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A KOROVKIN-TYPE THEOREM FOR SET-VALUED HAUSDORFF CONTINUOUS FUNCTIONS

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In this paper, we give a generalization of a Korovkin-type theorem for set-valued Hausdorff continuous functions (cf. [1] and [2]), by means of upper and lower envelopes.

1. Introduction.

Throughout this paper, X is a compact Hausdorff topological space and E is a real finite dimensional normed space. We denote by Conv(E) the set of all compact convex non empty subsets of E; Conv(E) is equipped with the natural addition and multiplication by scalars of sets; further, we consider on Conv(E) the Hausdorff distance defined by putting, for each $A, B \in \text{Conv}(E)$,

$$d(A,B) = \inf\{\varepsilon \in \mathbb{R}_+^* | A \subset B + \varepsilon \cdot \mathbb{B},\ B \subset A + \varepsilon \cdot \mathbb{B}\}$$

where $\mathbb{B}$ denotes the closed unit ball of E of center 0.

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Moreover we recall that a multivalued function $F : X \to \mathrm{Conv}(E)$ is said to be Hausdorff continuous (briefly $\mathcal{H}$-continuous) at the point $x_0 \in X$ if, for each $\varepsilon \in \mathbb{R}_+^*$, there exists a neighbourhood U of x_0 such that $d(F(x), F(x_0)) \leq \varepsilon$ for each $x \in U$; F is said to be $\mathcal{H}$-continuous if it is continuous at each $x_0 \in X$.

We denote by $\mathcal{S}(X, E)$ the set of all $\mathcal{H}$ continuous set-valued functions of X in $\mathrm{Conv}(E)$; this space is equipped with a natural addition and multiplication by scalars ($(F + G)(x) = F(x) + G(x)$ and $(\lambda F)(x) = \lambda F(x)$ for each $F, G \in \mathcal{S}(X, E)$, $\lambda \in \mathbb{R}$ and $x \in X$) and with the uniform distance $d : \mathcal{S}(X, E) \times \mathcal{S}(X, E) \to \mathbb{R}$ defined by putting, for all $F, G \in \mathcal{S}(X, E)$:

$$d(F, G) = \sup_{x \in X} d(F(x), G(x)).$$

In $\mathcal{S}(X, E)$ we also consider the following order relation; if $F, G \in \mathcal{S}(X, E)$, we say that $F \leq G$ if, for each $x \in X$, $F(x) \subset G(x)$.

Now, an operator $\mathcal{J} : \mathcal{S}(X, E) \to \mathcal{S}(X, E)$ is called linear if

$$\mathcal{J}(F + G) = \mathcal{J}(F) + \mathcal{J}(G), \quad \mathcal{J}(\lambda F) = \lambda \mathcal{J}(F)$$

for each $F, G \in \mathcal{S}(X, E)$ and $\lambda \in \mathbb{R}_+$ and is called monotone if the condition $F \leq G$ implies $\mathcal{J}(F) \leq \mathcal{J}(G)$ $(F, G \in \mathcal{S}(X, E))$.

Finally, we recall that a subset H of the space $C(X, \mathbb{R})$ of all continuous real functions defined in X is said to be a Korovkin system in $C(X, \mathbb{R})$ if, for each equicontinuous net $(T_\alpha)_{\alpha \in I}^{\leq}$ of positive linear operators of $C(X, \mathbb{R})$ into itself, the convergence to h of the net $(T_\alpha(h))_{\alpha \in I}^{\leq}$ for each $h \in H$ implies the convergence to f of the net $(T(f)_\alpha)_{\alpha \in I}^{\leq}$ for each $f \in C(X, \mathbb{R})$. This definition can be adjusted in the space $\mathcal{S}(X, E)$ in the following manner.

