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DECIDABILITY RESULTS FOR CLASSES OF PURELY UNIVERSAL FORMULAE AND QUANTIFIERS ELIMINATION IN SET THEORY

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A quantified theory of sets involving the boolean predicate Finite and a particular class of Δ_0 purely universal formulae are shown to have a decidable satisfiability problem. Also a technique for quantifiers elimination is given.

1. Introduction.

In this paper we present a decidability result for a quantified theory of sets involving the boolean predicate *Finite*. Also we will show a technique which, under certain conditions, allows the elimination of quantifiers from a given formula. So far many unquantified theories have been shown to be decidable and decision algorithms have been given for them (see for example [2]-[10] and [12]). Thus this procedure allows to lift to the quantified case several decidability results

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concerning unquantified sublanguages. Related results for quantified theories are also given in [11], [12]. Moreover a particular class of Δ_0 purely universal formulae is shown to be decidable by using a technique that also proves, as a by-product, a reflection result on the hereditarily finite sets for that class of formulae. Notice that this is not the case in general, as it has been shown in [14]. Related results for quantified theories are also given in [11], [12], [13] and [14]. For all the definitions and basic properties in set theory we refer to [1].

2. The predicate Finite.

Let T be the quantifier free theory in the language $\underline{\emptyset}, =, \in$, *Finite* where the constant $\underline{\emptyset}$ represents the empty set and the predicate $Finite(x)$ is true if and only if x represents a finite set.

DEFINITION 2.1. *A simple prenex formula on the theory T is a formula of the form*

$$Q_1 Q_2 \dots Q_n \varphi$$

where:

(a) φ is a boolean combination of literals of type $x \in y$, $x = y$, $Finite(x)$;

(b) the Q_i's are restricted quantifiers either all of the form $\exists x_i \in y_i$ or all of the form $\forall x_i \in y_i$;

(c) no x_i is a y_j, for any $i, j = 1, \dots, n$.

DEFINITION 2.2.

(i) By $SPF(T)$ we denote the class of simple prenex formulae of the theory T.

(ii) By $SPF^\star(T)$ we denote the subclass of $SPF(T)$ consisting of those formulae in which the predicate Finite can apply only to free variables.

(iii) *By $SPF^{\star}_{=}(T)$ we denote the subclass of $SPF^{\star}(T)$ consisting of those formulae in which the relation = never applies to pairs of bounded variables.*

This paper solves the decision problem for the class of formulae $SPF^{\star}_{=}(T)$. Namely, let Φ be a conjunction of formulae of $SPF^{\star}_{=}(T)$ and let $V = \{y_1, \ldots, y_n\}$ be the set of free variables occurring in Φ. Also, let $V_0 = V \cup \{\underline{\emptyset}\}$.

THEOREM 2.3. *Let D and F be two collections of set variables such that $|D| \leq n^2 - n$ and $|F| \leq n$. Let $W = V_0 \cup D \cup F$ and let Ψ be the formula obtained from Φ by recursively replacing each formula $(\forall x \in y)\varphi$ with the set of formulae $\{((x \in y) \to \varphi)^x_t | t \in W\}$ until no universal quantifiers are left. Then Φ is injectively satisfiable if and only if there exist*

- *a set Q of membership and inequalities relations on the elements of W and assignments of truth values to the predicate Finite ranging on V_0, such that*
  - *for each x, y distinct variables in W, either $x \in y$ or $x \notin y$ is in Q,*
  - *for each x, y distinct variables in $W, x{\neq}y$ is in Q, and*
  - *for each $x \in V_0$ either Finite (x) or $\neg$Finite(x) is in Q.*
- *a disjunct Ψ' of a disjunctive normal form of Ψ*

such that

1. *$\Psi' \wedge Q$ does not contain any explicit contradiction of the form $x{\neq}x$, or $x = y \wedge x{\neq}y$, or $x \in y \wedge x \notin y$, or Finite$(x) \wedge \neg$Finite$(x)$;*
2. *Q does not contain any cycle of memberships $x_0 \in x_1 \in \ldots \in x_h \in x_0$;*
3. *for each y_i, y_j in V either $y_i \equiv y_j$ or there exists d in W such that either $d \in y_i \wedge d \notin y_j$ or $d \notin y_i \wedge d \in y_i$ are in Q;*
4. *for each z in W, $z \notin \underline{\emptyset}$ is in Q;*
5. *for each y in V, $\neg$Finite(y) is in Q if and only if there exists f in*

F such that $f \in y$ *is in* Q.

Proof. $\Rightarrow$ Let us suppose that Φ has a model M, such that $M\underline{\emptyset} = \underline{\emptyset}$. Then for each $y \in V$, if My is finite put *Finite*(y) in Q otherwise put $\neg$*Finite*(y) in Q. Put also *Finite*$(\underline{\emptyset})$ in Q. Let A be a minimal set which intersects all nonempty sets of the form

$$(Mx \backslash My) \backslash \{Mz | z \text{ in } V_0\}$$

with x, y ranging over V_0. For each $a \in A$ introduce a new variable z_a and put $Mz_a = a$. Let $D = \{z_a | a \in A\}$. Clearly $|D| \leq n^2 - n$.

Let B be a minimal set which intersects all the infinite sets of the form

$$My \backslash \{Mz | x \text{ in } V \cup D\},$$

for y in V, and which has empty intersection with all the finite sets My', with y' in V. For each b in B, introduce a new variable f_b and put $Mf_b = b$. Let F be the set of these newly introduced variables. Then clearly $|F| \leq n$.

Finally for each pair of variables in $V_0 \cup D \cup F$, say z_1 and z_2, if $Mz_1 \in Mz_2$ put $z_1 \in z_2$ in Q, else put $z_1 \notin z_2$ in Q. Let then Ψ be a formula obtained from Ψ as described in the statement of the theorem.

LEMMA 2.4. *M satisfies* Ψ.

Proof. Let C be a conjunct in Ψ. Then C is logically equivalent to a formula of type

$$(w_{i_1} \in y_{i_1} \wedge \ldots \wedge w_{i_s} \in y_{i_s}) \rightarrow \varphi^{x_{i_1} \ldots x_{i_s}}_{w_{i_1} \ldots w_{i_s}},$$

for some conjunct $(\forall x_{i_1} \in y_{i_1}) \ldots (\forall x_{i_s} \in y_{i_s})\varphi$ in Φ. If $Mw_{i_j} \notin My_{i_j}$, for some $j = 1, \ldots, s$, then C is vacuously satisfied by M. On the other hand, if $Mw_{i_j} \in My_{i_j}$ for all $j = 1, \ldots, s$, then since M satisfies $(\forall x_{i_1} \in y_{i_1}) \ldots (\forall x_{i_s} \in y_{i_s})\varphi$, it clearly satisfies $\varphi^{x_{i_1} \ldots x_{i_s}}_{w_{i_1} \ldots w_{i_s}}$ also. ■

Let us bring Ψ to disjunctive normal form and let Ψ' be a disjunct satisfied by M. Since M satisfies Ψ' and Q, from the very construction of Q it follows that conditions 1-5 hold.

$\Leftarrow$ Conversely, assume that there exist D, F, Q, Ψ' such that conditions 1-5 are satisfied.

For every set s we define inductively

$$s^{(0)} = s, \; s^{(h+1)} = \{s^{(h)}\}.$$

Let $<$ be a total ordering of W such that

- $\underline{\emptyset}$ is the minimum of $<$;
- if $x \in y$ is in Q then $x < y$.

Notice that conditions 1, 2, 4, assure that such an ordering always exists.

