a partial order $\leq$. We denote by $G(P_n)$ the bipartite graph on $W\cup W'$, where $W=\{x_1,x_2,\dots,x_n\}$ and $W'=\{y_1,y_2,\dots,y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i,y_j\}$ with $p_i\leq p_j$. It is said that a bipartite graph G on $W\cup W'$ *comes from a poset*, if there exists a finite poset P_n on $\{p_1,p_2,\dots,p_n\}$ such that $p_i\leq p_j$ implies $i\leq j$, and after relabeling of the vertices of G one has $G=G(P_n)$. Herzog and Hibi proved in [3, Theorem 3.4] that a bipartite graph G on $W\cup W'$ is Cohen-Macaulay if and only if G comes from a poset. In this case, by [5], $A(G)$ is standard graded over S and it is the Rees algebra of the *cover ideal* I_G, which is generated by all monomials $x_1^{c_1}...x_n^{c_n}y_1^{c_{n+1}}...y_n^{c_{2n}}$, where $c=(c_1,\dots,c_{2n})$ is a 1-vertex cover of G. Thus

$$A(G)=S\oplus I_Gt\oplus\dots\oplus I_G^kt^k\oplus\dots$$

Let $\{m_1,m_2,...,m_l\}$ be the minimal system of generators of I_G. We view $A(G)$ as a standard graded K-algebra by assigning to each x_i and y_j, $1\leq i,j\leq n$, and to each m_kt, $1\leq k\leq l$, the degree 1. Since each monomial m_k corresponds to a minimal vertex cover of G of cardinality n, the Hilbert function of $A(G)$ is given by

$$H(A(G),k)=\sum_{j=0}^{k}\dim_K(I_G^j)_{jn+(k-j)},\text{ for all }k\geq 0.$$

Remark 1.1. The Hilbert function and series of $A(G)$ are invariant to poset isomorphisms. Indeed, if $P_n = \{p_1, p_2, ..., p_n\}$ and $P'_n = \{p'_1, p'_2, ..., p'_n\}$ are two isomorphic finite posets, then any poset isomorphism $f : P_n \to P'_n$ induces a permutation σ of $[n]$, $i \mapsto \sigma(i)$, defined by $p'_{\sigma(i)} = f(p_i)$, for every $i \in [n]$. The graph isomorphism $h : V(G(P_n)) \to V(G(P'_n))$, $h(x_i) = x_{\sigma(i)}$ and $h(y_j) = y_{\sigma(j)}$, $i, j \in [n]$, induces a K-automorphism of S which maps $I_{G(P_n)}$ onto $I_{G(P'_n)}$.

Let $P_n = \{p_1, p_2, ..., p_n\}$ be a poset such that $p_i \leq p_j$ implies $i \leq j$, let $\mathscr{J}(P_n)$ be the lattice of poset ideals of P_n and $G = G(P_n)$. By [4, Lemma 2.1] there exists a one-to-one correspondence between the set $\mathscr{M}(G)$ of minimal vertex covers of G and the lattice $\mathscr{J}(P_n)$. Hence the monomials $m_\alpha = (\prod_{p_i \in \alpha} x_i) \cdot (\prod_{p_j \notin \alpha} y_j)$, $\alpha \in \mathscr{J}(P_n)$, are the minimal generators of the cover ideal I_G.

We denote $B_G = K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}, \{u_\alpha\}_{\alpha \in \mathscr{J}(P_n)}]$. The *toric ideal* Q_G of $A(G)$ is the kernel of the surjective ring homomorphism $\xi : B_G \to A(G)$ defined by $\xi(x_i) = x_i$, $\xi(y_j) = y_j$, $1 \leq i, j \leq n$, $\xi(u_\alpha) = m_\alpha t$, $\alpha \in \mathscr{J}(P_n)$.

