AN ALMOST SURE INVARIANCE PRINCIPLE
FOR THE MAPS
\[ S_\alpha(x) = \alpha x(1 - x) \] IN THE INTERVAL \([0, 1]\)

MARIO ABUNDO (Roma) (*)

In this paper we consider the family of maps in the unit interval defined by \( S_\alpha(x) = \alpha x(1 - x), \ x \in [0, 1], \ \alpha \in [0, 4]. \) For a countable set of values of \( \alpha \) for which the a.c. \( S_\alpha \)-invariant measure \( \mu_\alpha \) exists, we show an almost sure invariance principle for the process \( \{f \circ S^i_\alpha\} \), with \( f \) a function of bounded \( p \)-variation; from this the previously proved central limit theorem with respect to \( \mu_\alpha \) follows again; moreover log-log-laws and weak invariance principles for the process \( \{f \circ S^i_\alpha\} \) follow.

1. Introduction.

We consider the family of maps in the unit interval defined by \( S_\alpha(x) = \alpha x(1 - x), \ x \in [0, 1], \ \alpha \in [0, 4]. \)

For such maps it is known that there exists a set \( \Theta \) of values of the parameter \( \alpha \) of power the continuum, for which \( S_\alpha \) has an

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absolutely continuous invariant measure ([4], [6]).

Moreover, a countable set of values of the parameter $a$ has been investigated by Pianigiani ([7]) for which an a.c. $S_a$-invariant measure exists. This measure is unique and ergodic ([4], [9], [1]).

In [1] a central limit theorem is proved for the map $S_a$ for the countable set of values of $a$ satisfying the conditions of Pianigiani ([7]), with respect to the a.c. $S_a$-invariant measure $\mu_a$.

In this paper we show an almost sure invariance principle (a.s.i.p.) for the map $S_a$ for such values of $a$.

We make use of an a.s.i.p. for maps of Misiurewicz ([6]) stated by K. Ziemia (5).

In [5] the a.s.i.p. is proved for maps satisfying a certain set of conditions, said «of Misiurewicz» (see [6]).

Indeed, we show that our maps $S_a$, for certain values of $a \in [0, 4]$, satisfy the set of conditions of Misiurewicz, and therefore the a.s.i.p. holds.

We remark that the set of values of $a$ satisfying the conditions of Pianigiani, is disjoint from the continuum set of values for which Misiurewicz showed the existence of $S_a^k$-invariant a.c. measures, for some integer $k$ ([6]).

From the a.s.i.p. log-log-laws, and weak invariance principles follows (see [8]). Also we obtain again the central limit theorem that was already proved in [1].

In section 2 we formulate the problem and state the main result. In section 3 we prove the result.

2. Formulation of the problem and main result.

Let us consider the family of maps $S_a : [0, 1] \rightarrow [0, 1]$, defined by $S_a(x) = ax(1 - x)$, where the parameter $a \in [0, 4]$ satisfies the conditions of Pianigiani [7]:

...
1) $\exists p > 1$ such that $S_a^p(1/2) = 1 - S_a^{2p}(1/2)$ and $S_a^p(1/2) < 1/2$;

2) $S_a^p(I) = I$ and $S_a^i(I) \cap (1/a, 1 - 1/a) = \emptyset$ for $i = 1, \ldots, p - 1$, where $I = (S_a^p(1/2), S_a^{2p}(1/2))$.

In [7] is proved that a countable set of values of the parameter $a = a_p > 2$ satisfying 1) and 2) exists. We say $\mathcal{A}$ this set. If $a \in \mathcal{A}$ a measure $\mu_a(dx) = \rho_a(x)dx$ invariant under $S_a$ and absolutely continuous with respect to the Lebesgue measure on $[0,1]$ exists [7] and this measure is unique and ergodic ([4], [9]).

The support of $\mu_a$ is not $[0,1]$, but the subset $A_I = \bigcup_{k=0}^{p-1} S_a^k(I)$. The density $\rho_a(x)$ of the $S_a$-invariant measure $\mu_a$ is not a bounded function, but it has singularities as $\text{const*}(|x - S_a^k(1/2)|)^{-1/2}$ in the points $S_a(1/2), S_a^{2}(1/2), \ldots, S_a^{2p}(1/2)$.