Let $\{i_x | x \text{ in } W \setminus V_0\}$ and $\{j_f | f \text{ in } F\}$ be two, respectively $O(n^2)$ and $O(n)$, collections of pairwise distinct, finite sets such that all the i_x's have the same rank ρ and cardinality $\gamma \geq n^2 + 1$ and all the j_f's have the same rank $\rho' \geq \rho + n^2$ and cardinality $\gamma' \geq \gamma$.

Following the ordering $<$ of variables, we put for all x in W:

$$Mx = I_x \cup F_x \cup \{Mx' | x' \in x \text{ is in } Q, x' \text{ in } W\} \tag{1}$$

where:

$$I_x = \begin{cases} \emptyset & \text{if } x \text{ is in } V_0 \\ \{i_x\} & \text{if } x \text{ is in } W \setminus V_0, \end{cases}$$

$$F_x = \begin{cases} \emptyset & \text{if } f \notin x \text{ is in } Q \text{ for all } f \text{ in } F \\ \{j_f^{(h)} \cup Mf | h \geq 1, f \text{ in } F, f \in x \text{ in } Q\} & \text{otherwise.} \end{cases}$$

LEMMA 2.5. *For all $z \in W$, $f \in F$, $h \geq 0$, the following assertions are true*

(i) $Mz \neq i_x, j_f$, *for all* $x \in D \cup F$.

(ii) $Mz \neq j_f^{(h)}$.

(iii) $Mz \neq j_f^{(h)} \cup Mf$.

Proof. (i) If $F_z \neq \emptyset$, then $|Mz| \geq \omega$. On the other hand if $F_z = \emptyset$ then $f \notin x$ is in Q for all f in F so that

$$|Mz| \leq 1 + |W \backslash F| = 1 + |V_0| + |D| = 1 + n + 1 + n^2 - n = n^2 + 2.$$

Therefore either $|Mz| \geq \omega$ or $|Mz| \leq n^2 + 2$ and in any case $Mz \neq i_x$ and $Mz \neq j_f$, since $n^2 + 3 \leq |i_x| \leq |j_f| < \omega$.

(ii) The case in which $h = 0$ has already been considered in (i). We can therefore suppose $h \geq 1$. We distinguish two cases according to whether there is a chain of membership relations $f' \in z_{i_1} \in \ldots \in z_{i_k} \in z$, with $k \geq 0$ and $f' \in F$, or not. In the first case, it follows easily that $rank(Mz) \geq \omega$. In the second case it can be proved by induction that $rank(Mz) \leq \rho + h_z + 1$ where h_z is the length of a longest chain of membership relations ending in z. Thus $h_z \leq |W \backslash F|$, which in turn implies $rank(Mz) \leq \rho + n^2 + 2$. In any case $Mz \neq j_f^{(h)}$, since $\rho + n^2 + 2 + h \leq rank(j_f^{(h)}) < \omega$ for all $h \geq 1, f \in F$.

(iii) Assume by contradiction that $Mz = j_f^{(h)} \cup Mf$, for some z in W, f in F, and $h \geq 1$. Then $j_f^{(h-1)} \in Mz$. Notice that $j_f^{(h-1)} \notin I_z$, since if $I_z \neq \emptyset$ then $rank(j_f^{(h-1)}) = \rho' + (h-1) > \rho = rank(i_z)$. In addition, by (ii), $j_f^{(h-1)} \notin \{Mz' | z' \in z \text{ is in } Q,\ z' \text{ in } W\}$. Therefore, if $Mz = j_f^{(h)} \cup Mf$, by (1) we necessarily must have $j_f^{(h-1)} \in F_z \neq \emptyset$, i.e., $j_f^{(h-1)} = j_g^{(k)}$, for some $k \geq 1$ and g in F such that $g \in z$ is in Q. We will show below that this is impossible, therefore proving that $Mz \neq j_f^{(h)} \cup Mf$, for all z in W, f in $F, h \geq 1$. Indeed, if $j_f^{(h-1)} = j_g^{(k)}$ then from the definition of the $j_{f'}$'s, f' in F, we must have $f \equiv g$ and $k = h - 1$. But then $f \in z$ would be in Q and $j_f^{(h)} \cup Mf \in Mz$, i.e., $Mz \in Mz$, which contradicts the well foundedness of membership. ■

LEMMA 2.6. *For all w_1, w_2 in W*

(i) if $w_1 \not\equiv w_2$ then $Mw_1 \neq Mw_2$;

(ii) $w_1 \in w_2$ is in Q if and only if $Mw_1 \in Mw_2$.

Proof. (i) Let $w_1 \not\equiv w_2$. Suppose first that w_1, w_2 are in V_0. Let $height_Q(w)$ be the length of a longest chain of membership relations in Q ending in w. We will proceed by induction on *max(height*$_Q(w_1)$, *height*$_Q(w_2)$). If *max(height*$_Q(w_1)$, *height*$_Q(w_2)$) $= 0$, then by condition 3 of the theorem $w_1 \equiv w_2$, which shows that the base case of the induction is vacuously true. Concerning the inductive step, assume by contradiction that $Mw_1 = Mw_2$. From condition 3 of the theorem, it follows that there exists a variable d in W such that either $d \in w_1 \wedge d \notin w_2$ is in Q, or $d \notin w_1 \wedge d \in w_2$ in Q. Suppose for defineteness that $d \in w_1$ and $d \notin w_2$ are in Q. The definition (1) of the model M implies that $Md \in Mw_1$. Thus our initial assumption yields $Md \in Mw_2$, which by (i) and (iii) of the preceding lemma implies $Md = Md'$ for some variable d' in W for which $d' \in w_2$ is in Q. But *max(height*$_Q(d)$, *height*$_Q(d')$) <*max(height*$_Q(w_1)$, *height*$_Q(w_2)$). Hence by induction $d \equiv d'$ and consequently both $d \notin w_2$ and $d \in w_2$ are in Q, contradicting condition 1 of the theorem. Thus (i) is proved.

(ii) If $w_1 \in w_2$ is in Q then from the definition (1) of the assignment M it follows that $Mw_1 \in Mw_2$. Conversely, assume that $Mw_1 \in Mw_2$. From Lemma 2.5, it follows that $Mw_1 \notin I_{w_2} \cup F_{w_2}$. Therefore $Mw_1 = Mw_1'$ for some w_1' in W such that $w_1' \in w_2$ is in Q. Thus from (i) $w_1 \equiv w_1'$, i.e. $w_1 \in w_2$ is in Q. ∎

LEMMA 2.7. *The following assertions are true for all* $h \geq 1$, *f in F, x in W:*

(i) $(j_f^{(h)} \cup Mf) \in Mx$ *if and only if* $Mf \in Mx$;

(ii) $Mx \in (j_f^{(h)} \cup Mf)$ *if and only if* $Mx \in Mf$;

Proof. (i) Let $(j_f^{(h)} \cup Mf) \in Mx$, for some f in F, x in W and $h \geq 1$. From (1), the definition of the i_x's and j_f's, and Lemma 2.5 it follows that $j_f^{(h)} \cup Mf = j_g^{(k)} \cup Mg$ for some g in W such that $g \in x$ is in Q. We will show that $f \equiv g$, thus proving that $f \in x$ is in Q and in turn by Lemma 2.6 (ii) that $Mf \in Mx$. We have $j_f^{(h-1)} \in j_g^{(k)} \cup Mg$. We show that $j_f^{(h-1)} \in j_g^{(k)}$. Indeed if $j_f^{(f-1)} \in Mg$, then as above, $j_f^{(h-1)} \in j_{g'}^{(k')} \cup Mg'$, for some g' in W such that $g' \in g$ is in Q, and for some $k' \geq 1$. But by Lemma 2.5(ii)

$|j_{g'}^{(k')} \cup Mg'| \geq 2$, thus $h = 1$, i.e., $j_f^{(h-1)} = j_f$. This is a contradiction since $rank(j_f) = rank(j_{g'}) < rank(j_{g'}^{(k')}) \leq rank(j_{g'}^{(k')} \cup Mg')$. Having proved that $j_f^{(h-1)} \in j_g^{(k)}$, from the definition of the j's it follows that $f \equiv g$ (and $h = k$).