Let $<_{lex}$ denote the lexicographic order on $K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}]$ induced by the ordering $x_1 > ... > x_n > y_1 > ... > y_n$ and $<^\#$ the reverse lexicographic order on $K[\{u_\alpha\}_{\alpha \in \mathscr{J}(P_n)}]$ induced by an ordering of the variables u_α's such that $u_\alpha > u_\beta$ if $\beta \subset \alpha$ in $\mathscr{J}(P_n)$. Let $<^\#_{lex}$ be the monomial order on B_G defined as the product of the monomial orders $<_{lex}$ and $<^\#$ from above. The reduced Gröbner basis $\mathscr{G}$ of the toric ideal Q_G of $A(G)$ with respect to the monomial order $<^\sharp_{lex}$ on B_G was computed in [3, Theorem 1.1]:

$$\mathscr{G} = \{\underline{x_j u_\alpha} - y_j u_{\alpha \cup \{p_j\}} \mid j \in [n], \alpha \in \mathscr{J}(P_n), p_j \notin \alpha, \alpha \cup \{p_j\} \in \mathscr{J}(P_n)\}$$
$$\cup \{\underline{u_\alpha u_\beta} - u_{\alpha \cup \beta} u_{\alpha \cap \beta} \mid \alpha, \beta \in \mathscr{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial of $\mathscr{G}$ is the first monomial.

Let $S_G = K[\{u_\alpha\}_{\alpha \in \mathscr{J}(P_n)}]$ denote the polynomial ring in $|\mathscr{J}(P_n)|$ variables over K, let $\bar{A}(G)$ be the basic cover algebra and let $\Delta(\mathscr{J}(P_n))$ be the order complex of the lattice $(\mathscr{J}(P_n), \subset)$ whose faces are the chains of P_n. (We refer the reader to [1], [4, Section 3] for the definition and properties of the basic cover algebra associated to a graph and [2, §5.1] for the definition and properties of the order complex of a poset.) The *toric ideal* $\bar{Q}_G$ of $\bar{A}(G)$ is the kernel of the surjective ring homomorphism $\pi : S_G \to \bar{A}(G)$, $\pi(u_\alpha) = m_\alpha$, $\alpha \in \mathscr{J}(P_n)$. The reduced Gröbner basis $\mathscr{G}_0$ of $\bar{Q}_G$ with respect to $<^\#$ on S_G was computed in [4, Theorem 3.1]:

$$\mathscr{G}_0 = \{\underline{u_\alpha u_\beta} - u_{\alpha \cup \beta} u_{\alpha \cap \beta} \mid \alpha, \beta \in \mathscr{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial of $\mathscr{G}_0$ is the first monomial.

Proposition 1.2. *The graded K-algebra $\bar{A}(G)$ and the order complex $\Delta(\mathscr{J}(P_n))$ have the same h-vector.*

Proof. By [4, Proposition 3.1] $\bar{Q}_G$ is a graded ideal (generated by binomials) and the initial ideal $\mathrm{in}_{<^\#}(\bar{Q}_G)$ of the toric ideal $\bar{Q}_G$ coincides with the Stanley-Reisner ideal $I_{\Delta(\mathscr{J}(P_n))}$, hence $S_G/\bar{Q}_G$ and $K[\Delta(\mathscr{J}(P_n))]$ have the same h-vector. Since $S_G/\bar{Q}_G \simeq \bar{A}(G)$ as graded K-algebras, the conclusion follows. □

Remark 1.3. Since $\mathscr{J}(P_n)$ is a full sublattice of the Boolean lattice $\mathscr{L}_n$ on the set $\{p_1, p_2, \dots, p_n\}$ ([4, Theorem 2.2.]), it follows that $\dim \Delta(\mathscr{J}(P_n)) = n$. Let $h = (h_0, h_1, ..., h_{n+1})$ be the h-vector of $\Delta(\mathscr{J}(P_n))$ and $\bar{A}(G)$. As we noticed above, the basic cover algebra $\bar{A}(G)$ can be identified with the Hibi ring $S_G/\bar{Q}_G$, which arises from the distributive lattice $\mathscr{J}(P_n)$. The i-th component h_i of the h-vector of $S_G/\bar{Q}_G$ and, consequently, of $\bar{A}(G)$ is equal to the number of linear extensions of P_n, which, seen as permutations of $[n]$, have exactly i descents ([6]). In particular,

$$h_i \geq 0, \text{ for all } 0 \leq i \leq n-1, \; h_0 = 1, \text{ and } h_n = h_{n+1} = 0. \tag{1}$$