DEFINITION 11. *A subset $\mathcal{G}$ of $\mathcal{S}(X, E)$ is said to be a Korovkin system in $\mathcal{S}(X, E)$ if, for each equicontinuous net $(\mathcal{J}_\alpha)_{\alpha \in I}^{\leq}$ of monotone linear operators of $\mathcal{S}(X, E)$ into itself such that the net $(\mathcal{J}_\alpha(G))_{\alpha \in I}^{\leq}$ converges to G for each $G \in \mathcal{G}$, it follows the convergence to F of the net $(\mathcal{J}_\alpha(F))_{\alpha \in I}^{\leq}$ for each $F \in \mathcal{S}(X, E)$.*

As observed in [1], if H (resp. $\mathcal{G}$) contains the constant functions, then the equicontinuity of the net $(T_\alpha)^{\leq}_{\alpha\in I}$ (resp. $(\mathcal{J}_\alpha)^{\leq}_{\alpha\in I}$) is superfluous.

2. The main theorem.

In [1] the following Korovkin type theorem has been proved.

THEOREM 2.1. *If H is a Korovkin system of positive functions in $C(X,\mathbb{R})$, then the subset $\mathcal{G}$ of $\mathcal{S}(X,E)$ whose elements are the constant functions and, for each $f\in H$, the function $x\mapsto f(x)\cdot \mathbb{B}$ ($\mathbb{B}$ denotes the closed unit ball of E) is a Korovkin system in $\mathcal{S}(X,E)$.*

In what follows, we denote by $\mathbb{B}$ the constant function in $\mathcal{S}(X,E)$ of constant value $\mathbb{B}$.

Our main result is the following theorem.

THEOREM 2.2. *If $\mathcal{G}$ is a subset of $\mathcal{S}(X,E)$ containing the constant functions and if there exists a Korovkin system H of positive functions in $C(X,\mathbb{R})$ such that, for each $f\in H$ and $x_0\in X$ and for each $\varepsilon\in\mathbb{R}^*_+$,*

$$(2.1)\qquad (f(x_0)+\varepsilon)\cdot\mathbb{B}=\overline{\bigcup_{\substack{\psi\in\mathcal{G}\\ \psi\leq(f+\varepsilon)\cdot\mathbb{B}}}\psi(x_0)},$$

$$(2.2)\qquad (f(x_0)+\varepsilon)\cdot\mathbb{B}=\bigcap_{\substack{\psi\in\mathcal{G}\\ (f+\varepsilon)\cdot\mathbb{B}\leq\psi}}\psi(x_0)$$

then $\mathcal{G}$ is a Korovkin system in $\mathcal{S}(X,E)$.

Proof. Let $(\mathcal{J}_\alpha)^{\leq}_{\alpha\in I}$ be a net of monotone linear operators of the space $\mathcal{S}(X,E)$ into itself such that net $(\mathcal{J}_\alpha(G))^{\leq}_{\alpha\in I}$ converges to G for each $G\in\mathcal{G}$. Thanks to Theorem 2.1, our proof will be accomplished if we will show that the net $(\mathcal{J}_\alpha(f\cdot\mathbb{B}))^{\leq}_{\alpha\in I}$ converges to $f\cdot\mathbb{B}$ for each $f\in H$. To this end, fix $f\in H$ and let $\varepsilon\in\mathbb{R}^*_+$ and $x_0\in X$; since

$\bigcup_{\substack{\psi\in\mathcal{G}\\ \psi\leq(f+\varepsilon)\cdot\mathbb{B}}} \psi(x_0)$ is precompact, by (2.1) there exist $\psi_1,\ldots,\psi_m \in \mathcal{G}$ and, for each $i=1,\ldots,m$, there exists $y_i \in E$ such that

$$\left(f(x_0)+\frac{\varepsilon}{4}\right)\cdot\mathbb{B} \subset \bigcup_{i=1}^{m}\left(y_i+\frac{\varepsilon}{4}\cdot\mathring{\mathbb{B}}\right) \tag{2.3}$$

