«LE MATEMATICHE»
Vol. XLIII (1988) - Fasc. I, pp. 297-302

A NOTE ON EXTENDING DECISION ALGORITHMS BY STABLE PREDICATES

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A general mechanism to extend decision algorithms to deal with additional predicates is described. The only condition imposed on the predicates is stability with respect to some transitive relations.

Introduction.

In 1979 Nelson and Oppen described a mechanism to combine the decision procedures of two independent theories into a single one for their union (see [5]). In this paper we describe a method for dealing with exstensions of decidable theories by suitable predicates.

Indeed, in automatic theorem proving, formulas for which a decision method is not known (or does not exist) are often considered. However it is sometimes possible to weaken the theory in a manner which pinpoints a useful decidable subclass of formulas.

(*) Entrato in Redazione il 14 novembre 1989

This happens, for example, in those cases where only stability (or invariance) properties of particular relations with respect to certain basic relations and operations of the background theory are used.

More precisely, we consider a decidable theory T extended by the presence of one or more additional predicates that satisfy certain axioms. Assume that the decidability of the extended theory T' is not known. On the other hand suppose that the stability of the aforementioned predicates with respect to certain transitive relations that are expressible in the language of T is known. Then it is possible to decide those sentences of T' which are consequences of the axioms of T together with these stability properties. To this class of theorems belong statements that result from properties of monotonicity, hereditarity, closure with respect to morphisms, and so on.

An example of an application of the present method to the automatic deduction of a set-theoretic statement is given at the end of the present note.

1. Extending decidable theories by stable predicates.

Let $\mathcal{L}$ be any first-order language with equality and $\mathcal{A}$ be a set of sentences of $\mathcal{L}$.

DEFINITION. *A formula φ of $\mathcal{L}$ is said to be $\mathcal{A}$-valid (satisfiable) if its universal (existential) closure is true in every (some) model of $\mathcal{A}$.*

If the $\mathcal{A}$-validity of any formula of a class C is computable, then the class C is called $\mathcal{A}$-decidable.

Let F be an $\mathcal{A}$-decidable class of formulas such that:

a) $x = y$ is in F for every pair x, y of variables;

b) F is closed with respect to the propositional connectives;

c) F is closed with respect to substitutions of variables by means of variables.

Extend the language $\mathcal{L}$ by adding new predicate symbols P_1, $P_2, \ldots$ and let m_i be the arity of P_i, $i = 1, 2, \ldots$

In what follows we will often use the abbreviated notation $\vec{x}$ to indicate some vector $(x_1, x_2, \ldots, x_m)$.

Let $\varphi_i(x_1, \ldots, x_{m_i}, y_1, \ldots, y_{m_i})$ be formulas of F with free variables belonging to the set $\{x_1, \ldots, x_{m_i}, y_1, \ldots, y_{m_i}\}$ such that the formula

$$(\forall \vec{x}, \vec{y}, \vec{z})[(\varphi_i(\vec{x}, \vec{y}) \& \varphi_i(\vec{y}, \vec{z})) \rightarrow \varphi_i(\vec{x}, \vec{z})]$$

is $\mathcal{A}$-valid, $i = 1, 2, \ldots$ (Transitivity of φ_i).

For $i = 1, 2, \ldots$ add the following stability (see Dantoni [2] and Rosenberg [6]) postulates:

$$(B_i) \qquad (\forall \vec{x}, \vec{y})[\varphi_i(\vec{x}, \vec{y}) \,\&\, P_i(\vec{x}) \rightarrow P_i(\vec{y})].$$

Let $\mathcal{A}^*$ be the extended set of sentences and let F^* be the smallest set of formulas containing F together with all atomic formulas of type $P_i(t_1, \ldots, t_{m_i})$, where $t_1, \ldots, t_{m_i}$ are terms of $\mathcal{L}$, and closed with respect to the logical connectives $\neg, \&, \vee, \rightarrow, \leftrightarrow$. Then we have the following

THEOREM 1.1. *F^* is $\mathcal{A}^*$-decidable.*

Proof. Since every formula can be put in disjunctive normal form, in order to prove the desired result it is sufficient to provide an algorithm to decide $\mathcal{A}^*$-satisfiability of any conjunction C of formulas of F together with literals of the form

$$(P_i) \qquad P_i(x_1, x_2, \ldots, x_{m_i})$$

$$(\neg P_i) \qquad \neg P_i(y_1, y_2, \ldots, y_{m_i}).$$

We proceeed as follows:

1) For every $(x_1, x_2, \ldots, x_{m_i})$ in (P_i) and $(y_1, y_2, \ldots, y_{m_i})$ in $(\neg P_i)$ add the formula

$$(*) \qquad \left(\bigvee_{j=1}^{m_i} x_j \neq y_j\right) \& \neg\varphi_i(x_1, x_2, \ldots, x_{m_i}, y_1, y_2, \ldots, y_{m_i})$$

for each (P_i) in C, $i = 1, 2, \ldots$

2) Drop the literals (P_i), $(\neg P_i)$ for every $i = 1, 2, \ldots$, and let C' be the remaining formula of F.

We prove our assertion by demostrating the following

LEMMA 1.2. *C is $\mathcal{A}^*$-satisfiable if and only if C' is $\mathcal{A}$-satisfiable.*

It is obvious that the restriction to the language of $\mathcal{A}$ of any model of $\mathcal{A}^* \cup \{C\}$ is a model of $\mathcal{A} \cup \{C'\}$.