For example, if $P_n'' = \{p_1'', ..., p_n''\}$ is an antichain, then each permutation of $[n]$ can be seen as a linear extension of P_n'', hence, for all $0 \leq i \leq n-1$, the i-th component h_i of the h-vector of $\Delta(\mathscr{J}(P_n''))$ is equal to the number of all permutations of $[n]$ with exactly i descents, which is the Eulerian number $A(n,i)$.

For each $\emptyset \neq F \subset [n]$ we denote by $P_n(F)$ the subposet of P_n induced by the subset $\{p_i | i \in F\}$. The main result of the paper relates the Hilbert series of $A(G)$ to the Hilbert series of $\bar{A}(G_F)$, for all $F \subset [n]$, where G_F denotes the bipartite graph that comes from the poset $P_n(F)$. If $F = \emptyset$, then, by convention, the Hilbert series of $\bar{A}(G_F)$ is equal to $\frac{1}{1-z}$.

In order to prove the main theorem we need a preparatory result.

Let $\emptyset \neq F \subsetneqq [n]$ and let α be a poset ideal of $P_n(\bar{F})$, where by $\bar{F}$ we mean the complement of F in $[n]$. We denote by δ_α the maximal subset of $P_n(F)$ such that $\alpha \cup \delta_\alpha \in \mathscr{J}(P_n)$. Note that

$$\delta_\alpha = \cup\{\gamma \mid \gamma \subset P_n(F), \alpha \cup \gamma \in \mathscr{J}(P_n)\}.$$

If we set $\beta = \alpha \cup \delta_\alpha$, then, by the definition of δ_α, β has the following property: for any $j \in F$, $p_j \notin \beta$ implies $\beta \cup \{p_j\} \notin \mathscr{J}(P_n)$.

Lemma 1.4. *Let $\emptyset \neq F \subsetneqq [n]$ and let $\mathscr{S}$ be the set of poset ideals β of P_n with the property that for any $j \in F$ such that $p_j \notin \beta$ we have $\beta \cup \{p_j\} \notin \mathscr{J}(P_n)$. Then the map $\varphi\colon \mathscr{J}(P_n(\bar{F})) \to \mathscr{S}$ defined by $\alpha \mapsto \beta = \alpha \cup \delta_\alpha$, is an isomorphism of posets.*

Proof. We show that φ is invertible. Indeed, the map $\psi : \mathscr{S} \to \mathscr{J}(P_n(\bar{F}))$ defined by $\psi(\beta) = \beta \cap P_n(\bar{F})$ is the inverse of φ since if $\alpha = \beta \cap P_n(\bar{F})$, then, by the property of β, we have $\delta_\alpha = \beta \setminus P_n(\bar{F})$.

Let $\alpha_1 \subsetneqq \alpha_2$ be poset ideals of $P_n(\bar{F})$ and $\beta_i = \varphi(\alpha_i) = \alpha_i \cup \delta_i$, $i = 1,2$. We only need to prove that $\beta_1 \subset \beta_2$ since the strict inclusion follows from the hypothesis $\alpha_1 \subsetneqq \alpha_2$. Let us assume that $\beta_1 \not\subset \beta_2$ and let p_a, $a \in F$, be a minimal element in $\beta_1 \setminus \beta_2$. Since $p_a \notin \beta_2$, it follows that $\beta_2 \cup \{p_2\}$ is not a poset ideal of P_n. Therefore, there exists some $p_b < p_a$ such that $p_b \notin \beta_2$. On the other hand, $p_b \in \beta_1$ since $\beta_1 \in \mathscr{J}(P_n)$, hence, $p_b \in \beta_1 \setminus \beta_2$, which is a contradiction with the choice of p_a.

