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TOTALLY REAL SUBMANIFOLDS OF GENERALIZED HOPF MANIFOLDS

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We classify totally-geodesic totally-real submanifolds of generalized Hopf manifolds, provided the normal connection is flat.

1. Introduction and statement of results.

Let (M^{2n}, g_0, J) be a *locally conformal Kaehler* (l.c.K.) manifold of complex dimension $n \geq 2$ with the complex structure J and the Hermitian metric g_0. Let ω_0 be the *Lee form* of M^{2n}; i.e.

$$\omega_0 = \frac{1}{n-1} i(\Omega_0) d\Omega_0 \tag{1.1}$$

Here $i(\Omega_0)$ denotes the adjoint of $e(\Omega_0)$, $e(\Omega_0)\emptyset = \Omega_0 \wedge \emptyset$, for any differential form $\emptyset$ on M^{2n}, while Ω_0 is the Kaehler 2-form. If ω_0 is parallel (with respect to the Riemannian connection of (M^{2n}, g_0)) then M^{2n} is a *generalized Hopf* (g.H.) manifold, cf. [11].

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Let M^{2n} be a g.H. manifold; we put $||\omega_0|| = 2c$, $c \geq 0$. Throughout we suppose $c \neq 0$, since otherwise M^{2n} falls into nothing but a Kaehler manifold. Let $B_0 = \omega_0\#$ be the *Lee field* of M^{2n}, where # denotes raising of indices by g_0. Let $\psi : M^m \longrightarrow M^{2n}$ be an isometric immersion of a real m-dimensional Riemannian manifold (M^m, g) into M^{2n}. Let $E \longrightarrow M^m$ be the normal bundle of the given immersion ψ. Let tan_x, nor_x be the natural projections associated with the direct sum decomposition $T_x(M^{2n}) = T_x(M^m) \oplus E_x$, $x \in M^m$. We set $B = \tan(B_0)$, $B^\perp = nor(B_0)$.

Let $\omega = \psi^*\omega_0$. If ω has no singular points, i.e. $\omega_x \neq 0$ at any $x \in M^m$, then M^m carries a canonical foliation $\mathcal{F}$ whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$.

The submanifold M^m of M^{2n} is *totally-real* if $J_x(T_x(M^m)) \subseteq E_x$, $x \in M^m$. The study of totally-real submanifolds in l.c.K. manifolds (in particular, in complex Hopf manifolds) has been initiated in [2]. Cf. also [3], [7]. We obtain the following result:

THEOREM. *Let M^n be a connected complete totally-geodesic totally-real submanifold (of real dimension $n \geq 2$) of the g.H. manifold M^{2n}; suppose M^n has a flat normal connection.*

i) If M^n is a simply-connected real surface $(n = 2)$ tangent to the Lee field $(B^\perp = 0)$ of the ambient g.H. manifold M^4 then it is isometric to $\mathbb{R}^2$ with the flat Euclidean metric;

ii) If $n > 2$, $B^\perp \neq 0$ and M^n is orientable then M^n is normal to the Lee field and its universal covering manifold is isometric to the sphere $S^n\left(\frac{1}{c}\right)$;

iii) If $n = 2$, $B^\perp \neq 0$ and M^2 is orientable then M^2 is isometric to $S^2\left(\frac{1}{c}\right)$;

iv) If $n > 2$ and $B^\perp = 0$ then M^n is quasi-Einstein and its Ricci form is positive semi-definite and degenerated only along the distribution *generated by the Lee field. Moreover, any leaf of the canonical foliation $\mathcal{F}$ of M^n is a totally-geodesic real hypersurface of M^n and its universal covering manifold is isometric to $S^{n-1}\left(\frac{1}{c}\right)$.*

Throughout $S^m(r)$ denotes the sphere of radius $r > 0$ and center the origin of $\mathbb{R}^{m+1}$.

2. Basic formulae.

Let M^{2n} be a g.H. manifold and $\psi : M^m \to M^{2n}$ an isometric immersion. As usual, we do not distinguish notationally between x and $\psi(x)$, respectively X and $\psi_* X$, for each $X \in T_x(M^m)$, $x \in M^m$. We recall the Gauss and Weingarten formulae, see e.g. [1]:

$$(2.1)\qquad \begin{aligned} \nabla^0_X Y &= \nabla_X Y + h(X,Y) \\ \nabla^0_X \xi &= -A_\xi X + \nabla^\perp_X \xi \end{aligned}$$

for any tangent vector fields X, Y on M^n, respectively any cross-section ξ in E. Here ∇, h, A_ξ and $\nabla^\perp$ denote the *induced connection*, the *second fundamental form* (of ψ), the *Weingarten operator* (associated with the normal section ξ) and the *normal connection* (in E).