Indeed, if $a \in \mathcal{A}$ ([7]) the map $S_a^p$ is conjugate, via the homeomorphism $F : I \to [0,1]$, to a piecewise $C^1$ expanding map $R_a : [0,1] \to [0,1]$ ($\|R_a'\| > \sqrt{8/a} > \sqrt{2}$); that is $R_a = F \circ S_a^p \circ F^{-1}$ where $F(x) = \int_{S_a^p(1/2)}^{x} h(t)dt$, $h(t) = \text{const*}(|S_a^p(1/2) - t| |S_a^{2p}(1/2) - t|)^{-1/2}$,

$$\int_{I} h(t)dt = 1.$$  

The expanding map $R_a$ has a $R_a$-invariant measure $\hat{\nu}_a$ and the measure $\nu_a$ defined by $\nu_a = \hat{\nu}_a \circ F$ is invariant under $S_a^p$.

Thus $\mu_a$ is obtained as:

$$\mu_a = (\nu_a + \nu_a \circ (S_a^{-1}) + \ldots + \nu_a \circ (S_a^{-p+1}))/p.$$  

In [1] a central limit theorem is proved for $S_a$, $a \in \mathcal{A}$ with respect to the a.c. $S_a$-invariant measure $\mu_a$.

Now, we state theorem 1., that is an almost sure invariance principle, from which log-log-laws, weak invariance principles and the central limit theorem follow.

Let $f : [0,1] \to \mathbb{R}$ be a function with bounded $p$-variation, $p \geq 1$.
that is
\[ V^p f = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p < \infty \]

where the supremum is taken over all finite subsets \{x_0, \ldots, x_n\} \ni x_0 < x_1 < \ldots < x_n \text{ of the interval } [0,1].

Let \( s^a_t = \sum_{n \leq t} f \circ (S^n_a), a \in A \).

**Theorem 1.** If \( a \in [0,4] \) satisfies conditions 1) and 2), then for every \( f : [0,1] \to \mathbb{R} \) with bounded \( p \)-variation such that \( \int_0^1 f d\mu_a = 0 \) the series
\[ \sigma^2_a = \int f^2 d\mu_a + 2 \sum_{i=1}^{\infty} \int f \circ S^i_d d\mu_a \]
is absolutely convergent;
\[ \int (s^a_n)^2 d\mu_a = n\sigma^2_a + o(1) \]
and if \( \sigma^2_a \neq 0 \), then without changing its distribution we can redefine the process \( (s^a_t)_{t \geq 0} \) on a richer probability space together with a standard Brownian motion \( (B^a_t)_{t \geq 0} \) such that:
\[ |\sigma^{-1}_a s^a_t - B^a_t| = o(t^{1/2-\lambda_a}) \quad \mu_a - a.s. \]
for some \( \lambda_a > 0 \).

3. **Proofs.**

First, we recall the set of conditions of Misiurewicz which are needed for the proof of the w.i.p. in [5].

Let \( J \) be a closed interval, \( g : J \to J \) a continuous map monotonic on the connected components of the set \( J \setminus P(g) \), where \( P(g) \) is a finite subset of \( J \) containing the ends of \( J \), and consider the following assumptions ([6], [5]) for \( g|J \setminus P(g) \):
(i) $g$ is of class $C^3$;

(ii) $g' \neq 0$;

(iii) the Schwarzian derivative $S(g)$ of $g$ is non positive, i.e.

$$S(g) = g''''/g' - (3/2) \cdot (g'''/g') \leq 0$$

(iv) if $g^n(x) = x$, then $|(g^n)'(x)| > 1$

(v) there exists a neighbourhood $U$ of the set $P(g)$ such that for all $t \in P(g)$, $n \geq 0$ $g^n(t) \in P(g)$ or $g^n(t) \notin U$ for all $m \geq n$

(vi) for all $t \in P(g)$ there exists a neighbourhood $U_t$ of $t$ and constants $\alpha, \omega, \delta > 0$, $u \geq 0$ such that:

(I) $\alpha|x-t|^u \leq |g'(x)| \leq \omega|x-t|^u$

(II) $|g''(x)| \leq \delta|x-t|^{u-1}$ for all $x \in U_t$.

We recall that assumptions (i)-(iv) imply theorem 1., that is the a.s.i.p., as stated by Ziemian in [5].

Indeed, we will show that the above assumptions are true for the maps considered (this will be done in lemma 2).