Now let $\beta_1, \beta_2 \in \mathscr{S}$ with $\beta_1 \subsetneqq \beta_2$, and let us assume that $\alpha_1 = \alpha_2$, where $\alpha_1 = \beta_1 \cap P_n(\bar{F})$ and $\alpha_2 = \beta_2 \cap P_n(\bar{F})$. Then $\delta_1 = \beta_1 \setminus P_n(\bar{F}) \subsetneqq \delta_2 = \beta_2 \setminus P_n(\bar{F})$. But this is impossible since δ_1 is maximal among the subsets $\gamma \subset P_n(F)$ with $\alpha_1 \cup \gamma \in \mathscr{J}(P_n)$. □

We can state now the main theorem which relates the Hilbert series of the vertex cover algebra $A(G)$ to the Hilbert series of the basic cover algebras $\bar{A}(G_F)$ for all $F \subset [n]$.

Theorem 1.5. *For $F \subset [n]$ let $H_{\bar{A}(G_F)}(z)$ be the Hilbert series of $\bar{A}(G_F)$ and let $H_{A(G)}(z)$ be the Hilbert series of $A(G)$. Then:*

$$H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z} \right)^{n-|F|}. \tag{2}$$

In particular, if $h(z) = \sum_{j \geq 0} h_j z^j$ and $h^F(z) = \sum_{j \geq 0} h_j^F z^j$, where $h = (h_j)_{j \geq 0}$ and $h^F = (h_j^F)_{j \geq 0}$ are the h-vectors of $A(G)$, and, respectively, $\bar{A}(G_F)$, then

$$h(z) = \sum_{F \subset [n]} h^F(z) z^{n-|F|}. \tag{3}$$

Proof. Let $J_G = \mathrm{in}_{<^{\#}_{lex}}(Q_G)$. It is known that B_G/Q_G and B_G/J_G have the same Hilbert series. Let $B'_G = K[\{x_i\}_{1 \leq i \leq n}, \{u_\alpha\}_{\alpha \in \mathscr{J}(P_n)}]$. By using the following K-vector space isomorphism

$$B_G/J_G \simeq K[y_1, y_2, \ldots, y_n] \otimes_K B'_G/(J_G \cap B'_G),$$

we get

$$H_{A(G)}(z) = H_{B_G/Q_G}(z) = H_{B_G/J_G}(z) = \frac{1}{(1-z)^n} H_{B'_G/(J_G \cap B'_G)}(z).$$

We need to compute the Hilbert series of $B'_G/(J_G \cap B'_G)$. To this aim we show that we have an isomorphism of K-vector spaces

$$B'_G/(J_G \cap B'_G) \simeq \bigoplus_{F \subset [n]} \bar{A}(G_{\bar{F}}) \otimes_K x_F K[\{x_i\}_{i \in F}]. \tag{4}$$

For $\emptyset \neq F \subset [n]$ let J_F be the initial ideal with respect to $<^{\#}$ of the toric ideal of $\bar{A}(G_F)$. Then $J_F = (u_\alpha u_\beta - u_{\alpha\cup\beta} u_{\alpha\cap\beta} \mid \alpha, \beta \in \mathscr{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha)$. If $F = \emptyset$, we put by convention $J_F = (0)$.

The basic vertex cover algebra $\bar{A}(G_{\bar{F}})$ can be decomposed as a K-vector space as $\bar{A}(G_{\bar{F}}) \simeq \bigoplus_{w \notin J_{\bar{F}}} Kw$. We notice that $w \notin J_{\bar{F}}$ if and only if $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_s\}$, $s \geq 0$, where $\alpha_1 \subsetneqq \ldots \subsetneqq \alpha_s$ is a chain in $\mathscr{J}(P_n(\bar{F}))$. It follows that for $F \subset [n]$ we have

$$V_F = \bar{A}(G_{\bar{F}}) \otimes_K x_F K[\{x_i\}_{i \in F}] \simeq \bigoplus Kvw,$$

where the direct sum is taken over all monomials vw with v a monomial in the variables x_i's such that $\mathrm{supp}(v) = F$ and w a monomial in the variables u_α's such that $w \notin J_{\bar{F}}$.

