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FURTHER PROPERTIES OF THE ZEROS OF BESSEL FUNCTIONS

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New monotonicity and convexity properties for the zeros $c_{\nu k}$ $(k = 1, 2, \ldots)$ of the Bessel functions are proved. New inequalities for $c_{\nu k}$ are also given. These inequalities are useful for small values of ν.

1. Introduction.

For $\nu \geq 0$ let $j_{\nu k}$ and $c_{\nu k}$ be the k-th positive zeros of the Bessel function $J_\nu(x)$ of the first kind and of the general cylinder function

$$C_\nu(x, \alpha) = C_\nu(x) = J_\nu(x) \cos\alpha - Y_\nu(x) \sin\alpha, \quad 0 \leq \alpha < \pi$$

respectively, where $Y_\nu(x)$ denotes the Bessel function of the second kind.

In [6] the authors introduced the notation $j_{\nu\kappa}$ by $c_{\nu k} = j_{\nu\kappa}$, where $\kappa = k - \alpha/\pi$. The usefulness of this notation has been shown in the papers [1-7] where *monotonicity, concavity, convexity* and *asymptotic*

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properties of $j_{\nu\kappa}$ have been investigated. We shall use the notation $j_{\nu\kappa}$ or $c_{\nu k}$ indifferently.

In this paper we are concerned with some further properties and inequalities for $j_{\nu\kappa}$.

The results are motivated by the fact that monotonicity, concavity and convexity properties of $j_{\nu k}$ arise in the quantum mechanical explanation for the origin of the vortex lines produced in superfluid helium when its container is rotated. This explanation has been proposed by Putterman, Kac and Uhlenbeck [15].

The proofs of the results of the next section are based on the Sturm comparison theorem [17, p. 19] and on a lower bound for the second derivative of $j_{\nu\kappa}$. We observe that monotonicity results on $j_{\nu\kappa}$ (for example the decreasing character of $j_{\nu k}/\nu, \nu > 0$) have been proved by Lewis and Muldoon [12] as a consequence of the Hellman-Feynman theorem of quantum chemistry [8].

Finally we find new inequalities and approximations for $j_{\nu\kappa}$. These results are stringent for small values of ν.

2. Monotonicity and Convexity results.

THEOREM 2.1. *For $\nu > 0$ let $c_{\nu k}$ be the k-th zero of the cylinder function $C_\nu(x, \alpha)$. Then for $0 \leq \alpha < \pi$.*

$$\left(\frac{c_{\nu,k+1}}{c_{\nu,k}}\right)^{2\nu} \quad \textit{increases with } \nu > 0,\ k = 2, 3, \ldots$$

When $0 \leq \alpha \leq \dfrac{\pi}{2}$ the result holds even in the case $k = 1$.

Proof. The function $y_\nu(x) = \sqrt{x}C_\nu(c_{\nu k}x^{1/(2\nu)})$ satisfies the differential equation

$$y'' + p_\nu(x)y = 0 \tag{2.1}$$

where

$$p_\nu(x) = \left(\frac{c_{\nu k}}{2\nu}x^{1/(2\nu)-1}\right)^2.$$

Besides (2.1) consider the differential equation

$$z'' + p_{\nu+\varepsilon}(x)z = 0, \quad \varepsilon > 0$$

satisfied by $z_\nu(x) = y_{\nu+\varepsilon}(x)$. Clearly the function $y_\nu(x)$ and $z_\nu(x)$ have a common zero at $x = 1$. The function $x^{\frac{1}{2\nu}-1}$ decreases with $\nu > 0$, for $x > 1$. Moreover $c_{\nu k}/\nu$ decreases with $\nu > 0$, $k = 2, 3, \dots$ [11, p. 471].

Therefore $p_{\nu+\varepsilon} < p_\nu$ and an application of the Sturm comparison theorem [17, p. 19] gives that the next zero of $y_\nu(x)$ occurs before the next zero of $z_\nu(x)$; this leads to

$$\left(\frac{c_{\nu,k+1}}{c_{\nu,k}}\right)^{2\nu} < \left(\frac{c_{\nu+\varepsilon,k+1}}{c_{\nu+\varepsilon,k}}\right)^{2(\nu+\varepsilon)}, \nu > 0, k = 2, 3, \dots$$

In order to complete the proof of the theorem we have to consider the case $k = 1, 0 \le \alpha \le \pi/2$. In general it is not true that $c_{\nu 1}/\nu$ decreases with $\nu > 0$, but when $0 \le \alpha \le \pi/2$ it is so. In fact we know that under this restriction on $\alpha, c_{\nu 1}$ is concave for $\nu > 0$, [11].

