LE MATEMATICHE
Vol. LXIII (2008) – Fasc. II, pp. 185–189

COMPONENTWISE LINEARITY OF IDEALS ARISING FROM GRAPHS

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Let G be a simple undirected graph on n vertices. Francisco and Van Tuyl have shown that if G is chordal, then $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle$ is componentwise linear. A natural question that arises is for which $t_{ij}>1$ the intersection ideal $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle^{t_{ij}}$ is componentwise linear, if G is chordal. In this report we show that $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle^{n-1}$ is componentwise linear for all $n\geq 3$, if G is a complete graph. We give also an example where G is chordal, but the intersection ideal is not componentwise linear for any $t>1$.

1. Introduction

Let G be a simple graph on n vertices, E_G the edge set of G and V_G the vertex set of G. Let $R=k[x_1,\dots,x_n]$ be the polynomial ring over a field k. The *edge ideal* of G is the quadratic squarefree monomial ideal $\mathscr{I}(G)=\langle\{x_ix_j\}\mid\{x_i,x_j\}\in E_G\rangle\subset R$. Then we define the *squarefree Alexander dual* of $\mathscr{I}(G)$ as $\mathscr{I}(G)^{\vee}=\cap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle$. To call $\mathscr{I}(G)^{\vee}$ the squarefree Alexander dual of $\mathscr{I}(G)$ is natural since it is the Stanley–Reisner ideal of the simplicial complex $\Delta^{\vee}$ that is the Alexander dual simplicial complex of Δ, where Δ in turn is the simplicial complex whose Stanley–Reisner ideal is $\mathscr{I}(G)$.

Entrato in redazione: 2 gennaio 2009

AMS 2000 Subject Classification: Primary 13C05, 13D40; Secondary 13A30, 20M14
Keywords: Linear resolution, componentwise linearity, chordal graphs
The first author was partly sponosered by The Royal Swedish Academy of Sciences

In [4] Herzog and Hibi give the following definition. Given a graded ideal $I \subset R$, we denote by $I_{\langle d \rangle}$ the ideal generated by the elements of degree d that belong to I. Then we say that a (graded) ideal $I \subset R$ is *componentwise linear* if $I_{\langle d \rangle}$ has a linear resolution for all d.

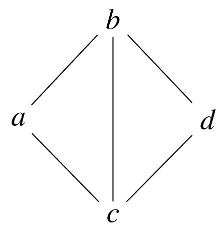
If the graph G is chordal, that is, every cycle of length $m \geq 3$ in G has a chord, then it is proved by Francisco and Van Tuyl in [2] that $\mathscr{I}(G)^V$ is componentwise. (The authors then use the result to show that all chordal graphs are sequentially Cohen-Macaulay.)

In this report we examine componentwise linearity of ideals arising from complete graphs and of the form $\bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle^{n-1}$.

2. A counterexample

There exists a chordal graph G such that $\bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle^t$ is not componentwise linear for any $t > 1$.

Let G be the chordal graph



Denote the intersection $\bigcap_{\{i,j\} \in E_G} \langle i, j \rangle^t$ by $I_4^{(t)}$. We have that

$$I_4^{(1)} = \bigcap_{\{i,j\} \in E_G} \langle i, j \rangle = \langle bc, abd, acd \rangle$$

and

$$I_4^{(2)} = \bigcap_{\{i,j\} \in E_G} \langle i, j \rangle = \langle b^2c^2, abcd, a^2b^2d^2, a^2c^2d^2 \rangle.$$

We claim that for $t > 1$ the ideal has the form

$$I_4^{(t)} = \langle b^t c^t, b^{t-1} c^{t-1} ad \rangle + J_t,$$

where J_t is an ideal generated of elements of degree at least $2t+1$. This is evidently true for $t = 1$. Now, for $t+1$ we may write the ideal as

$$I_4^{(t+1)} = \langle a,b \rangle \langle a,b \rangle^t \cap \langle a,c \rangle \langle a,c \rangle^t \cap \langle b,c \rangle \langle b,c \rangle^t \cap \langle b,d \rangle \langle b,d \rangle^t \cap \langle c,d \rangle \langle c,d \rangle^t.$$

Assuming our claim holds for $I_4^{(t)}$, it is clear that no generator of $I_4^{(t+1)}$ has degree strictly less than $2t+2$. Furthermore one sees that the only generators of degree equal to $2t+2$ are $b^{t+1}c^{t+1}$ and b^tc^tad. This proves our claim.

Consider the minimal free resolution of $I_4^{(t)}$. Its degree $2t$-part is

$$0 \to R(-(2t+2)) \to R^2(-2t) \to (b^tc^t, b^{t-1}c^{t-1}ad) \to 0,$$

which clearly is non-linear.