As a K-vector space, $B'_G/(J_G \cap B'_G)$ has the decomposition

$$B'_G/(J_G \cap B'_G) \simeq \bigoplus_{F \subset [n]} W_F,$$

where $W_F = \bigoplus Kvw'$ and the direct sum is taken over all monomials v with $\mathrm{supp}(v) = F$ and all monomials w' in the variables u_α's with $\alpha \in \mathscr{J}(P_n)$ such that $vw' \neq 0$ modulo $J_G \cap B'_G$.

Thus in order to prove (4), we only need to show that for each $F \subset [n]$, the K-vector spaces V_F and W_F are isomorphic. This is obvious for $F = \emptyset$ and $F = [n]$.

Let us consider now $\emptyset \neq F \subsetneqq [n]$. Based on the previous lemma, we are going to prove that there exists a one-to-one correspondence between the monomials of the K-bases of V_F and W_F.

Let vw be a monomial of the K-basis of V_F. This means that $\mathrm{supp}(v) = F$ and w is of the form $w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s}$ for some chain $\alpha_1 \subsetneqq \ldots \subsetneqq \alpha_s$ in $\mathscr{J}(P_n(\bar{F}))$, $s \geq 1$. For each $1 \leq i \leq s$, let $\beta_i = \varphi(\alpha_i) \in \mathscr{J}(P_n)$ as it was defined in Lemma 1.4. We map vw to the monomial vw', where $w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s}$. By Lemma 1.4, we have that $\beta_1 \subsetneqq \ldots \subsetneqq \beta_s$ is a chain in $\mathscr{J}(P_n)$. Moreover, for any $j \in F$ and any β_i such that $p_j \notin \beta_i$, we have $\beta_i \cup \{p_j\} \notin \mathscr{J}(P_n)$. Therefore, vw' is a monomial in the K-basis of W_F.

Conversely, let vw' be a monomial of the K-basis of W_F, where $\mathrm{supp}(v) = F$ and $w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s}$, with $\beta_1 \subsetneqq \ldots \subsetneqq \beta_s$ a chain in $\mathscr{J}(P_n)$. Let $\alpha_i = \beta_i \cap P_n(\bar{F})$, for $1 \leq i \leq s$. Then we associate to vw' the monomial vw in the K-basis of V_F, where $w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s}$.

By using again Lemma 1.4 it follows that the above defined maps between the K-bases of V_F and W_F are inverse.

By (4) we get

$$H_{B'_G/(J_G \cap B'_G)}(z) = \sum_{F \subset [n]} H_{\bar{A}(G_{\bar{F}})}(z) \left(\frac{z}{1-z}\right)^{|F|} = \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|},$$

which ends the proof. □

Corollary 1.6. *For all $0 \leq j \leq n-1$, the j-th component h_j of the h-vector of $A(G)$ is equal to the number of all linear extensions of all $n-l$-element subposets of P_n, which, seen as permutations of $[n-l]$, have exactly $j-l$ descents, for all $0 \leq l \leq j$.*

Proof. It follows immediately from (3) and Remark 1.3. □

Corollary 1.7. *The h-vector of $A(G)$ is unimodal.*

Proof. By (3) we get $h_{n+1} = \sum\limits_{F \subset [n]} h_{|F|+1}^F$ and $h_n = \sum\limits_{F \subset [n]} h_{|F|}^F$. By using (1) from Remark 1.3, we have $h_{|F|}^F = h_{|F|+1}^F = 0$, for all $\emptyset \neq F \subset [n]$. Hence $h_{n+1} = h_1^{\emptyset} = 0$ and $h_n = h_0^{\emptyset} = 1$. In [5, Corollary 4.4] it was proved that $A(G)$ is a Gorenstein ring, hence, by [2, Corollary 4.3.8 (b) and Remark 4.3.9 (a)], $h_i = h_{n-i}$, for all $0 \leq i \leq n$. We denote by $\nu(l,j)$ the number of all linear extensions of all $n-l$-element subposets of P_n which, seen as permutations of $[n-l]$, have exactly $j-l$ descents. Hence, by Corollary 1.6, $h_j = \sum\limits_{l=0}^{j} \nu(l,j)$.