Differentiating $c_{\nu 1}/\nu$ we have to study the sign of the function

$$f(\nu) = \nu \frac{d}{d\nu} c_{\nu 1} - c_{\nu 1}, \quad \nu > 0.$$

Clearly $f(0) < 0$ and using the concavity of $c_{\nu 1}$ we have $f'(\nu) < 0$, $\nu > 0$. Thus we can conclude that $f(\nu) < 0$ and $c_{\nu 1}/\nu$ decreases with $\nu > 0$.

Therefore $p_{\nu+\varepsilon} < p_\nu$ also for $k = 1, 0 \le \alpha \le \pi/2$ and using the Sturm theorem again we have the conclusion of the Theorem. The proof of the Theorem 2.1 is complete.

Remark 2.1. The equation (2.1) was the starting point of the proof given by Makai [13] that $j_{\nu k}/\nu$ decreases with $\nu > 0$. Here $j_{\nu k}$ is the k-th positive zero of $J_\nu(x)$.

Remark 2.2. The proof of the theorem 2.1. cannot be extended to $k = 1$ for any $\alpha \in [\pi/2, \pi]$. In fact for these values of α we can have $c_{\nu 1} < \nu$. On the other hand from a formula by Spigler [16] we have

$\lim_{\nu\to+\infty} c_{\nu 1}/\nu = 1$ and we cannot conclude that $c_{\nu 1}/\nu$ decreases with $\nu > 0$.

Remark 2.3. We observe that $0 \leq \alpha \leq \pi/2$ encludes the important case of the zeros $j_{\nu k}$ and $y_{\nu k}$ of $J_\nu(x)$ and $Y_\nu(x)$ corresponding to $\alpha = 0$ and $\alpha = \pi/2$, respectively.

We proved in [1] that the function $j_{\nu\kappa}$ is concave in $\nu \geq 0$ for $\kappa \geq \kappa_0 = 0.344\ldots$. Here we study the behaviour of the function $j_{\nu\kappa}+\nu^2$. The result is given by the following theorem.

THEOREM 2.2. *For* $\nu \geq 0$ *and* $\kappa \geq \kappa_0 = 0.7070\ldots$ *the function* $j_{\nu\kappa} + \nu^2$ *is convex with respect to* ν.

Proof. We use the lower bound [3, p. 74]

$$j''_{\nu\kappa} > \frac{\nu j'^2_{\nu\kappa} - j_{\nu\kappa} j'_{\nu\kappa})}{(\nu + j_{\nu\kappa}) j_{\nu\kappa}}$$

where $' = d/d\nu$. Thus we need only to show that

$$\nu j'^2_{\nu\kappa} + 2\nu j_{\nu\kappa} > j_{\nu\kappa} j'_{\nu\kappa} - 2j^2_{\nu\kappa}.$$

Since the left-hand side is clearly positive $\left(\nu > 0, j_{\nu\kappa} > 0, j'_{\nu\kappa} = \frac{d}{d\nu} j_{\nu\kappa} > 0\right)$ hence it is sufficient to show that

$$j_{\nu\kappa} j'_{\nu\kappa} - 2j^2_{\nu\kappa} < 0.$$

This is true because [3, p. 76]

$$j'_{\nu\kappa} < \frac{j_{\nu\kappa}}{\nu + 1} < 2j_{\nu\kappa}, \ \nu \geq 0, \ \kappa \geq \kappa_0 = 0.7070\ldots$$

This completes the proof of the Theorem.

Remark 2.4. Theorem 2.2 shows immediately the convexity of the function $j_{\nu\kappa} + a\nu^2$ when $a > 1$. The investigation of similar properties when $a < 1$ is more difficult and it seems not possible to deal with this problem with the current results of the literature.

3. Inequalities for $j_{\nu\kappa}$.

The results of this section require the following classical Jensen's inequality [14, p. 12]

$$\log f(x_1) + \log f(x_2) + \ldots + \log f(x_n) \geq n \log f(x)$$

(3.1)

$$x = (x_1 + x_2 + \ldots + x_n)/n, \quad x_i > 0, i = 1, 2, \ldots, n$$

where $f(x)$ is a log-convex function.