Since M^{2n} is l.c.K., there exists an open covering $(U_i)_{i\in I}$ of M^{2n} and a family $(f_i)_{i\in I}$ of smooth real-valued functions $f_i \in C^\infty(U_i)$ such that the metrics $g_i = \exp(-f_i)g$, $i \in I$, are Kaehler. The Levi-Civita connections of the (local) metrics g_i are known to glue up to a globally-defined torsion-free linear connection on M^{2n}, namely the *Weyl connection*, i.e.

$$(2.2)\qquad D^0_X Y = \nabla^0_X Y - \frac{1}{2}\{\omega_0(X)Y + \omega_0(Y)X - g_0(X,Y)B_0\}$$

Throughout ∇^0 denotes the Riemannian connection of (M^{2n}, g_0). Cf. [10], p. 441, the curvature tensor fields K_0, R_0 of D^0, ∇^0 are related by:

$$K_0(X,Y)Z = R_0(X,Y)Z - \tag{2.3}$$
$$- \frac{1}{4}\{[\omega_0(X)Y - \omega_0(Y)X]\omega_0(Z) +$$
$$+ [g_0(X,Z)\omega_0(Y) - g_0(Y,Z)\omega_0(X)]B_0\} -$$
$$- c^2\{g_0(Y,Z)X - g_0(X,Z)Y\}$$

for any tangent vector fields X, Y, Z on M^{2n}, provided that $\nabla^0\omega_0 = 0$. We shall also need the Gauss and Ricci equations:

$$g_0(R_0(Z,U)Y,X) = g(R(Z,U)Y,X) + \tag{2.4}$$
$$+ g_0(h(X,Z),h(Y,U)) - g_0(h(X,U),h(Y,Z))$$

$$g_0(R_0(X,Y)\xi,\eta) = \tag{2.5}$$
$$= g_0(R^{\perp}(X,Y)\xi,\eta) - g([A_\xi, A_\eta]X,Y)$$

Here $R, R^{\perp}$ stand respectively for the curvature tensor fields of ∇, $\nabla^{\perp}$. Let ω be the 1-form induced on M^n by the Lee form of the ambient g.H. manifold. Note that:

$$(\nabla_X\omega)Y = \omega_0(h(X,Y)) \tag{2.6}$$

Therefore, if M^n is totally-geodesic then ω is parallel, too.

Let $\lambda \in C - \{0\}$, $|\lambda| \neq 1$ and denote by G_λ the discrete group of analytic transformations of $C^n - \{0\}$ generated by $z \to \lambda z$, $z \in C^n - \{0\}$. Then G_λ acts freely and property discontinuously on $C^n - \{0\}$ such that the factor space $CH^n = \dfrac{C^n - \{0\}}{G_\lambda}$ inherits a natural structure of complex manifold. This is the well known *complex Hopf manifold*. The Hermitian metric $ds^2 = |z|^{-2}\delta_{ij}dz^i \otimes d\bar{z}^j$ on $C^n - \{0\}$ is G_λ-invariant, thus defining a global l.c.K. metric g_0 on CH^n.

Let $\pi : C^n - \{0\} \to CH^n$ be the natural surjection. Let $j : M^m \to (C^n, \delta_{ij})$ be an isometric immersion of a Riemannian manifold

(M^m, g) into C^n (endowed with the flat Euclidean structure) such that M^n does not pass through the origin. Then $\psi : M^m \to CH^n$, $\psi = \pi \circ j$, is an isometric immersion of (M^m, g) into (CH^n, g_0). The second fundamental forms of the immersion ψ, j are related by $h(\psi) = h(j) + \frac{1}{2} g \otimes B^{\perp}$. Consequently, if j is totally-geodesic, then ψ is totally-umbilical (totally-geodesic, provided that M^m is tangent to the Lee field of H^n). Let $U \subseteq \mathbb{R}^2 - \{0\}$ be open and $\psi : U \to CH^2$, $\psi(x, y) = \pi(x, 0, y, 0)$, for any $(x, y) \in U$. Then U is a totally-real totally-geodesic (immersed) submanifold of CH^2 with a flat normal connection.

3. Proof of the theorem.

Let M^n be a totally-real submanifold of the g.H. manifold M^{2n}. If Z is a tangent vector field on M^n then JZ is a normal section. Therefore, our (2.3) yields:

$$\begin{aligned} K_0(X,Y)JZ = R_0(X,Y)JZ - \\ - \frac{1}{4}[\omega(X)Y - \omega(Y)X]\omega_0(JZ) \end{aligned} \tag{3.1}$$

for any X, Y, Z tangent to M^n. Since D^0 is the Riemannian connection of the (local) Kaehler metrics g_i, $i \in I$, it is almost complex, i.e. $D^0 J = 0$. Thus $K_0(X,Y)J = JK_0(X,Y)$, cf. [5], p. 149, vol. II. Then using again (2.3), the formula (3.1) turns into:

$$\begin{aligned} JR_0(X,Y)Z = R_0(X,Y)JZ + \\ + \frac{1}{4}\{[\omega(X)JY - \omega(Y)JX]\omega(Z) + \\ + [g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]JB_0\} + \\ + c^2\{g(Y,Z)JX - g(X,Z)JY\} + \\ + \frac{1}{4}\{\omega(X)Y - \omega(Y)X\}\omega_0(JZ) \end{aligned} \tag{3.2}$$