Let $0 \leq j < j+1 \leq \lfloor \frac{n}{2} \rfloor$. Then $\nu(l,j) \leq \nu(l+1,j+1)$, for all $0 \leq l \leq j$, which implies that $h_{j+1} = \nu(j+1,0) + \sum\limits_{l=0}^{j} \nu(j+1,l+1) \geq \sum\limits_{l=0}^{j} \nu(l,j) = h_j$. □

Remark 1.8. The a-invariant of $A(G)$ is equal to $-n-1$.

Corollary 1.9. *Let $e(A(G))$ be the multiplicity of $A(G)$ and let $e(\bar{A}(G_F))$ be the multiplicity of $\bar{A}(G_F)$ for $F \subset [n]$. Then*

$$e(A(G)) = \sum_{F \subset [n]} e(\bar{A}(G_F)).$$

Proof. It follows immediately from (3). □

Let $P_3 = \{p_1, p_2, p_3\}$ be the poset with $p_1 \leq p_2$ and $p_1 \leq p_3$ and $G = G(P_3)$. By applying our formula (2) we get:

$$H_{A(G)}(z) = \frac{1}{(1-z)^3} \sum_{F \subset [3]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{3-|F|} = \frac{1+4z+4z^2+z^3}{(1-z)^7}.$$

Hence $h_0 = h_3 = 1$, $h_1 = h_2 = 4$, $h_4 = 0$. We can also compute the h-vector of $A(G)$ by using Corollary 1.6. The poset P_3 has two linear extensions, which, seen as permutation of $[3]$, are equal to id_3 and (23). Hence $h_0 = 1$, since there exists only one linear extension of P_3, which, seen as a permutation of $[3]$, has exactly 0 descents. Furthermore, P_3 has three 2-element subposets, the chains $P_3(\{1,2\})$ and $P_3(\{1,3\})$ with a linear extension corresponding to id_2, and the antichain $P_3(\{2,3\})$ with two linear extensions corresponding to id_2 and (12). Thus $h_1 = 4$, since there exists one linear extension of P_3, which, seen as a permutation of $[3]$, has exactly 1 descent and each of the subposets $P_3(\{1,2\})$, $P_3(\{1,3\})$ and $P_3(\{2,3\})$ has one linear extension, which, seen as a permutation of $[2]$, has exactly 0 descents.

2. Bounds for the components of the h-vector and for the multiplicity of $A(G)$

Let $\mathscr{L}_n$ be the Boolean lattice on $\{p_1, p_2, ..., p_n\}$, $n \geq 1$, and let $A(p,q)$ be the Eulerian number for $1 \leq q \leq n$ and $0 \leq p < q$. By convention we put $A(0,0) = 1$ and $A(q,q) = 0$, for all $1 \leq q \leq n$. We compute the Hilbert series of the vertex cover algebra of the Cohen-Macaulay bipartite graphs that come from a chain and an antichain.

Proposition 2.1. *Let G', respectively, G'' be bipartite graphs on $W \cup W'$ that come from a chain, respectively, an antichain with n elements, $n \geq 1$. Then we have:*

(i) $H_{A(G')}(z) = \frac{(1+z)^n}{(1-z)^{2n+1}}$. *In particular,* $e(A(G')) = 2^n$.

(ii) $H_{A(G'')}(z) = \frac{\sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{l} A(n-l, j-l) z^j}{(1-z)^{2n+1}}$. *In particular,* $e(A(G'')) = n! \cdot \sum_{l=0}^{n} \frac{1}{l!}$.

Proof. (i) We may assume that $G' = G(P'_n)$, where $P'_n = \{p'_1, p'_2, ..., p'_n\}$ is the chain with $p'_1 \leq p'_2 \leq ... \leq p'_n$. P'_n as well as all its subposets have a unique linear extension. Therefore, the h-vector of $A(G')$ is $(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n})$.