We know [1, p. 276] that the function $j_{\nu\kappa}$ is concave in $\nu \geq 0$ for $\kappa \geq 0.344\ldots$. Since $j_{\nu\kappa}$ is also log-concave we can apply (3.1) to the function $f(\nu) = 1/j_{\nu\kappa}$ which clearly is log-convex. We obtain

$$j_{\nu_1\kappa} j_{\nu_2\kappa} \cdots j_{\nu_n\kappa} \leq (j_{\bar{\nu}\kappa})^n \tag{3.2}$$

where

$$\bar{\nu} = (\nu_1 + \nu_2 + \ldots + \nu_n)/n, \ \nu_i > 0, \ i = 1, 2, \ldots, n, \ \kappa \geq 0.344\ldots$$

Similarly a result established in [7] ensures that for $\kappa \geq \kappa_0 = 0.7070\ldots$ the function $\log(j_{\nu\kappa}/\nu)$ is convex with respect to $\nu > 0$.

Thus applying Jensen's inequality (3.1) to $f(\nu) = j_{\nu\kappa}/\nu$ we find

$$\frac{j_{\nu_1\kappa}}{\nu_1} \frac{j_{\nu_2\kappa}}{\nu_2} \cdots \frac{j_{\nu_n\kappa}}{\nu_2} \geq \left(\frac{j_{\bar{\nu}\kappa}}{\bar{\nu}}\right)^2, \quad \kappa \geq \kappa_0 = 0.7070\ldots$$

(3.3)

$$\bar{\nu} = (\nu_1 + \nu_2 + \ldots + \nu_n)/n, \ \nu_i > 0, \ i = 1, 2, \ldots, n.$$

Combining (3.2) and (3.3) we obtain

$$\frac{\nu_1 + \nu_2 \ldots \nu_n}{(\bar{\nu})^n} \leq \frac{j_{\nu_1\kappa} j_{\nu_2\kappa} \cdots j_{\nu_n\kappa}}{(j_{\bar{\nu}\kappa})^n} \leq 1, \ \nu > 0, \ \bar{\nu} = (\nu_1 + \nu_2 + \ldots + \nu_n)/n. \tag{3.4}$$

where the upper bound holds for $\kappa \geq 0.344\ldots$, while the lower bound has been proved only for $\kappa \geq 0.7070,\ldots$.

A numerical example shows the interst of these inequalities. For $n = 2$, $\kappa = 1$, $\nu_1 = 1/3$, $\nu_2 = 2/3$ we find

$$\frac{8}{9}\pi^2 \leq j_{\frac{1}{3},1} j_{\frac{2}{3},1} \leq \pi^2 \tag{3.5}$$

where we have used $j_{\frac{1}{2},1} = \pi$.

This result can be useful to obtain a first approximation for $j_{\frac{2}{3},1}$. Indeed we know [18, p. 751] that $j_{\frac{1}{3},1} = 2.9025862$ and by (3.5) we find

$$3.02 < j_{\frac{2}{3},1} < 3.41$$

More generally when $(\nu_1 + \nu_2 + \ldots + \nu_n)/n = 1/2$ and $\kappa = 1$, we can write

$$(2\pi)^n(\nu_1\nu_2\ldots\nu_n) \leq j_{\nu_1,1} j_{\nu_2,1} \ldots j_{\nu_n,1} \leq \pi^n$$

Anyway the (3.4) gives more and more stringent results the closer the arithmetical mean of $\nu_1\nu_2\ldots\nu_n$ is to their geometrical mean, as we can see in the following example. For $\kappa = 1, \nu_1 = 0,249$, $\nu_2 = 0,251$. $\frac{\sqrt{\nu_1, \nu_2}}{\bar{\nu}} = 0,9999920$ using the value $y_{\frac{1}{4},1} = 1.241662$, we have

$$1,241652 \leq \sqrt{y_{\nu_1,1} y_{\nu_2,1}} \leq 1.241662$$

Remark 3.1. An interesting particular case of (3.2) is when $n = 2$, $\nu_1 = \nu - \delta > 0$, $\nu_2 = \nu + \delta$. In this case (3.2) becomes

$$\frac{j_{\nu+\delta,\kappa}}{j_{\nu,\kappa}} < \frac{j_{\nu,\kappa}}{j_{\nu-\delta,\kappa}}$$

a result already pointed out in [4] where more general properties of the ratio between two zeros of the Bessel functions have been investigated.

4. Approximations of $j_{\nu\kappa}$ for small values of ν.

The first purpose of this section is to prove some new inequalities for $j_{\nu\kappa}$. The results are stringent for ν close to zero.