Conversely, assume that $Mf \in Mx$. Then the preceding lemma yields that $f \in x$ is in Q which by (1) in turn implies that $(j_f^{(h)} \cup Mf) \in Mx$, for all $h \geq 1$.

(ii) In order to prove (ii) it is enough to show that $Mx \notin j_f^{(h)}$ for all f in F and $h \geq 1$. But this follows immediately from Lemma 2.5(ii), since if $Mx \in j_f^{(h)}$ then we would have $Mx = j_f^{(h-1)}$, which is a contradiction. ■

LEMMA 2.8. *M is a model for* Ψ'.

Proof. Notice that by condition 1 of the theorem:

- if $x \in y$ (resp. $x \notin y$) is in Ψ' then $x \in y$ (resp. $x \notin y$) is in Q;
- if $x = y$ is in Ψ' then $x \equiv y$;
- if $x \neq y$ is in Ψ' then $x \neq y$ is in Q;
- if *Finite*(x) (resp. $\neg$*Finite*(x)) is in Ψ' then *Finite*(x) (resp. $\neg$*Finite*(x)) is in Q.

Thus in view of Lemma 2.6, in order to show that M models correctly every conjunct of Ψ' it is enough to show that if *Finite*(x) (resp. $\neg$*Finite*(x)) is in Ψ', then $|Mx| < \omega$ (resp. $|Mx| \geq \omega$). But if *Finite*(x) is in Ψ' then from conditions 1,5 of the theorem and the definition of Q it follows that $F_x = \emptyset$, so that, by(1), $|Mx| = |I_x \cup \{Mx' | x' \in x \text{ in } Q\}| < \omega$. And also, if $\neg$*Finite*(x) is in Ψ', then again from conditions 1,5 of the theorem it follows that $\{j_f^{(h)} \cup Mf | h \geq 1\} \subseteq F_x$, for some f in F. Therefore $|Mx| \geq |F_x| \geq \omega$. ■

Now we are ready to show that M, as defined by (1), satisfies Φ. Since all unquantified conjuncts of Φ are also present in Ψ', we only need to prove that M satisfies all conjuncts in Φ of the form

$$(\forall x_1 \in y_{i_1}) \ldots (\forall x_k \in y_{i_k})\varphi \tag{2}$$

where $y_{i_1}, \ldots, y_{i_k}$ are in V and φ is an unquantified formula of T. To show that M satisfies (2) we must prove that for all $s_1 \in My_{i_1}, \ldots, s_k \in My_{i_k}$, the assignment $M[x_1/s_1] \ldots [x_k/s_k]$(1) satisfies φ.

Let

$$\tilde{s}_j = \begin{cases} s_j & \text{if } s_j \in \{Mz \mid z \in y_{i_j} \text{ is in } Q\} \\ Mf & \text{if } s_j = j_f^{(h)} \cup Mf \text{ for some } h \geq 1, \\ & \quad f \text{ in } F \text{ such that } f \in y_{i_j} \text{ is in } Q. \end{cases}$$

Thus $\tilde{s}_j = Mz_j$ for some $z_1, \ldots, z_k$ in W. Since M satisfies Ψ' (cf. Lemma 2.8), it follows that M satisfies Ψ too. Consequently, since $Mz_j \in My_{i_j}$, for all $j = 1, \ldots, k$, M satisfies $\varphi^{x_1,\ldots,x_k}_{z_1,\ldots,z_k}$. Let l be any literal in φ. From our assumption that φ belongs to the class $SPF^{\star}_{=}(T)$ it follows that l can be neither of type $x_{i'} = x_{i''}$ nor of type $x_{i'} \neq x_{i''}$. By considering all remaining possibilities for the literal l, Lemma 2.7 yields $M[x_1/s_1] \ldots [x_k/s_k](l) = M[x_1/\tilde{s}_1] \ldots [x_k/\tilde{s}_k](l) = M(l^{x_1,\ldots,x_k}_{z_1,\ldots,z_k})$. Therefore $M[x_1/s_1] \ldots [x_k/s_k](\varphi) = M(\varphi^{x_1,\ldots,x_k}_{z_1,\ldots,z_k})$ =*true*. For the arbitrariness of $s_1 \in My_{i_1}, \ldots, s_k \in My_{i_k}$, this in turn implies that M satisfies (2), concluding the proof of the theorem. ∎

Remark Notice that the preceding algorithm fails to detect unsatisfiability of certain conjunctions of formulae of $SPF^{\star}(T)$. Consider for example the formula

$$\neg \mathit{Finite}(y) \wedge (\forall x_1 \in y)(\forall x_2 \in y)(x_1 = x_2)$$

which is clearly unsatisfiable. It is quite easy to see that by choosing $D = \emptyset$, $F = \{f\}$, $Q = \{f \in y, f \notin \emptyset, \emptyset \notin y, \emptyset \in f\}$, all conditions of the theorem are satisfied. ∎

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(1) Given an assignment A over sets, a variable x, and a set s, by $A[x/s]$ we mean the assignment B such that $By = Ay$, for all $y \not\equiv x$ and $Bx = s$.

3. Elimination of quantifiers in set theory.

In this section we present a result of elimination of quantifiers in set theory.

Consider the class of formulae in the language $\mathcal{L}$ consisting of

(i) a denumerable infinity of set variables;

(ii) set operators $(\emptyset, \cup, \cap, \setminus, \{\cdot, \ldots, \cdot\}$, pow, $Un, \times, \ldots)$;

(iii) set predicates $(=, \in, \ldots)$;

(iv) boolean connectives $(\wedge, \vee, \neg, \rightarrow, \leftrightarrow)$;

(v) quantifiers $(\forall, \exists)$.

DEFINITION 3.1. *A formula φ in $\mathcal{L}$ is 0-flat if each quantified variable x in φ appears only within atoms of type*

$$x = t \text{ or } x \in t,$$

where t is any term of $\mathcal{L}$ not containing x.

Remark Notice that the preceding definition implies that in a 0-flat formula φ:

(a) no composed term occurring in φ can contain quantified variables;

(b) if $x \in y$ occurs in φ, then y is free;

(c) if $x = y$ is in φ, then either x or y must be free.

THEOREM 3.2. *Let φ be a 0-flat formula of $\mathcal{L}$. Then there is an algorithm to constrct a quantifier-free formula ψ such that*

$$\vdash_{ZF} (\varphi \leftrightarrow \psi).$$

Proof. Below is the algorithm:

Step 1. Bring φ in prenex normal form, denoted by φ_1.

Step 2. Let φ_2 be obtained by replacing in φ_1 every universal quantifier $(\forall x)$ by the expression $\neg(\exists x)\neg$. Let p be the matrix of

φ_2 and let x be the innermost quantified variable (i.e., $(\exists x)p$ is a subformula of φ_2). Bring p into disjnctive normal form

$$(p_{11}\wedge\ldots\wedge p_{1,h_1}\wedge p_{1,h_1+1}\wedge\ldots\wedge p_{1,k_1})\vee\ldots\vee(p_{n,1}\wedge\ldots\wedge p_{n,h_n}\wedge p_{n,h_n+1}\wedge\ldots\wedge p_{n,k_n}),$$

where x does occur in $p_{j,1},\ldots,p_{j,h_j}$ and does not occur in $p_{j,h_j+1},\ldots,p_{j,k_j}$, $j=1,\ldots,n$.

