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AN ALMOST SURE INVARIANCE PRINCIPLE FOR THE MAPS $S_a(x) = ax(1-x)$ IN THE INTERVAL $[0,1]$

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In this paper we consider the family of maps in the unit interval defined by $S_a(x) = ax(1-x)$, $x \in [0,1]$, $a \in [0,4]$. For a countable set of values of a for which the a.c. S_a-invariant measure μ_a exists, we show an almost sure invariance principle for the process $\{f \circ S_a^i\}$, with f a function of bounded p-variation; from this the previously proved central limit theorem with respect to μ_a follows again; moreover log-log-laws and weak invariance principles for the process $\{f \circ S_a^i\}$ follow.

1. Introduction.

We consider the family of maps in the unit interval defined by $S_a(x) = ax(1-x)$, $x \in [0,1]$, $a \in [0,4]$.

For such maps it is known that there exists a set Θ of values of the parameter a of power the continuum, for which S_a has an

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absolutely continuous invariant measure ([4], [6]).

Moreover, a countable set of values of the parameter a has been investigated by Pianigiani ([7]) for which an a.c. S_a-invariant measure exists. This measure is unique and ergodic ([4], [9], [1]).

In [1] a central limit theorem is proved for the map S_a for the countable set of values of a satisfying the conditions of Pianigiani ([7]), with respect to the a.c. S_a-invariant measure μ_a.

In this paper we show an almost sure invariance principle (a.s.i.p.) for the map S_a for such values of a.

We make use of an a.s.i.p. for maps of Misiurewicz ([6]) stated by K. Ziemian ([5]).

In [5] the a.s.i.p. is proved for maps satisfying a certain set of conditions, said «of Misiurewicz» (see [6]).

Indeed, we show that our maps S_a, for certain values of $a \in [0,4]$, satisfy the set of conditions of Misiurewicz, and therefore the a.s.i.p. holds.

We remark that the set of values of a satisfying the conditions of Pianigiani, is disjoint from the continuum set of values for which Misiurewicz showed the existence of S_a^k-invarinat a.c. measures, for some integer k ([6]).

From the a.s.i.p. log-log-laws, and weak invariance principles follows (see [8]). Also we obtain again the central limit theorem that was already proved in [1].

In section 2 we formulate the problem and state the main result.

In section 3 we prove the result.

2. Formulation of the problem and main result.

Let us consider the family of maps $S_a : [0,1] \to [0,1]$, defined by $S_a(x) = ax(1-x)$, where the parameter $a \in [0,4]$ satisfies the conditions of Pianigiani [7]:

1) $\exists p > 1$ such that $S_a^p(1/2) = 1 - S_a^{2p}(1/2)$ and $S_a^p(1/2) < 1/2$;

2) $S_a^p(I) = I$ and $S_a^i(I) \cap (1/a, 1 - 1/a) = \emptyset$ $i = 1, \ldots, p-1$, where $I = (S_a^p(1/2),\ S_a^{2p}(1/2))$.

In [7] is proved that a countable set of values of the parameter $a = a_p > 2$ satisfying 1) and 2) exists. We say $\mathcal{A}$ this set. If $a \in \mathcal{A}$ a measure $\mu_a(dx) = \rho_a(x)dx$ invariant under S_a and absolutely continuous with respect to the Lebesgue measure on [0,1] exists [7] and this measure is unique and ergodic ([4], [9]).

The support of μ_a is not [0,1], but the subset $A_I = \bigcup_{k=0}^{p-1} S_a^k(I)$. The density $\rho_a(x)$ of the S_a-invariant measure μ_a is not a bounded function, but it has singularities as const$*(|x - S_a^k(1/2)|)^{-1/2}$ in the points $S_a(1/2),\ S_a^2(1/2), \ldots, S_a^{2p}(1/2)$.

Indeed, if $a \in \mathcal{A}$ ([7]) the map S_a^p is conjugate, via the homeomorphism $F : I \to [0,1]$, to a piecewise C^1 expanding map $R_a : [0,1] \to [0,1]$ $(|R_a'| > \sqrt{8/a} > \sqrt{2})$; that is $R_a = F \circ S_a^p \circ F^{-1}$ where $F(x) = \int_{S_a^p(\frac{1}{2})}^{x} h(t)dt$, $h(t) =$const$*(|S_a^p(1/2) - t|\ |S_a^{2p}(1/2) - t|)^{-1/2}$, $\int_I h(t)dt = 1$.