(ii) Let $G'' = G(P''_n)$, where $P''_n = \{p''_1, ..., p''_n\}$ is an antichain. If $F = [n]$, then, by convention, $A(0,0) = 1 = h_0^{\bar{F}}$. If $F \subsetneqq [n]$, then $\mathscr{J}(P''_n(\bar{F}))$ is a Boolean lattice on the set $P''_n(\bar{F})$, which implies that $\mathscr{J}(P''_n(\bar{F}))$ is isomorphic to $\mathscr{L}_{n-l}$, where $l = |F|$. Hence, by Remark 1.3, $h_i^{\bar{F}} = A(n-l,i)$, for all $0 \le i \le n-l-1$. If $i = n-l$, then $A(n-l,i) = 0$ (by convention) and $h_i^{\bar{F}} = 0$ (by Remark 1.3), which implies that $A(n-l,i) = h_i^{\bar{F}}$. By (2) we have $h''_j = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F|=l}} h_{j-l}^{\bar{F}}$, hence $h''_j = \sum_{l=0}^{j} \binom{n}{l} A(n-l, j-l)$, for all $0 \le j \le n$.

We get $e(A(G'')) = \sum_{j=0}^{n} h''_j = \sum_{j=0}^{n-1} h''_j + 1 = \sum_{j=0}^{n-1} \sum_{l=0}^{j} \binom{n}{l} A(n-l, j-l) + 1 = \sum_{l=0}^{n-1} \binom{n}{l} \sum_{j=0}^{n-l-1} A(n-l, j) + 1$. We obviously have $\sum_{j=0}^{n-l-1} A(n-l, j) = (n-l)!$, for all $0 \le l \le n-1$. Therefore, $e(A(G'')) = \sum_{l=0}^{n-1} \binom{n}{l} \cdot (n-l)! + 1 = n! \cdot \sum_{l=0}^{n} \frac{1}{l!}$. $\square$

Remark 2.2. The reduced Gröbner basis $\mathscr{G}'$ of the toric ideal $Q_{G'}$ of $A(G')$, where G' comes from the chain $P'_n = \{p'_1, p'_2, ..., p'_n\}$, $p'_1 \le p'_2 \le ... \le p'_n$, with respect to the monomial order $<_{lex}^{\sharp}$ on the polynomial ring $B_{G'}$ is:

$$\mathscr{G}' = \{\underline{x_j u_{\{p'_1,\dots,p'_{j-1}\}}} - y_j u_{\{p'_1,\dots,p'_j\}} | j \in [n]\},$$

where the initial monomial of each binomial of $\mathscr{G}'$ is the first monomial.

We notice that the initial ideal $\mathrm{in}_{<_{lex}^{\sharp}}(Q_{G'}) = (x_j u_{\{p'_1,\dots,p'_{j-1}\}} | j \in [n])$ is a complete intersection, which implies that the toric ideal $Q_{G'}$ is a complete intersection. Hence $A(G')$ has a pure resolution given by the Koszul complex.

Proposition 2.3. *Let G be a Cohen-Macaulay bipartite graph on $W \cup W'$. Then the following assertions hold:*

(i) G comes from a chain if and only if $H_{A(G)}(z) = \frac{(1+z)^n}{(1-z)^{2n+1}}$;

(ii) G comes from an antichain if and only if $H_{A(G)}(z) = \frac{h''_0 + h''_1 z + \dots + h''_n z^n}{(1-z)^{2n+1}}$, where $h'' = (h''_0, h''_1, \dots, h''_n)$ is the h-vector of the vertex cover algebra $A(G'')$ of the bipartite graph G'' that comes from an antichain $P''_n = \{p''_1, p''_2, ..., p''_n\}$.