Comment: Notice that

$$\vdash_{ZF} (\exists x)p \leftrightarrow \left[\bigvee_{j=1}^{n}((\exists x)(p_{j,1}\wedge\ldots\wedge p_{j,h_j})\wedge(p_{j,h_j+1}\wedge\ldots\wedge p_{j,k_j}))\right].$$

Therefore, for all $j=1,\ldots,n$, it is enought to construct a quantifier-free formula ψ_j, such that

$$\vdash_{ZF} (\exists x)(p_{j,1}\wedge\ldots\wedge p_{j,h_j})\leftrightarrow\psi_j.$$

Step 3. If $x=t$ is one of the conjuncts $p_{j,1},\ldots,p_{j,h_j}$, then we put

$$\psi_j\equiv(p_{j,1}\wedge\ldots\wedge p_{j,h_j})^x_t.$$

So, assume that no p_{jl} is of type $x=t$. Let

$$\begin{array}{ccc} x\in t_1, & \ldots, & x\in t_{m_1}, \\ x\notin t_{m_1+1}, & \ldots, & x\notin t_{m_2}, \\ x\neq t_{m_2+1}, & \ldots, & x\neq t_{m_3} \end{array}$$

be the conjuncts $p_{j,1},\ldots,p_{j,h_j}$.

Then we put

$$\psi_j\equiv\begin{cases}((t_1\cap\ldots\cap t_{m_1})\backslash(t_{m_1+1}\cup\ldots\cup t_{m_2}\cup\{t_{m_2+1},\ldots,t_{m_3}\})\neq\emptyset) & \text{if } m_1\geq 1\\ true & \text{if } m_1=0.\end{cases}$$

Let φ_3 be the formula obtained by replacing in φ_2 the subformula $(\exists x)p$ by the subformula $\bigvee_{j=1}^{n}\psi_j$.

Comment: We have $\vdash_{ZF} (\varphi_2 \leftrightarrow \varphi_3)$. *Moreover,* φ_3 *has one quantifier less than* φ_2 *and it is still a 0-flat formula.*

By applying repeatedly steps 2 and 3, in a finite number of iterations a quantifier-free formula ψ is obtained. Then, it follows immediately from the above observations that

$$\vdash_{ZF} (\varphi \leftrightarrow \psi). \qquad \blacksquare$$

Remark If the underlying unquantified theory is decidable, then so is the corresponding quantified class of 0-flat formulae.

EXAMPLE 1. Let $\mathcal{L}_1$ be the language whose set operators are $\emptyset, \cup, \cap, \setminus, \{\cdot, \ldots, \cdot\}$, *pow* and whose predicate symbols are $=, \in$. In [3] this theory has been shown to be decidable. Then the class of 0-flat formulae over $\mathcal{L}_1$ is decidable.

EXAMPLE 2. Let $\mathcal{L}_2$ be the language whose set operators are $\emptyset, \cup, \cap, \setminus$, *Un* and whose predicate symbols are $=, \in$. In [6] this theory has been shown to be decidable. Then the class of *restricted* 0-flat formulae over $\mathcal{L}_2$ is decidable, where a formula is *restricted 0-flat* if all quantified variables can occur only within atoms of the form $x \in t$.

3.1. Examples of elimination of quantifiers.

In this subsection we show how the algorithm works on two examples.

EXAMPLE 1. Consider the following formula φ

$$(\forall x)(\exists z)((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w)).$$

φ is already in prenex normal form so the first step of the algorithm can be skipped. Application of step 2 leads to

$$\neg(\exists x)\neg(\exists z)((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w))$$

whose matrix is already in disjunctive normal form. By distributing the existential quantifier over the disjunction we then get

$$\neg(\exists x)\neg((\exists z)(z \in w \wedge x \notin w) \vee (\exists z)(z \notin w \wedge x \in w)),$$

which is logically equivalent to

$$\neg(\exists x)\neg(((\exists z)(z \in w) \wedge (x \notin w)) \vee ((\exists z)(z \notin w) \wedge (x \in w)).$$

Application of step 3 then produces the formula

$$\neg(\exists x)\neg((w{\neq}\emptyset \wedge x \notin w) \vee (w{\neq}\emptyset \wedge x \in w),$$

i.e.,

$$\neg(\exists x)\neg((w{\neq}\emptyset \wedge x \notin w) \vee x \in w).$$

Bringing in the negation we obtain

$$\neg(\exists x)(\neg(w{\neq}\emptyset \wedge x \notin w) \wedge x \notin w),$$

i.e.,

$$\neg(\exists x)((w = \emptyset \vee x \in w) \wedge x \notin w),$$

which is obviously logically equivalent to

$$\neg(\exists x)(w = \emptyset \wedge x \notin w).$$

Finally, execution of step 3 gives

$$\neg(w = \emptyset \wedge \ \mathit{true}),$$

i.e.,

$$w{\neq}\emptyset.$$

So we can conclude that

$$\vdash_{ZF} (\forall x)(\exists z)((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w)) \leftrightarrow w{\neq}\emptyset.$$

EXAMPLE 2. Consider now the following formula φ'

$$(\exists z)(\forall x)((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w))$$

(which is obtained from the previous one by simply inverting the order of the quantifiers). Applying step 2 we get

$$(\exists z)\neg(\exists x)\neg((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w)),$$

which is logically equivalent to

$$(\exists z)\neg(\exists x)((z \notin w \vee x \in w) \wedge (z \in w \vee x \notin w)),$$

and also to

$$(\exists z)\neg(\exists x)((z \notin w \wedge x \notin w) \vee (x \in w \wedge z \in w)).$$

Thus by eliminating the quantifier $(\exists x)$, we obtain

$$(\exists z)\neg((z \notin w \wedge \textit{ true}) \vee (w \neq \emptyset \wedge z \in w)),$$

which is logically equivalent to

$$(\exists z)((z \in w) \wedge (w = \emptyset \vee z \notin w))$$

and also to

$$(\exists z)((z \in w \wedge w = \emptyset) \vee (z \in w \wedge z \notin w)).$$

This last formula obviously simplifies to

$$(\exists z)((z \in w \wedge w = \emptyset),$$

so that by applying again steps 2 and 3 we finally get

$$w \neq \emptyset \wedge w = \emptyset \; (\equiv \textit{ false}).$$

Thus we can conclude that

$$(\exists z)(\forall x)((z \in w \wedge x \notin w) \vee (z \notin w \wedge x \in w))$$

is unsatisfiable.

4. The $(\forall)_{0-}1$-formulae.

Let $(\forall)_0$ be the class of formulae such that

$$\varphi \in (\forall)_0$$

if and only if

$$\varphi = \bigwedge_{i=1,\ldots,n} \varphi_i$$

and

$$\varphi_i = (\forall x_{i,1} \in y_{i,1}) \ldots (\forall x_{i,m_i} \in y_{i,m_i})(l_{i,1} \vee \ldots \vee l_{i,p_i})$$

where $m_i \geq 0$, $p_i \geq 1$, and $l_{i,j}$ for $1 \leq j \leq p_i$ is a literal in the language $\emptyset, \in, =$.

A complete description of and an introduction to the class $(\forall)_0$ can be found in [13], where is also shown that the decision problem for simple prenex formulae (section 2 of this work, [11], [12]) can be seen as a subproblem of the decision problem for the $(\forall)_0$-formulae.

In [14] is presented a $(\forall)_0$-formulae which is satisfiable but not finitely satisfiable and in the present section we introduce a syntactic restriction that will allow us to obtain a reflection result over the hereditarily finite sets as well as a decidability result for a subclass of the $(\forall)_0$-formulae.