3. Intersections for complete graphs

Let K_n be a complete graph on n vertices, that is, $\{x_i, x_j\} \in E_{K_n}$ for all $1 \le i \neq j \le n$. We write $K_n^{(n-1)} = \bigcap_{\{x_i,x_j\}\in E_{K_n}} \langle x_i, x_j\rangle^{n-1}$. We show that the ideal $K_n^{(n-1)}$ is componentwise linear for all $n \ge 3$. Recall that a *vertex cover* of a graph G is a subset $A \subset V_G$ such that every edge of G is incident to at least one vertex of A. One can show that $\mathscr{I}(G)^V = \langle x_{i_1}\cdots x_{i_k} \mid \{x_{i_1},\dots,x_{i_k}\}$ a vertex cover of $G\rangle$. A *t-vertex cover* (or a *vertex cover of order t) of* G is a vector $\mathbf{a} = (a_1,\dots,a_n)$ with $a_i \in \mathbb{N}$ such that $a_i + a_j \ge t$ for all $\{x_i, x_j\} \in E_G$.

In the proof of the theorem below, we use the following definition and proposition.

Definition 3.1. A monomial ideal I is said to have *linear quotients*, if for some degree ordering of the minimal generators $f_1,\dots,f_r$ and all $k > 1$, the colon ideals $\langle f_1,\dots,f_{k-1}\rangle : f_k$ are generated by a subset of $\{x_1,\dots,x_n\}$.

Proposition 3.2 (Proposition 2.6 in [3] and Lemma 4.1 in [1])**.** *If I is a homogeneous ideal with linear quotients, then I is componentwise linear.*

Theorem 3.3. *The ideal $K_n^{(n-1)}$ is componentwise linear for all $n \ge 3$.*

Proof. For calculating an explicit generating system of $K_n^{(n-1)}$ we will use t-vertex covers. Pick any monomial m in $K_n^{(n-1)}$ and, for some k and l, consider the maximal t_k, t_l such that $x_k^{t_k}x_l^{t_l}$ is a factor in m. As m is contained in $\langle x_k, x_l\rangle^{n-1}$ we must have $t_k + t_k \ge n-1$. Hence, $K_n^{(n-1)}$ is generated by the monomials of the form $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a}$ is an $(n-1)$-cover of K_n. That is, the sum of the two lowest exponents in every (monomial) generator of $K_n^{(n-1)}$ is at least $n-1$.

Now assume that $n-1 = 2m+1$ is odd. Using the degree lexicographic ordering $x_1 \prec x_2 \prec \cdots \prec x_n$ on the the minimal generators we get

$$\begin{aligned} K_n^{(n-1)} = K_n^{(2m+1)} = \quad & \Big\langle x_1^m \textstyle\prod_{i\neq 1} x_i^{m+1}, \quad \dots \quad , x_n^m \prod_{i\neq n} x_i^{m+1}, \\ & x_1^{m-1} \textstyle\prod_{i\neq 1} x_i^{m+2}, \quad \dots \quad , x_n^{m-1} \prod_{i\neq 1} x_i^{m+2}, \\ & \qquad \vdots \\ & \textstyle\prod_{i\neq 1} x_i^{2m+1}, \quad \dots \quad , \prod_{i\neq n} x_i^{2m+1} \Big\rangle. \end{aligned}$$

This order on the minimal generators satisfies the condition in Definition 3.1. Hence, $K_n^{(n-1)}$ has linear quotients and is componentwise linear by Proposition 3.2.

If $n-1=2m$ is even, then the degree lexicographic ordering yields the sequence

$$\begin{aligned} K_n^{(n-1)} = K_n^{(2m)} = \quad \langle \textstyle\prod_{i=1}^{2m} x_i^m, \quad & x_1^{m-1}\textstyle\prod_{i\neq 1} x_i^{m+1}, \quad \dots \quad , x_n^{m-1}\textstyle\prod_{i\neq n} x_i^{m+1}, \\ & x_1^{m-2}\textstyle\prod_{i\neq 1} x_i^{m+2}, \quad \dots \quad , x_n^{m-2}\textstyle\prod_{i\neq 1} x_i^{m+2}, \\ & \vdots \\ & \textstyle\prod_{i\neq 1} x_i^{2m}, \quad \dots \quad , \textstyle\prod_{i\neq n} x_i^{2m} \rangle, \end{aligned}$$

which also satisfies the condition in Definition 3.1, and the same result follows. □

Example 3.4.

$$K_6^{(5)} = \left\langle \{x_j^2 \prod_{i\neq j} x_i^3\}_{1\leq j\leq 6},\ \{x_j \prod_{i\neq j} x_i^4\}_{1\leq j\leq 6},\ \{\prod_{i\neq j} x_i^5\}_{1\leq j\leq 6} \right\rangle$$

and

$$K_7^{(6)} = \langle \prod_{i=1}^{7} x_i^3,\ \{x_j^2 \prod_{i\neq j} x_i^4\}_{1\leq j\leq 7},\ \{x_j \prod_{i\neq j} x_i^5\}_{1\leq j\leq 7},\ \{\prod_{i\neq j} x_i^6\}_{1\leq j\leq 7} \rangle.$$

4. Problems and generalizations

We want to check whether the result in Section 3 is valid for complete hypergraphs. We would also like to investigate the relation between sequentially Cohen-Macaulayness and componentswise linearity for non-squarefree ideals.

Acknowledgements

First of all we would like to thank the Università di Catania and the organizers of the PRAGMATIC summer school 2008, especially Alfio Ragusa and Giuseppe Zappalà. We are deeply greatful to Jürgen Herzog and Volkmar Welker for their excellent lectures, interesting problems and thorough guidance.

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