Proof. Let us suppose that G comes from a poset $P_n = \{p_1, p_2, ..., p_n\}$, $n \ge 1$, and let $h = (h_0, h_1, ..., h_n)$ be the h-vector of $A(G)$. In the first place we need to compute the component h_1. By using (3), we get $h_1 = h_1^{[n]} + n$. But $h_1^{[n]}$ is the component of rank 1 in the h-vector of $\bar{A}(G)$. By using the formula which relates the h-vector to the f-vector of the order complex $\Delta(\mathscr{J}(P_n))$, we immediately get $h_1^{[n]} = |\mathscr{J}(P_n)| - n - 1$, which implies that $h_1 = |\mathscr{J}(P_n)| - 1$.

(i) Let h_1=$\binom{n}{1}=n$. Then $|\mathscr{J}(P_n)|=n+1$, which implies that P_n is a chain.
(ii) Let $h_1=h''_1=|\mathscr{J}(P''_n)|-1=2^n-1$. Then $|\mathscr{J}(P_n)|=2^n$, which implies that P_n is an antichain.

In both cases the converse follows from Proposition 2.1. $\square$

Proposition 2.4. *Let G be a Cohen-Macaulay bipartite graph on $W\cup W'$. If $h=(h_0,h_1,...,h_n)$ is the h-vector of $A(G)$, then $\binom{n}{j}\le h_j\le h''_j$, for all $0\le j\le n$, where G'' comes from an antichain with n elements and $h''=(h''_0,h''_1,...,h''_n)$ is the h-vector of $A(G'')$.*

Proof. Let $P'_n=\{p'_1,p'_2,...,p'_n\}$ be the chain with $p'_1\le p'_2\le...\le p'_n$ and let $P''_n=\{p''_1,p''_2,...,p''_n\}$ be an antichain. We may assume that $G=G(P_n)$, where $P_n=\{p_1,p_2,...,p_n\}$ is a poset such that $p_i\le p_j$ implies $i\le j$.

By using (3) and (1), we get $h_0=1=\binom{n}{0}=h''_0$ and $h_n=1=\binom{n}{n}=h''_n$. Let $1\le j\le n-1$. By Corollary 1.6, h_j is equal to the number of all linear extensions of all $n-l$-element subposets, which, seen as permutations of $[n-l]$, have exactly $j-l$ descents, for all $0\le l\le j$. Each $n-l$-element subposet of P'_n, respectively, P''_n is a chain, respectively, an antichain, hence it has only one linear extension which corresponds to id_{n-l}, respectively, it has $(n-l)!$ linear extensions which correspond to all permutations of $[n-l]$. Therefore, we get $\binom{n}{j}\le h_j\le h''_j$, for all $1\le j\le n-1$. $\square$

Corollary 2.5. *Let G be a bipartite graph on $W\cup W'$ that comes from a poset with n elements, $n\ge 1$. Then $2^n\le e(A(G))\le n!\sum\limits_{l=0}^{n}\frac{1}{l!}$. The left equality holds if and only if the poset is a chain and the right equality holds if and only if the poset is an antichain.*

Proof. Let $G=G(P_n)$, $G'=G(P'_n)$ and $G''=G(P''_n)$, where $P_n=\{p_1,p_2,...,p_n\}$ is a poset such that $p_i\le p_j$ implies $i\le j$, $P'_n=\{p'_1,p'_2,...,p'_n\}$ is the chain with $p'_1\le p'_2\le...\le p'_n$ and $P''_n=\{p''_1,p''_2,...,p''_n\}$ is an antichain. Let h, h' and h'' be the h-vectors of $A(G)$, $A(G')$ and $A(G'')$. By summing up the inequalities $h'_j\le h_j\le h''_j$ from Proposition 2.4 we get $e(A(G'))\le e(A(G))\le e(A(G''))$. Thus the desired inequalities follow from Proposition 2.1. Moreover, the left equality, respectively, the right equality holds if and only if $h'_j=h_j$, respectively, $h_j=h''_j$, for all $0\le j\le n$, therefore, by using Proposition 2.3, this is equivalent to $P_n\simeq P'_n$, respectively, $P_n\simeq P''_n$. $\square$

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