DEFINITION 4.1. *φ is a$(\forall)_{0-}1$-formula if and only if:*

1. *φ is a $(\forall)_0$-formula.*
2. *if $\varphi = \bigwedge_{i=1,\ldots,n} \varphi_i$ then any φ_i is of the form*

$$\varphi_i = (\forall x_{1,i} \in y_{1,i}) \ldots (\forall x_{m_i,i})(x \diamond y)$$

where $\diamond$ can be any one of the predicates $\notin, \in, \neq, =$.

EXAMPLE:

1. $x \in y \wedge y \in w \wedge x \neq \emptyset \wedge (\forall z_1 \in y)(\forall z_2 \in z_1)(z_2 \in w)$ is a $(\forall)_0$-1-formula.
2. $(\forall x \in y)(\forall z \in x)(z \in y) \wedge y \neq \emptyset$ is a $(\forall)_0$-1-formula.

Remark Any decision algorithm for subclasses of the $(\forall)_0$-formulae can have, as a preprocessing step, the decision algorithm for the unquantified formulae of the language $\emptyset, \in, =$ presented in [11] that forbids to continue and declare the whole formula unsatisfiable if its unquantified part is unsatisfiable. ■

First we prove the following lemma that will allow us to put any $(\forall)_0$-1-formula in a suitable form.

Let $(\forall)_{0,l}$-1 the subclass of $(\forall)_0$-1-formulae of maximum nesting level l, where the nesting level of a formula φ is the maximum K in any chain of the form

$$(\forall x_1 \in y)(\forall x_2 \in x_1)(\forall x_3 \in x_2)\ldots(\forall x_K \in x_{K-1})$$

in the quantified prefix of one of the conjuncts in φ.

Both the examples presented above are $(\forall)_{0,2}$-1 formulae (see [13] for more information).

In the following we will denote by $\bigcup x$ the set of all elements of elements of x and by $\bigcap x$ the intersection of all elements of x.

Moreover $\bigcup^{n} x = \bigcup \left(\bigcup^{n-1} x \right)$ and $\bigcap^{n} x = \bigcap \left(\bigcap^{n-1} x \right)$, where $\bigcup^{0} x = \bigcap^{0} x = x$.

Let us start by showing that it is general enough to consider only a rather simplified form of $(\forall)_0$-1-formulae.

LEMMA 4.2. *Given any $\varphi \in (\forall)_{0,l}$-1 it is always possible to build a ψ such that:*

i) φ and ψ are equisatisfiable (provably in a suitable axiomatic set theory).

ii) ψ is a propositional combination of literals of the form:

1. $\bigcup^{m} x \subseteq \bigcap^{n} y$,
with $l > m > 0$, $l > n \geq 0$.

2. $\bigcup^{m} x \cap \bigcap^{n} y = \emptyset$,
with $l > m > 0$ *and* $l > n > 0$.

3. $x \neq \emptyset$.

4. $(\forall x_1 \in x)(\forall x_2 \in x_1)\ldots(\forall x_p \in x_{p-1})L$
where L *is of the form*

$$\bigcup^{m} x_p \subseteq \bigcap^{n} x_p \text{ with } l > m + p > 0, l > n + p \geq 0$$

or

$$\bigcup^{m} x_p \cap \bigcup^{n} x_p = \emptyset \text{ with } l > m + p, n + p > 0.$$

Proof. We will describe a sequence of algorithmic transformations of the formula φ such that the formula obtained at the end will satisfy i), ii).

Step 0. Apply the test described in [11] and if the answer is *unsatisiable* set ψ equal to $\emptyset \neq \emptyset$.(2)

Step 1. Consider all the unquantified conjuncts in φ of the form:

$$x \neq y$$

and substitute them with the following formula:

$$(z_{xy} \in x \wedge z_{xy} \notin y) \vee (z_{xy} \notin x \wedge z_{xy} \in y)$$

with z_{xy} a new variable.

Let φ_1 be the formula obtained after this step.

Step 2. Bring φ_1 in disjunctive normal form and perform the following actions on each disjunct: substitute each conjunct of the form:

(2) see previous remark.

1. $x \in y$
2. $x \notin y$
3. $x = y$
4. $(\forall w \in y)(x \in w)$ (analogous for $x \notin w$)

respectively with the formula:

1. $(\forall z \in x')(z = x) \wedge (\forall z \in x')(z \in y) \wedge x' \neq \emptyset$
2. $(\forall z \in x')(z = x) \wedge (\forall z \in x')(z \notin y) \wedge x' \neq \emptyset$
3. $(\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x)$
4. $(\forall z \in x')(z = x) \wedge (\forall z \in x')(\forall w \in y)(z \in w) \wedge x' \neq \emptyset$ (analogous for $x \notin w$)

where x' is a newly introduced variable.

Let φ_2 be the formula obtained after this step.

In the following steps litersals of the form $z = y$, $z \neq y$ are eliminated from the matrices of quantified conjuncts.

Spep 3. Eliminate any conjunct in φ_2 of the form:

$$(\forall z \in x)(z \neq y)$$

by substituting it with

$$y \notin x$$

and then perform the substitution described in step 2 case to eliminate this last unquantified conjuncts.

Eliminate any conjunct of the form

$$(\forall z \in x)(z = y)$$

by substituting it with

$$(\forall z \in x)(\forall z_1 \in z)(z_1 \in y) \wedge (\forall z \in x)(\forall y_1 \in y)(y_1 \in z)$$

This transformation can be justified by observing that $(\forall z \in x)$ $(z = y)$ is equisatisfiable with

$$(\forall z \in x)((\forall z_1 \in z))(z_1 \in y) \wedge (\forall y_1 \in y)(y_1 \in z)).$$

Let φ_3 be the formula obtained after this step.

Notice that at this point any unquantified conjunct in φ_3 is of the form: $x \neq \emptyset$.

Step 4. Substitute any conjunct in φ_3 of the form

$$(\forall z \in x)(\forall w \in y)(z = w)$$

with

$$(\forall z \in x)(\forall z' \in z)(\forall w \in y)(z' \in w) \wedge (\forall z \in x)(\forall w \in y)(\forall w' \in w)(w' \in z),$$

where this formula can be obtained as in the previous case in step 3 by distributing the bounded quantifiers.

Perform an analogous substitution in the cases in which the number of quantifiers of the quantified conjunct is greater than two.

Substitute any conjunct of the form:

$$(\forall z \in x)(\forall w \in y)(z \neq w)$$

with

$$(\forall z \in x)(z \notin y)$$

and perform an analogous substitution in the case the number of quantifiers of the quantified conjunct is greater.

Let φ_4 be the formula obtained after this step.

Notice that at this point any quantified conjunct has a literal of the form $x \in y$ or $x \notin y$ as matrix.

Step 5. Substitute any conjunct of the form

$$(\forall x_1 \in x)(\forall x_2 \in x_1) \ldots (\forall x_m \in x_{m-1})$$

$$(\forall y_1 \in y)(\forall y_2 \in y_1) \ldots (\forall y_n \in y_{n-1})(x_m \diamond y_n)$$

(notice that we can assume $m \geq 1$ because of step 2 case 4 with

- $\bigcup^{m-1} x \subseteq \bigcap^{n} y$

if $\diamond$ is $\in$

- $\bigcup^{m-1} x \cap \bigcup^{n} y = \emptyset$

if $\diamond$ is $\notin$

Let ψ be the formula obtained after this step.