The expanding map R_a has a R_a-invariant measure $\hat{\nu}_a$ and the measure ν_a defined by $\nu_a = \hat{\nu}_a \circ F$ is invariant under S_a^p.

Thus μ_a is obtained as:

$$\mu_a = (\nu_a + \nu_a \circ (S_a^{-1}) + \ldots + \nu_a \circ (S_a^{-p+1}))/p.$$

In [1] a central limit theorem is proved for S_a, $a \in \mathcal{A}$ with respect to the a.c. S_a-invariant measure μ_a.

Now, we state theorem 1., that is an almost sure invariance principle, from which log-log-laws, weak invariance principles and the central limit theorem follow.

Let $f : [0,1] \to \mathbb{R}$ be a function with bounded p-variation, $p \geq 1$

that is

$$V^p f = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p < \infty$$

where the supremum is taken over all finite subsets $\{x_0, \ldots, x_n\}$ $x_0 < x_1 < \ldots < x_n$ of the interval [0,1].

Let $s_t^a = \sum_{n \leq t} f \circ (S_a^n)$, $a \in \mathcal{A}$.

THEOREM 1. *If $a \in [0,4]$ satisfies conditions 1) and 2), then for every $f : [0,1] \to R$ with bounded p-variation such that $\int_0^1 f d\mu_a = 0$ the series*

$$\sigma_a^2 = \int f^2 d\mu_a + 2 \sum_{i=1}^{\infty} \int f(f \circ S_a^i) d\mu_a$$

is absolutely convergent;

$$\int (s_n^a)^2 d\mu_a = n\sigma_a^2 + 0(1)$$

and if $\sigma_a^2 \neq 0$, then without changing its distribution we can redefine the process $(s_t^a)_{t \geq 0}$ on a richer probability space together with a standard Brownian motion $(B_t^a)_{t \geq 0}$ such that:

$$|\sigma_a^{-1} s_t^a - B_t^a| = 0(t^{1/2 - \lambda_a}) \quad \mu_a - a.s.$$

for some $\lambda_a > 0$.

3. Proofs.

First, we recall the set of conditions of Misiurewicz which are needed for the proof of the w.i.p. in [5].

Let J be a closed interval, $g : J \to J$ a continuous map monotonic on the connected components of the set $J \backslash P(g)$, where $P(g)$ is a finite subset of J containing the ends of J, and consider the following assumptions ([6], [5]) for $g|J \backslash P(g)$:

(i) g is of class C^3;

(ii) $g' \neq 0$;

(iii) the Schwarzian derivative $\mathcal{S}(g)$ of g is non positive, i.e.

$$\mathcal{S}(g) = g'''/g' - (3/2) \cdot (g''/g') \leq 0$$

(iv) if $g^n(x) = x$, then $|(g^n)'(x)| > 1$

(v) there exists a neighbourhood U of the set $P(g)$ such that for all $t \in P(g)$, $n \geq 0$ $g^n(t) \in P(g)$ or $g^n(t) \notin U$ for all $m \geq n$

(vi) for all $t \in P(g)$ there exists a neighbourhood U_t of t and constants $\alpha, w, \delta > 0$, $u \geq 0$ such that:

(I) $\alpha|x-t|^u \leq |g'(x)| \leq \omega|x-t|^u$

(II) $|g''(x)| \leq \delta|x-t|^{u-1}$ for all $x \in U_t$.

We recall that assumptions (i)-(iv) imply theorem 1., that is the a.s.i.p., as stated by Ziemian in [5].

Indeed, we will show that the above assumptions are true for the maps considered (this will be done in lemma 2).

LEMMA 1. *If* $T^n(x) = x$, *then* $|(T^n)'(x)| > 1$, *where* $T = S_a^p$ *and* $n \in \mathbb{N}$.