(where $\mathring{\mathbb{B}}$ denotes the open unit ball in E) and

$$\psi_i \leq \left(f+\frac{\varepsilon}{4}\right)\cdot\mathbb{B},\ y_i\in\psi_i(x_0). \tag{2.4}$$

Since the net $(\mathcal{I}_\alpha(\psi_i))^{\leq}_{\alpha\in I}$ converges to ψ_i for each $i=1,\ldots,m$, we may found $\iota_1\in I$ such that, for each $\alpha\in I$, $\alpha\geq\iota_1$ and for each $i=1,\ldots,m$, $d(\psi_i,\mathcal{I}_\alpha(\psi_i))\leq\frac{\varepsilon}{8}$, which implies, for each $x\in X$,

$$\psi_i(x)\subset\mathcal{I}_\alpha(\psi_i)(x)+\frac{\varepsilon}{8}\cdot\mathbb{B}.$$

Further, for each $i=1,\ldots,m$ (cf. (2.4)) and $\alpha\geq\iota_1$,

$$\mathcal{I}_\alpha(\psi_i)\leq\mathcal{I}_\alpha\left(\left(f+\frac{\varepsilon}{4}\right)\cdot\mathbb{B}\right)=\mathcal{I}_\alpha(f\cdot\mathbb{B})+\frac{\varepsilon}{4}\cdot\mathcal{I}_\alpha(\mathbb{B});$$

it follows, for each $x\in X$,

$$\bigcup_{i=1}^{m}\psi_i(x)\subset\mathcal{I}_\alpha(f\cdot\mathbb{B})(x)+\frac{\varepsilon}{4}\cdot\mathcal{I}_\alpha(\mathbb{B})(x)+\frac{\varepsilon}{8}\cdot\mathbb{B}. \tag{2.5}$$

Since $\mathbb{B}\in\mathcal{G}$, the net $(\mathcal{I}_\alpha(\mathbb{B}))_{\alpha\in I}$ converges to $\mathbb{B}$ and therefore there exists $\iota_2\in I$ such that, for each $\alpha\geq\iota_2$

$$\mathcal{I}_\alpha(\mathbb{B})(x)\subset\mathbb{B}+\frac{\varepsilon}{4}\cdot\mathbb{B}. \tag{2.6}$$

By (2.3) and (2.4), it results

$$\left(f(x_0)+\frac{\varepsilon}{4}\right)\cdot\mathbb{B}\subset\bigcup_{i=1}^{m}\left(y_i+\frac{\varepsilon}{4}\cdot\mathring{\mathbb{B}}\right)\subset\bigcup_{i=1}^{m}\left(\psi_i(x_0)+\frac{\varepsilon}{8}\cdot\mathring{\mathbb{B}}\right);$$

since $f \cdot \mathbb{B}$ and each ψ_i are Hausdorff continuous, there exists a neighbourhood U_1 of x_0 such that, for each $x \in U_1$,

$$\left(f(x) + \frac{\varepsilon}{4}\right) \cdot \mathbb{B} \subset \bigcup_{i=1}^{m} \left(\psi_i(x) + \frac{\varepsilon}{4} \cdot \mathbb{B}\right);$$

by (2.5) and (2.6), we have, for each $x \in U_1$ and $\alpha \in I$ with $\alpha \geq \iota_1$, $\alpha \geq \iota_2$,

$$\begin{aligned}\left(f(x) + \frac{\varepsilon}{4}\right) \cdot \mathbb{B} \subset \bigcup_{i=1}^{m} \left(\psi_i(x) + \frac{\varepsilon}{4} \cdot \mathbb{B}\right) \subset \\ \subset \mathcal{J}_\alpha(f \cdot \mathbb{B})(x) + \frac{\varepsilon}{4} \cdot \mathcal{J}_\alpha(\mathbb{B})(x) + \frac{\varepsilon}{8} \cdot \mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B} \subset \\ \subset \mathcal{J}_\alpha(f \cdot \mathbb{B})(x) + \frac{\varepsilon}{4} \cdot \left(\mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B}\right) + \frac{\varepsilon}{8} \cdot \mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B}\end{aligned}$$