Step 6. Substitute any conjunct of the form

$$(\forall x_1 \in x)(\forall x_2 \in x_1)\ldots(\forall x_p \in x_{p-1})$$

$$(\forall z_1 \in x_p)\ldots(\forall z_m \in z_{m-1})(\forall y_1 \in x_p)\ldots(\forall y_n \in y_{n-1})(z_m \diamond y_n)$$

with

- $(\forall x_1 \in x), \ldots, (\forall x_p \in x_{p-1}) \left(\bigcup^{m-1} x_p \subseteq \bigcap^{n} x_p \right)$

if $\diamond$ is $\in$

- $(\forall x_1 \in x)\ldots(\forall x_p \in x_{p-1}) \left(\bigcup^{m-1} x_p \cap \bigcup^{n} x_p = \emptyset \right)$

if $\diamond$ is $\notin$

It is now straightforward to check that i) and ii) hold ■

Now we prove a decidability result for the class C (included in $(\forall)_0$–1) of formulae which are conjunction of literals of the form:

$$x \neq \emptyset, \ \bigcup^{m} x \subseteq \bigcap^{n} y$$

The class C is a proper subclass of the $(\forall)_0$–1-formulae and is a class in which both the unary operators $\bigcup$ and $\bigcap$ and the binary operator $\subseteq$ occur in a particularly restricted form.

It is not known if the general case, in which occurrences of $\bigcup$, $\bigcap$ and $\subseteq$ are allowed unrestrictedly, is decidable.

LEMMA 4.3. *Given any formula $\psi \in C$ it is possible to determine a number p such that if ψ is satisfiable then it is satisfied by hereditarily finite sets of rank less than p.*

From this lemma it is immediate to conclude.

COROLLARY 4.4. *The class C is decidable.*

Proof of Lemma 4.3. To prove our lemma we will consider a formula $\varphi \in C$ and we will assume φ to be satisfiable, then we will assume $Mx_1, \ldots, Mx_k$ to be a *model* for φ (that is a tuple of sets satisfying the formula) and we will give a method to build another (simpler) model for φ.

The following two facts will hold:

1. the model M and the new model will satisfy the same set of formulae in the class C;
2. the model we will build will be a tuple of *hereditarily finite sets.*

Let us define l to be the maximum m or n appearing in a conjunct of the form

$$\bigcup^{m} x \subseteq t,$$

or

$$t \subseteq \bigcap^{n} x$$

for t any term.

Considering all the terms than can possibly appear in φ, we have that any such term is of the following form:

$$\bigcup^{0} x, \bigcup^{1} x, \ldots, \bigcup^{l} x, \bigcap^{0} x, \bigcap^{1} x, \ldots, \bigcap^{l} x, \tag{0}$$

where x is a free variable.

We will use our model M as an *oracle* that will tell us some of the features that we will reproduce in the hereditarily finite model we are going to build.

At this point we use M simply to establish which of the terms of the form (0) is non-empty, and to consider, from now on, only those terms associated to non-empty sets by M.

Analogously we will consider only those terms which are actually associated to sets by M, that is we will ignore those terms of the form $\bigcap t$ such that the empty set is associated to t by M.

Here and in the following, we will look at any information *retrieved* from M as a (correct) guess of a nondeterministic algorithm implementing the procedure we are outlining.

Besides the information directly contained in φ (such as inclusion of some term in some other and non-emptiness of some term) it is easy to verify that a certain amount of information is *hidden* in φ.

The following algorithm having φ as input and $\tilde{\varphi}$ as output, modifies φ in such a way that some of these hidden constraints appear explicitly. It is straightforward to check that all the conjuncts added to φ by the algorithm are a consequence of the usual meaning we give to the set-theoretic constructs involved.

Algorithm A.1.

step 1. set $\tilde{\varphi} := \varphi$;

Notice that the intersection of a set a it is always included in the union of a since $x \in \bigcap a$ iff $(\forall z \in a)(x \in z)$, whereas $x \in \bigcup a$ iff $(\exists z \in a)(x \in z)$. This fact generalizes and it is straightforward to check that it is always $\bigcap^{n} a \subseteq \bigcup^{n} a$. This justifies the following step:

step 2. for each free variable x and each $n \leq l$, add

$$\bigcap^{n} x \subseteq \bigcup^{n} x$$

to $\tilde{\varphi}$, if the two terms $\bigcup^{n} x$ and $\bigcap^{n} x$ are not associated to the empty set by M;

The feature relative of the semantics of our set-theoretic operators which is used in the following step is basically this: given two sets a and b, if $a \subseteq b$, then $\bigcap b \subseteq \bigcap a$. This is easily checked and justifies step 3:

step 3. for each inclusion of the form $\bigcup^{m} x \subseteq \bigcap^{n} y$ in $\tilde{\varphi}$, add

$$\bigcap^{n+1} y \subseteq \bigcup^{m+1} x$$

to $\tilde{\varphi}$ if it is not already there;

step 4. propagate all the inclusions in $\tilde{\varphi}$, that is for any three terms t_1, t_2, t_3 such that $t_1 \subseteq t_2$ and $t_2 \subseteq t_3$ are in $\tilde{\varphi}$, add $t_1 \subseteq t_3$ to $\tilde{\varphi}$ if it is not already there;

step 5. go back to step 3 until no inclusion is added to $\tilde{\varphi}$. ❑

At this point we have in $\tilde{\varphi}$ all the constraints contained in φ together with other inclusions forced by φ itself.

There is another form of constraints forced by the choices we made so far: consider, for example, the case in which the term $\bigcup t$ was intended to be interpreted as the empty set. In this case if the term t appears in $\tilde{\varphi}$, then t must be interpreted as the set $\{\emptyset\}$. Analogously, in this situation we would be forced to interpret all the terms of the form $\bigcup^{n} t$, as well as those terms which must be included in some of these terms, as the empty set.

The following algorithm will determine those terms which are *trapped* in the previous sense.

Algorithm A.2.

A term is trapped if we can give an upper bound to its rank (and therefore is a term that we can guess). Hence any term representing a set that must have rank less than or equal to the rank of a trapped term must be trapped as well. The rank of $\bigcup^{n} x$ *is less than or equal to the rank of* $\bigcup^{m} x$ *if* $n \geq m$.

Morever, if $\bigcup^{n} x$ *is trapped it is going to be interpreted by a hereditarily finite set of rank, say,* ρ_x, *so* $\bigcup^{m} x$ *with* $m \leq l$ *can only be* *interpreted by a set of rank less than or eqal to* $\rho_x + l$.

This justifies the following step:

step 1. if $\bigcup^{n} x$ is interpreted as the empty set by M, or is trapped, then $\bigcup^{m} x$ for $1 \leq m \leq l$ is trapped;

Since rank $\left(\bigcap^{m} x\right)$ $\leq$*rank* $\left(\bigcap^{n} x\right)$ *when* $m \geq n$ *we have also the following step:*

step 2. if $\bigcap^{n} x$ is interpreted as the empty set by M, or is trapped, then $\bigcap^{m} x$ for $n \leq m \leq l$ is trapped;

Since rank(t_0) $\leq$*rank*(t_1) *if* $t_0 \subseteq t_1$ *we then have:*

step 3. if $t_0 \subseteq t_1$ is in $\tilde{\varphi}$ and t_1 is trapped, then t_0 is trapped;

At this point we might have a larger set of trapped terms, in which case we must start the procedure again from the beginning:

step 4. go back to step 1 until no new term is marked trapped.❑

Notice that the maximum rank of a set interpreting a term which is trapped is certainly less than the number of trapped terms, therefore we can safely assume that we are able to substitute any trapped term t by a closed term t_h in the language having only $\emptyset$ as constant and $\{\ldots\}$ as term constructors, and representing the set used to interpret t by M.

Again, one can see the previous argument as a nondeterministic step in the procedure we are describing.