The behaviour of $j_{\nu\kappa}$ for small values of ν has been investigated by A. Laforgia and M.E. Muldoon [10]. In particular for $\kappa = 1$ they found the asymptotic formula

$$j_{\nu 1} = j_{01} + a_1\nu + a_2\nu^2 + O(\nu^3), \quad \nu \to 0 \tag{4.1}$$

where

$$a_1 = \left.\frac{dj_{\nu 1}}{d\nu}\right|_{\nu=0} = 1.542889743, \quad a_2 = \left.\frac{d^2 j_{\nu 1}}{d\nu^2}\right|_{\nu=0} = -0.175493592.$$

Now, using the concavity of $j_{\nu 1}$ we have that the first two terms in (4.1) give an upper bound for $j_{\nu 1}$. Similarly using the property $j'''_{\nu\kappa} > 0$, κ fixed, established in [3] for $\nu \geq 0$, we get that the first three terms of (4.1) give a lower bound for $j_{\nu 1}$. Then we have the inequalities

$$j_{01} + 1.542889743\nu - 0.175493592\nu^2 < j_{\nu 1} < j_{01} + 1.542889743\nu \tag{4.2}$$

Now we want to use the result $d^3 j_{\nu 1}/d\nu^3 > 0$ already mentioned and to consider polynomial interpolations of second degree for $j_{\nu 1}$.

We know the particular values

$$j_{01} = 2.40482556, \; j'_{01} = \left.\frac{d}{d\nu} j_{\nu 1}\right|_{\nu=0} = 1.542889743, \; j_{\frac{1}{2},1} = 3.14159265$$

Therefore we can write the polynomial of second degree $p_2(\nu)$ which satisfies the conditions

$$p_2(0) = j_{01}, \; p'_2(0) = j'_{01}, \; p_2(1/2) = j_{\frac{1}{2},1}$$

that is

$$p_2(\nu) = j_{01} + 1.542889743\nu - 0.138711126\nu^2.$$

The error $E_2(\nu) = j_{\nu 1} - p_2(\nu)$ is given by

$$E_2(\nu) = \frac{\nu^2(\nu - 1/2)}{3!}[j'''_{\nu 1}]_{\nu=\xi}, \quad 0 < \xi < 1/2.$$

Since $j'''_{\nu 1} > 0$, $\nu \geq 0$, we have

$$j_{\nu 1} - p_2(\nu) < 0, \quad 0 < \nu < 1/2$$

and the inequality

$$(4.3) \qquad j_{\nu 1} < j_{01} + 1.542889743\nu - 0.138711126\nu^2, \; 0 < \nu < 1/2$$

follows. The improvement of this inequality with respect to the upper bound in (4.2) is evident.

In [9] we compared some inequalities for $j_{\nu 1}$ on the interval $[0, 1/2]$ and we found that these should be used in the following way

$$j_{\nu 1} < j_{01} + \frac{\pi}{2}\nu, \; 0 < \nu \leq 0.042$$

$$j_{\nu 1} < [2(\nu + 1)(\nu + 5)(5\nu + 11)/(7\nu + 19)]^{1/2}, \; 0.043 \leq \nu \leq 0.459$$

$$j_{\nu 1} < [j_{01}^2 + 2\nu(\pi^2 - j_{01}^2)]^{1/2}, \; 0.460 \leq \nu < 0.5.$$

Comparing these inequalities with (4.3), by evaluating the approximations of $j_{\nu 1}$ with step 0.001, $\nu \in [0, 1/2]$, we find that (4.3) is the closest.

In order to improve the bound (4.3) in the middle part of the interval $[0, 1/2]$ we interpolate $j_{\nu 1}$ at the points $\nu = 0$, $\nu = 1/4$, $\nu = 1/2$.

By means of straighforward calculations we are led to the bound

$$(4.4) \qquad j_{\nu 1} > j_{01} + 1.534963132\nu - 0.122857904\nu^2, \; 0 < \nu < 1/4.$$

The inequality is reversed for $1/4 < \nu < 1/2$. The upper bound established now for $\nu \in (1/4, 1/2)$ is more stringent than the one in (4.3). For example from (4.3) we have

$$j_{\frac{1}{3},1} < 2.9037098$$

while (4.4) reversed gives

$$j_{\frac{1}{3},1} < 2.9028291.$$

The «exact» value is [18, p. 751] $j_{\frac{1}{3},1} = 2.902586\ldots$.

Further inequalities can be derived interpolating $j_{\nu 1}$ in different intervals at different points $\nu_i (i = 0, 1, \ldots, n)$. In order to do this we need only to know the values of $j_{\nu 1}$ at $\nu = \nu_i$ $(i = 0, 1, \ldots, n)$.

On the other hand it is not possible to extend inequality (4.3) to any zero $j_{\nu k}$ of $J_\nu(x)$. In fact it is not known the value of $\left.\frac{d}{d\nu} j_{\nu k}\right|_{\nu=0}$, when $k = 2, 3, \ldots$.

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