Let U be a tangent vector field on M^n. Taking the inner product of (3.2) with JU and substituting from (2.4)-(2.5) into the resulting equation, one obtains:

$$\begin{aligned}R(X,Y)Z = &\frac{1}{4}\{[\omega(X)Y - \omega(Y)X]\omega(Z)+\\ &+[g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\}+\\ &+c^2\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{3.3}$$

provided that $h = 0$, $R^{\perp} = 0$. Further contraction of indices in (3.3) furnishes the expression of the Ricci form of (M^n, g), i.e.

$$\begin{aligned}Ric(X,Y) = &\left\{(n-1)c^2 - \frac{1}{4}||\omega||^2\right\} g(X,Y)-\\ &-\frac{n-2}{4}\omega(X)\omega(Y)\end{aligned} \tag{3.4}$$

Let now M^2 be a totally-geodesic totally-real surface in a g.H. manifold M^4 of complex dimension 2. By (3.4) one has $Ric = \frac{1}{4}||B^{\perp}||^2 g$. Therefore, if M^2 is tangent to B_0, then it is Ricci flat, and thus flat, (for surfaces the two notions are known to coincide). Suppose from now on that $n > 2$. If X is tangent to M^n, (3.4) leads to:

$$\begin{aligned}Ric(X,X) = &\frac{1}{4}\{(n-1)||B^{\perp}||^2 + (n-2)||\omega||^2\}||X||^2-\\ &-\frac{n-2}{4}\omega(X)^2\end{aligned} \tag{3.5}$$

Since $||\omega|| = const.$, either $\omega = 0$ and then (3.3) shows that M^n is a Riemannian manifold of constant sectional $c^2 > 0$, or ω is nowhere vanishing. The last step in the proof of ii) consists in showing that actually the second possibility does not occur. To this end, suppose $\omega \neq 0$. Let $x \in M^n$, $X \in T_x(M^n)$. There exists a unique $Y \in T_x(M^n)$ perpendicular on B_x such that $X = Y + \lambda B_x$, for some $\lambda \in \mathbb{R}$. Substitution into (3.5) leads to

$$\begin{aligned}Ric(X,X) = &\frac{1}{4}[(n-1)||B^{\perp}||^2 + (n-2)||\omega||^2]||Y||^2+\\ &+\frac{1}{4}\lambda^2(n-1)||B^{\perp}||^2||\omega||^2, \quad \text{i.e.} \quad Ric(X,X) \geq 0.\end{aligned}$$

Also $Ric(X,X) = 0$ iff $Y = 0$, $\lambda = 0$, since $B^\perp \neq 0$. Thus Ric is positive-definite. By a result of [6], M^n follows to be compact. Since ω is parallel, by (2.12.4) in [4], p. 78, it is harmonic. Consequently, the first Betti number $b_1(M^n)$ of M^n is ≥ 1. This is a contradiction, since by a result of S. Bochner ([4], th. 3.2.1., p. 87), $b_1(M^n) = 0$, Q.E.D. Let us prove iii). If $n = 2$, $B^\perp \neq 0$, then by (3.4) it follows that M^2 is a compact Einstein surface (with $b_1(M^2) = 0$). But ω is harmonic hence $\omega = 0$. Consequently, by (3.3) and a result of [8], $\pi_1(M^2) = 0$. At this point th. 7.10. of [5], p. 265, vol. I, yields iii). Moreover, if $n > 2$, $B^\perp = 0$, then $Ric = \frac{n-2}{4}[||\omega||^2 g - \omega \otimes \omega]$, i.e. M^n is quasi-Einstein, cf. the terminology in [9]. Also if $X = Y + \lambda B$, $g(Y, B) = 0$, then $Ric(X,X) = 0$ iff $Y = 0$. To prove the last part of the statement iv) let M^{n-1} be a leaf of $\mathcal{F}$. As M^{2n} is non-Kaehler and $B^\perp = 0$, the induced form ω has no singular points. Then $U = \frac{1}{2c}B$ is a unit normal on M^{n-1}. Let ∇', h' be respectively the induced connection and the second fundamental form of M^{n-1} in M^n. The Gauss formula $\nabla_X Y = \nabla'_X Y + h'(X,Y)$ and $\nabla_X \omega = 0$ yield $h' = 0$, i.e. M^{n-1} is totally-geodesic; then, on one hand completeness of M^n implies completeness of M^{n-1}. On the other, our (3.3) combined with the Gauss equation (e.g. (2.6) in [1], p. 45) of M^{n-1} in M^n, shows that M^{n-1} is a Riemannian manifold of constant sectional curvature $c^2 > 0$.

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