Clearly we will assume that all the guesses made up to this point are consistent with the semantics of the set-theoretic constructors involved in $\tilde{\varphi}$. If, for example, t_{h_1} is used in place of a term t_1 and t_{h_2} is used in place of $\bigcup t_1$, then we will assume that the unary union of the set represented by t_{h_1} is the set represented by t_{h_2}.

Because of the previous assumption we drop all the inclusions in $\tilde{\varphi}$ of the form

$$t_{h_1} \subseteq t_{h_2}$$

with t_{h_1} and t_{h_2} representing hereditarily finite sets, since they are clearly satisfied and therefore give no information to us.

Let us prove that inclusion of the following form

$$\bigcup^{n} x \subseteq \bigcup^{m} x \tag{1}$$

with $m > n$ can appear in $\tilde{\varphi}$.

By contradiction let us suppose that an inclusion of type (1) is in $\tilde{\varphi}$.

$\bigcup^{n} x \subseteq \bigcup^{m} x$ with $m > n$ is an inclusion in a form that is not in φ and is not introduced by any step but the propagation step in the algorithm generating $\tilde{\varphi}$. Therefore we can conclude that for some term t the following inclusions

1. $\bigcup^{n} x \subseteq t$

2. $t \subseteq \bigcup^{m} x$

are in φ.

Any inclusion having $\bigcup^{m} x$ as right hand side and not introduced in $\tilde{\varphi}$ by the propagation step, must satisfy one of the following two conditions:

a) is of the form $\bigcap^{h+1} y \subseteq \bigcup^{m} x$

b) is of the form $\bigcap^{m} x \subseteq \bigcup^{m} x$.

In case a) it must be that either $t \subseteq \bigcap^{h+1} y$ is in $\tilde{\varphi}$ or t is itself $\bigcap^{h+1} y$.

In both cases, using 1 and 2 and recalling step 3 and 4 of the algorithm generating $\tilde{\varphi}$, we can conclude that both the inclusions

i) $\bigcup^{n} x \subseteq \bigcap^{h+1} y$

ii) $\bigcup^{m-1} x \subseteq \bigcap^{h} y,$

are in $\tilde{\varphi}$.

Now recalling that $n \leq m-1$, it is easy to see that i) and ii) cannot be satisfiable, contradicting the fact that M is a model for φ and $\tilde{\varphi}$. In fact: $rank\left(\bigcup^{n} x\right) \geq rank\left(\bigcup^{m-1} x\right)$ for any x whereas from i) and ii) it would follow that

$$rank\left(\bigcup^{n} x\right) \leq rank\left(\bigcap^{h+1} y\right) \leq$$

$$\min\left\{rank(z) | z \in \bigcup^{m-1} x\right\} < rank\left(\bigcup^{m-1} x\right)$$

In case b) from 1. we would have $\bigcup^{n} x \subseteq \bigcap^{m} x$ is in $\tilde{\varphi}$ and also in this case we would reach a contradiction since $\bigcup^{n} x \subseteq \bigcap^{m} x$ with $m > n$ is unsatisfiable, in fact:

it is always the case that $\bigcap^{n} x \subseteq \bigcup^{n} x$, hence $rank\left(\bigcap^{n} x\right) \leq rank\left(\bigcup^{n} x\right)$;

moreover $rank\left(\bigcap^{m} x\right) < rank\left(\bigcap^{n} x\right)$ if $m > n$ and therefore

$$rank\left(\bigcap^{m} x\right) < rank\left(\bigcup^{n} x\right),$$

whereas from $\bigcup^{n} x \subseteq \bigcap^{m} x$ it would follow that $rank\left(\bigcup^{n} x\right) \leq rank\left(\bigcap^{m} x\right)$

As a particular case of (1) we have that non inclusion of the form

(2) $$\bigcup^{n} x \subseteq \bigcup^{n+1} x$$

is in $\tilde{\varphi}$.

Notice that inclusions of the form (2) are satisfied by infinite sets.

At this point if there is a cycle of inclusions of the form

$$t_{j_1} \subseteq t_{j_2} \wedge t_{j2} \subseteq t_{j_3} \wedge \ldots \wedge t_{j_{p-1}} \subseteq t_{j_p} \wedge t_{j_p} \subseteq t_{j_1},$$

in $\tilde{\varphi}$, let us substitute all the terms $t_{j_1}, \ldots, t_{j_p}$ in $\tilde{\varphi}$ by a new variable $x_{j_1,\ldots,j_p}$ (note that in this case the model M must associate the same set to all terms in the cycle) and let us eliminate the inclusions of the form

$$x_{j_1,\ldots,j_p} \subseteq x_{j_1,\ldots,j_p}$$

from $\tilde{\varphi}$.

Let us now consider an ordering $t_1, \ldots, t_s$ of the terms appearing in $\tilde{\varphi}$ which complies with rank comparison between sets associated to them by the model M and, moreover, satisfies the following conditions:

i) if $t_i \subseteq t_j$ is in $\tilde{\varphi}$ then $i < j$;

ii) if t_i is $\bigcup^{n+1} x$ and t_j is $\bigcup^{n} x$ then $i < j$;

iii) if t_i is $\bigcap^{n+1} x$ and t_j is $\bigcap^{n} x$ then $i < j$;

iv) if for all z in Mt_j, $z \supseteq t_i$ then $i < j$.

As a matter of fact *any* ordering of the terms which is compatible with the ranks would satisfy iii) and iv), the only problem is to show that it is possible to arrange the elements of the same ranks in such a way that both i) and ii) are satisfied, but this is guaranteed by (2).

At this point we are ready to define a first approximation of the sought hereditarily finite model $M^\star$ that we will call M'.

M' will be defined by induction on the ordering $t_1, \ldots, t_s$.

Let us define $M't_i$ by cases and using the following inductive hypotheses:

1) if $Mt_i \subseteq Mt_j$ and $i < j$ then $M't_i \subseteq M't_j$;

2) $M'\left(\bigcap^{n+1} y\right) = \bigcap M'\left(\bigcap^{n} y\right)$,

3) if for all $z \in Mt_j$, $z \supseteq Mt_i$ and $i < j$, then for all $z \in M't_j$, $z \supseteq M't_i$.

Case a): t_i is a tapped term.

In this case we put $M't_i = Mt_i$, that is $M't_i$ is defined to be exactly the hereditarily finite set defined by t_i.

In this case it is obvious that the inductive hypotheses continue to hold.

Case b): t_i is $\bigcap^{l} y$ or $\bigcup^{l} y$ (recall: l is the maximum integer for which $\bigcap^{n} y$, $\bigcup^{n} y$ can possibly appear in $\tilde{\varphi}$).

Let

$$M't_i = \bigcup\{M't_j | Mt_j \subseteq Mt_i \wedge j < i\} \cup A,$$

where A is different from the empty set only in the case in which there is no $j < i$ such that $Mt_j \subseteq Mt_i$; moreover, when A is non-empty it is of the form $\{a\}$ with $a \supseteq Mt_j$ if for all $z \in Mt_i$, $z \subseteq Mt_j$.

The set A is introduced in the definition of $M't_i$ (in this case) only to ensure that $M't_i$ is not empty when Mt_i is not empty.

Inductive hypothesis 2) continues to hold straightforwardly, and the very definition of $M't_i$ guarantees that hypothesis 1) continues to hold too.

Hypothesis 3): let us suppose that for all $z \in Mt_i$, $z \supseteq Mt_j$, then if $A \neq \emptyset$ we required that $A = \{a\}$ and $a \supseteq M't_j$; on the other hand if $A = \emptyset$ then $z \in M't_k$ with $Mt_k \subseteq Mt_i$ and $k < i$, hence for all $w \in Mt_k$, $w \supseteq Mt_j$ and therefore inductive hypothesis 3) on $M't_k$ allows one to conclude that $z \supseteq M't_j$.