Conversely, let M be a model of $\mathcal{A} \cup \{C'\}$. Let

$$V_i = \{\vec{x} | P(\vec{x}) \text{ is in } C\} \text{ for every } i = 1, 2, \ldots$$

Extend M to a model of $\mathcal{A}^* \cup \{C\}$ by putting:

$$(**) \qquad MP_i(s_1, s_2, \ldots, s_{m_i}) \leftrightarrow$$

$$\bigvee_{\vec{x} \in V_i} \left[\left(\mathop{\&}_{j=1}^{m_i} s_j = Mx_j\right) \vee \varphi_i(Mx_1, Mx_2, \ldots, Mx_{m_i}, s_1, s_2, \ldots, s_{m_i})\right]$$

for every m_i-tuple $(s_1, s_2, \ldots, s_{m_i})$ belonging to the domain of the model M.

Indeed, to see that M is a model of $\mathcal{A}^*$ we have to verify axioms (B_i) for each $i = 1, 2, \ldots$.

Assume that $\varphi_i(s_1, \ldots, s_{m_i}, t_1, \ldots, t_{m_i})$ and $MP_i(s_1, \ldots, s_{m_i})$ both hold.

Then, by definition of MP_i, for some $\vec{x} \in V_i$ either

$$(a) \qquad s_1 = Mx_1 \,\&\, s_2 = Mx_2 \,\&\, \ldots \,\&\, s_{m_i} = Mx_{m_i}$$

or

$$(b) \qquad \varphi_i(Mx_1, \ldots, Mx_{m_i}, s_1, \ldots, s_{m_i}) \text{ is true}$$

In case (a) we have $\varphi_i(Mx_1, \ldots, Mx_{m_i}, t_1, \ldots, t_{m_i})$ which by (**) yields $MP_i(t_1, \ldots, t_{m_i})$.

In case (b) we still get $\varphi_i(Mx_1, \ldots, Mx_{m_i}, t_1, \ldots, t_{m_i})$ by the transitivity of φ and hence again $MP_i(t_1, \ldots, t_{m_i})$.

This shows that M is a model of A^*. In order to finish the proof of our lemma we have only to show that literals (P_i) and $(\neg P_i)$ in C are correctly modeled. But this is an immediate consequence of (*) and of the definition of MP_i, completing the proof of the lemma and of our main theorem.

2. An example of an application to automated deduction of set-theoretic statements.

Let *MLSP* (multilevel syllogistic with powerset) be the unquantified theory MLS in the language: = (set equality), $\in$ (membership), $\subseteq$ (inclusion), $\setminus$ (set difference), pow (powerset operator). This is known to be decidable (see [1]). Add to $MLSP$ the predicate finite(x), asserting that x is a finite set. It is not known if this new theory $MLSPF$ is decidable. However, by our decidability criterion, we can say that any theorem of $MLSPF$, which is a consequence of $MLSP$ together with the monotonicity of the finiteness predicate with respect to $\subseteq$ is decidable. Let us describe a simple scenario of an interactive proof in which such result can be used.

Let us consider any statement of type

$$(x \subseteq z \setminus u \,\&\, z = \text{pow}(v) \,\&\, l = v \cup m \,\&\, w = \text{pow}(l) \,\&\, \text{finite}(l)) \rightarrow \text{finite}(x)$$

and assume we have already proved the following

LEMMA 2.1. $(\text{finite}(x) \,\&\, y = \text{pow}(x)) \rightarrow \text{finite}(y)$.

By taking $\varphi(x,y)$ to be $y \subseteq x$ and applying our main theorem to the decidable theory $MLSP$ with the extra-predicate finite(x), the procedure described in the preceding section can be used to prove the following

LEMMA 2.2. $(x \subseteq z \backslash u \;\&\; z = pow(v) \;\&\; l = v \cup m \;\&\; w = pow(l) \;\&$ finite$(w)) \rightarrow$ finite(x)

Indeed this lemma is a consequence of $MLSP$ together with the monotonicity of the finiteness predicate ($\varphi(x,y)$ & finite(x) →finite(y)).

Notice that lemma 2.2 contains the additional hypothesis finite(w) which can be discarded by lemma 2.1. This completes the automatic proof of our set-theoretic statement.

BIBLIOGRAPHY

[1] Cantone D., Ferro A., Schwartz J.T., *Decision procedures for elementary sublanguages of set theory,* VI. *Multilevel syllogistic extended by the powerset operator.* Comm. Pure Appl. Math., **38**, (1985), 549-571.

[2] Dantoni G., *Relazioni invarianti di un'algebra universale ed algebre con il sistema di operazioni completo rispetto ad una famiglia di relazioni invarianti,* Le Matematiche, Vol XXIV, **1**, (1969), 187-216.

[3] Ershov Y.L., Lavrov I.A., Taimanov A.D., Taitslin M.A., *Elementary theories,* Russian Mathematical Surveys, 20, no. **4**, (1965), 35-105.

[4] Ferro A., Omodeo E., Schwartz J.T., *Decision procedures for elementary sublanguages of set theory,* I. *Multilevel syllogistic and some extensions,* Comm. Pure Appl. Math., **33**, (1980), 599-608.

[5] Nelson G., Oppen D., *Simplification by cooperating decision procedures,* ACM Transactions Prog. Lang. Syst. **1**, (1979), 245-257.

[6] Rosenberg I., *La structure des functions de plusieurs variables sur un ensemble fini,* C. R. Acad. Sc. Paris **260**, (1965), 3817-3819.

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