**Lemma 1.** If $T^n(x) = x$, then $|(T^n)'(x)| > 1$, where $T = S^p_a$ and $n \in \mathbb{N}$.

**Proof.** We have:

$$(T^n)'(x) = [F'(F^{-1} \circ R_a^n \circ F(x))]^{-1} \cdot (R_a^n)'(F(x)) \cdot F'(x);$$

since $T^n x = x$, then $F^{-1} \circ R_a^n \circ F(x) = x$. Therefore: $(T^n)'(x) = (R_a^n)'(F(x))$. Thus: $|(T^n)'(x)| = |(R_a^n)'(F(x))| \geq \alpha > 1$, where $\alpha > \sqrt{2}$ is the expanding coefficient of $R_a$.

**Lemma 2.** If $a \in A$, $S_a$ satisfies the above conditions (i)-(vi) of Misiurewicz

**Proof.** (i), (ii) and (iii) are trivial, since $S_a^{iii}(x) = 0$.

To prove (iv) we proceed as follows.
Observe that $\forall n$ there exist two integers $m$ and $K$ such that $nK = mp$. Then, if $S_a^n(x) = x$ we have:

$$x = S_a^{nk}(x) = S_a^{mp}(x) = T^m(x)$$

and so for lemma 1.

$$|(S^{nk})'(x)| > 1$$

that is:

$$\underbrace{|(S^n)'(x) \cdot (S^n)'(x) \cdots (S^n)'(x)|}_{k \text{ times}} > 1 \quad \text{i.e.} \quad |(S^n)'(x)|^k > 1$$

which implies $|(S^n)'(x)| > 1$ and so (iv) holds.

To prove (v) we recall that $P(S_a) = \{0, 1/2, 1\}$ and $S_a^n(0) = S_a^n(1) = 0 \ \forall n > 0$. If $I = (S_a^p(1/2), S_a^{2p}(1/2)) \ni 1/2$, we have $S_a(I) \subset (1 - 1/a, 1)$, $S_a^l(I) \subset (0, 1/a)$, $l = 2, \ldots, p - 1$ and $S_a^p(I) = I$; moreover the right endpoint of the interval $S_a^l(I)$ increases with $l = 2, \ldots, p - 1$ (see [2]).

So:

$$S_a(1/2) \in (1 - 1/a, 1); \quad S_a^l(1/2) \in (0, 1/a), \quad l = 2, \ldots, p - 1; \quad S_a^p(1/2) < 1/2.$$ 

Now, we set

$$U_0 = (0, (1/2) \cdot S_a^2(1/2));$$

$$U_1 = ((1/4) + (1/2) \cdot S_a^p(1/2), (3/4) - (1/2) \cdot S_a^p(1/2));$$

$$U_2 = ((1/2) + (1/2) \cdot S_a(1/2), 1).$$

Then $U = U_0 \cup U_1 \cup U_2$ is a neighbourhood of $P(S_a)$ such that $S_a^m(1/2) \notin U$ for any $m$.

To prove (vi) is enough to choose

$U_t = (0, 1/4)$ and $u = 0, \ \alpha = 1, \ \omega = 4, \ \delta = 2$ for $t = 0$;

$U_t = (3/4, 1)$ and $u = 0, \ \alpha = 1/2, \ \omega = 4, \ \delta = 1$ for $t = 1$;
\[ U_t = ((1/2) - r, (1/2) + r) \text{ for small enough } r \text{ and } u = 1, \alpha = 1, \omega = 2, \delta = 8 \text{ for } t = 1/2. \]

Now we can prove theorem 1.

Proof of th. 1. Thanks to lemma 2., for \( a \in A \), \( S_a \) satisfies all the assumptions of theorem 1. in [5]. Then, our theorem follows from this.

Remark 1. If we denote \( P \) the partition into the intervals of monotonicity of \( S_a \), that is \( P = \{(0, 1/2), (1/2, 1)\} \), an exponential rate of decreasing of the diameter of the partition \( V_{0}^{n-1} S_{a}^{-1}P \) in the sense of Lebesgue measure and of the measure \( \mu_a \) can be proved by using the properties of \( S_a \). Analogous estimates are proved in [5] starting from assumption (i)-(vi).

Remark 2. From the proof of the a.s.i.p. the central limit theorem follows again for \( S_a, a \in A \) (for instance, see [8], [3]); already it was proved with different arguments in [1].

Moreover, the a.s.i.p. implies log-log-laws, and weak invariance principles (see [8]) for the process \( \{f \circ S_a^t\} \).

REFERENCES


*Dipartimento di Matematica*

*II Università degli Studi di Roma*

*Via Orazio Raimondo*

*00173 Roma (Italia)*