Case c): t_i is of the form $\bigcap^{n} y$ with $l > n > 0$.

In this case property iii) of the well ordering $t_1, \ldots, t_s$ guarantees that $M'\left(\bigcap^{n+1} y\right)$ has already been defined, hence we can put

$$M't_i = \bigcup\{M't_j | Mt_j \subseteq Mt_i \wedge j < i\} \cup \left\{M'\left(\bigcap^{n+1} y\right)\right\}.$$

Inductive hypothesis 1) continues to hold as before.

Inductive hypothesis 2): notice that if $Mt_j \subseteq M\left(\bigcup^{n} y\right)$ and $j < i$, then for all $z \in Mt_j, z \supseteq M\left(\bigcap^{n+1} y\right) = \bigcap M\left(\bigcap^{n} y\right)$, therefore from inductive hypothesis 3) it follows that for all $z \in M't_i, z \supseteq M'\left(\bigcap^{n+1} y\right)$.

Moreover the fact that $M'\left(\bigcap^{n+1} y\right)$ itself is among the elements of $M't_i$ allows us to conclude that:

$$\bigcap M't_i = \bigcap M'\left(\bigcap^{n} y\right).$$

Inductive hypothesis 3) is proved as in the previous case, with the additional observation that, if for all $z \in Mt_i$, $z \supseteq Mt_j$ then also $\bigcap Mt_i = M\left(\bigcap^{n+1} y\right) \supseteq Mt_j$ and therefore from hypothesis 1), $M'\left(\bigcap^{n+1} y\right) \supseteq M't_j$.

Case d): t_i is of the form $\bigcup^{n} y$ with $l > n > 0$.

Let

$$M't_i = \bigcup\{M't_j | Mt_j \subseteq Mt_i \wedge j < i\} \cup \left\{M'\left(\bigcup^{n+1} y\right)\right\}.$$

Inductive hypotheses are proved analogously as in the previous cases.

Case e): t_i is of the form $\bigcup^0 y$ or $\bigcap^0 y$

$$M' t_i = \bigcup \{M' t_j | M t_j \subseteq M t_i \wedge j < i\} \cup \{M' \bigcap y\} \cup \{M' \bigcup y\}.$$

Inductive hypotheses are proved as before.

Case f): t_i is of the form $x_{j_1}, \ldots, x_{j_p}$.

$$M' t_i = \bigcup \{M' t_j | M t_j \subseteq M t_i \wedge j < i\} \cup$$

$$\left\{ M' t^-_{j_h} | 1 \le h \le p \wedge t_{j_h} \text{ is } \bigcap^n y \wedge t^-_{j_h} \text{ is } \bigcap^{n+1} y \right\}$$

$$\cup \left\{ M' t^-_{j_h} | 1 \le h \le p \wedge t_{j_h} \text{ is } \bigcup^n y \wedge t^-_{j_h} \text{ is } \bigcup^{n+1} y \right\}.$$

Inductive hypotheses are proved as before.

Notice that $M' t_1, \ldots, M' t_s$ are already hereditarily finite sets and it is clear that they would constitute the model we are seeking if we had the following property

2') $M' \left(\bigcup^{n+1} y \right) = \bigcup M' \left(\bigcup^{n} y \right)$

in addition to properties 1), 2) and 3), whereas up to this point we can only conclude that $M' \left(\bigcup^{n+1} y \right) \subseteq \bigcup M' \left(\bigcup^{n} y \right)$.

The following algorithm modifies the hereditarily finite sets $M' t_1, \ldots, M' t_s$ in such a way that when it terminates the tuple $M^\star t_1, \ldots, M^\star t_s$ produced as output will satisfies 1), 2), 2') and 3).

Algorithm A.3.

step 0: *initialize* $M^\star$ *to* M'

```
for   i = 1   to s do
              M* := M't_i;
end   for;
```

step 1: *guarantee property 2')*

for $i = s$ to 1 do

if $t_i = \bigcup^{n} y \wedge n \geq 1$ then

$M^\star t_i := \bigcup M^\star \left(\bigcup^{n-1} y \right)$ or

$M^\star t_i := \bigcup M^\star x_{j_1,\ldots,j_p}$ when $\bigcup^{n-1} y$

has been substituted by $x_{j_1,\ldots,j_p}$;

else if $t_i = x_{j_1,\ldots,j_p}$ then

$M^\star x_{j_1,\ldots,j_p} =$

$$\bigcup \left\{ \bigcup M^\star \left(\bigcup^{n_r - 1} y \right) | 1 \leq r \leq p \wedge t_{j_r} = \bigcup^{n_r} y \wedge n_r > 1 \right\}$$

end if;

end for;

step 2: *restore properties 1), 2) and 3)*

for $i = 1$ to s do

if $t_i = \bigcap^{l} y$ or $t_i = \bigcup^{l} y$ then

if $M^\star t_i = \{a\}$ then $M^\star t_i = \{a'\}$ where

$a' = a \cup \bigcup \{ M^\star t_i | (\forall z \in M t_i)(z \supseteq M t_j) \}$;

else $M^\star t_i := M^\star t_i \cup \bigcup \{ M^\star t_j | M t_j \subseteq M t_i \wedge j < i \}$;

end if;

else if $t_i = \bigcap^{n} y \wedge l > n > 0$ then

$M^\star t_i := M^\star t_i \cup \bigcup \{ M^\star t_j | M t_j \subseteq M t_i \wedge j < i \} \cup$

$\left\{ M^\star \left(\bigcap^{n+1} y \right) \right\}$;

else if $t_i = \bigcup^{n} y \wedge l > n > 0$ then

$M^\star t_i := M^\star t_i \cup \bigcup \{ M^\star t_j | M t_j \subseteq M t_i \wedge j < i \} \cup$

$\left\{ M^\star \left(\bigcup^{n+1} y \right) \right\}$;

else if $t_i = \bigcup^{0} y$ or $t_i = \bigcap^{0} y$ then

$M^\star t_i := M^\star t_i \cup \bigcup \{ M^\star t_j | M t_j \subseteq M t_i \wedge j < i \} \cup$

$\{ M^\star (\bigcap y) \} \cup \{ M^\star (\bigcup y) \}$;

else if $t_i = x_{j_1,\ldots,j_p}$ then
$M^\star t_i := M^\star t_i \cup \bigcup\{M^\star t_j | Mt_j \subseteq Mt_i \wedge j < i\} \cup$
$\left\{ M^\star t^-_{j_h} | 1 \leq h \leq p \wedge t_{j_h} \text{ is } \bigcap^n y \wedge t^-_{j_h} \text{ is } \bigcap^{n+1} y \right\} \cup$
$\left\{ M^\star t^-_{j_h} | 1 \leq h \leq p \wedge t_{j_h} \text{ is } \bigcup^n y \wedge t^-_{j_h} \text{ is } \bigcup^{n+1} y \right\} \cup$
end if;
end for;

step 3. go back to step 1 until no modification to $M^\star$ is made. ■

Notice that the algorithm terminates in a finite number of steps since the maximum rank of the $M^\star t_i$'s does not increase in a cycle (this is proved by induction on $t_1, \ldots, t_s$) and any $M^\star t_i$ can only grow at any given cycle; therefore if we let r =max{rank($M' t_i$)$|1 \leq i \leq s$}, and $r^\star$ be the number of sets of rank less than r, in at most $r^\star \cdot s$ steps the algorithm will terminate.

Moreover at the end of step 1 hypothesis 2') always holds, and at the end of step 2, 1), 2) and 3) hold. Hence on exit we have that 1), 2), 2') and 3) hold and $M^\star$ is a model of $\tilde{\varphi}$ and φ since M is a model of $\tilde{\varphi}$ and φ. ■

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