Proof. We have:

$$(T^n)'(x) = [F'(F^{-1} \circ R_a^n \circ F(x))]^{-1} \cdot (R_a^n)'(F(x)) \cdot F'(x);$$

since $T^n x = x$, then $F^{-1} \circ R_a^n \circ F(x) = x$. Therefore: $(T^n)'(x) = (R_a^n)'(F(x))$. Thus: $|(T^n)'(x)| = |(R_a^n)'(F(x))| \geq \alpha > 1$, where $\alpha > \sqrt{2}$ is the expanding coefficient of R_a

LEMMA 2. *If* $a \in \mathcal{A}$, S_a *satisfies the above conditions (i)-(vi) of Misiurewicz*

Proof. (i), (ii) and (iii) are trivial, since $S_a''''(x) = 0$.

To prove (iv) we proceede as follows.

Observe that $\forall n$ there exist two integers m and K such that $nK = mp$. Then, if $S_a^n(x) = x$ we have:

$$x = S_a^{nk}(x) = S_a^{mp}(x) = T^m(x)$$

and so for lemma 1.

$$|(S^{nk})'(x)| > 1$$

that is:

$$\underbrace{|(S^n)'(x) \cdot (S^n)'(x) \ldots (S^n)'(x)|}_{k \text{ times}} > 1 \quad \text{i.e.} \quad |(S^n)'(x)|^k > 1$$

which implies $|(S^n)'(x)| > 1$ and so (iv) holds.

To prove (v) we recall that $P(S_a) = \{0, 1/2, 1\}$ and $S_a^n(0) = S_a^n(1) = 0 \ \forall n > 0$. If $I = (S_a^p(1/2),\ S_a^{2p}(1/2)) \ni 1/2$, we have $S_a(I) \subset (1 - 1/a, 1)$, $S_a^l(I) \subset (0, 1/a)$, $l = 2, \ldots, p-1$ and $S_a^p(I) = I$; moreover the right endpoint of the interval $S_a^l(I)$ increases with $l = 2, \ldots, p-1$ (see [2]).

So:

$$S_a(1/2) \in (1 - 1/a, 1); \ S_a^l(1/2) \in (0, 1/a), \ l = 2, \ldots, p-1; \ S_a^p(1/2) < 1/2.$$

Now, we set

$$U_0 = (0, (1/2) \cdot S_a^2(1/2));$$

$$U_1 = ((1/4) + (1/2) \cdot S_a^p(1/2), (3/4) - (1/2) \cdot S_a^p(1/2));$$

$$U_2 = ((1/2) + (1/2) \cdot S_a(1/2), 1).$$

Then $U = U_0 \cup U_1 \cup U_2$ is a neighbourhood of $P(S_a)$ such that $S_a^m(1/2) \notin U$ for any m.

To prove (vi) is enough to choose

$U_t = (0, 1/4)$ and $u = 0$, $\alpha = 1$, $\omega = 4$, $\delta = 2$ for $t = 0$;

$U_t = (3/4, 1)$ and $u = 0$, $\alpha = 1/2$, $\omega = 4$, $\delta = 1$ for $t = 1$;

$U_t = ((1/2) - r, (1/2) + r)$ for small enough r and $u = 1$, $\alpha = 1$, $\omega = 2$, $\delta = 8$ for $t = 1/2$.

Now we can prove theorem 1.

Proof of th. 1. Thanks to lemma 2., for $a \in \mathcal{A}$, S_a satysfies all the assumptions of theorem 1. in [5]. Then, our theorem follows from this.

Remark 1. If we denote $\mathcal{P}$ the partition into the intervals of monotonicity of S_a, that is $\mathcal{P} = \{(0, 1/2), (1/2, 1)\}$, an exponential rate of decreasing of the diameter of the partition $V_0^{n-1} S_a^{-i} \mathcal{P}$ in the sense of Lebesgue measure and of the measure μ_a can be proved by using the properties of S_a. Analogous estimates are proved in [5] starting from assumption (i)-(vi).

Remark 2. From the proof of the a.s.i.p. the central limit theorem follows again for S_a, $a \in \mathcal{A}$ (for istance, see [8], [3]); already it was proved with different arguments in [1].

Moreover, the a.s.i.p. implies log-log-laws, and weak invariance principles (see [8]) for the process $\{f \circ S_a^i\}$.

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