and finally, for each $x \in U_1$ and $\alpha \in I$ with $\alpha \geq \iota_1$, $\alpha \geq \iota_2$,

$$f(x) \cdot \mathbb{B} \subset \mathcal{J}_\alpha(f \cdot \mathbb{B})(x) + \varepsilon \cdot \mathbb{B}. \tag{2.7}$$

Now, put

$$A = \left(\left(f(x_0) + \frac{\varepsilon}{2}\right) \cdot \mathbb{B}\right) \setminus \left(\left(f(x_0) + \frac{\varepsilon}{4}\right) \cdot \mathring{\mathbb{B}}\right);$$

for each $z \in A$, we have $z \notin \left(f(x_0) + \frac{\varepsilon}{8}\right) \cdot \mathbb{B}$ and therefore (cf. (2.2)) there exists $\chi \in \mathcal{G}$ such that $\left(f + \frac{\varepsilon}{8}\right) \cdot \mathbb{B} \leq \chi$ and $z \notin \chi(x_0)$; since $\chi(x_0)$ is compact, there exists a neighbourhood of z disjoint from $\chi(x_0)$.

The set A is compact and therefore, we may found $\chi_i, \dots, \chi_p \in \mathcal{G}$ such that

$$\bigcap_{i=1}^{p} \chi_i(x_0) \subset \left(f(x_0) + \frac{\varepsilon}{4}\right) \cdot \mathbb{B} \tag{2.8}$$

and, for each $i = 1, \dots, p$,

$$\left(f + \frac{\varepsilon}{8}\right) \cdot \mathbb{B} \leq \chi_i. \tag{2.9}$$

Since $f \cdot \mathbb{B}$ is Hausdorff continuous, there exists a neighbourhood U_0' of x_0 such that, for each $x \in U_0'$,

$$f(x_0) \cdot \mathbb{B} \subset f(x) \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B};$$

moreover, each χ_i is Hausdorff continuous, and therefore, for each $i = 1, \dots, p$, there exists a neighbourhood U_i' of x_0 such that, for each $x \in U_i'$, $\chi_i(x) \subset \chi_i(x_0) + \frac{\varepsilon}{8} \cdot \mathbb{B}$; now, put $U_2 = \bigcap_{i=0}^{p} U_i'$; for each $x \in U_2$, it results (cf. (2.8))

$$\begin{aligned} \bigcap_{i=1}^{p} \chi_i(x) &\subset \bigcap_{i=1}^{p} \chi_i(x_0) + \frac{\varepsilon}{8} \cdot \mathbb{B} \subseteq \\ &\subset \left(f(x_0) + \frac{\varepsilon}{4}\right) \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B} \subset \\ &\subset f(x) \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B} = \\ &= \left(f(x) + \frac{\varepsilon}{2}\right) \cdot \mathbb{B}. \end{aligned} \tag{2.10}$$

For each $i = 1, \dots, p$, the net $(\mathcal{J}_\alpha(\chi_i))_{\alpha \in I}$ converges to χ_i and therefore there exists $\iota_3 \in I$ such that, for each $\alpha \geq \iota_3$ and $x \in X$,

$$d(\mathcal{J}_\alpha(\chi_i(x)), \chi_i(x)) \leq \frac{\varepsilon}{8}$$

from which

$$\mathcal{J}_\alpha(\chi_i(x)) \subset \chi_i(x) + \frac{\varepsilon}{8} \cdot \mathbb{B}. \tag{2.11}$$

By (2.10) and (2.11), we obtain, for each $\alpha \geq \iota_3$ and $x \in U_2$,

$$\bigcap_{i=1}^{p} \mathcal{J}_\alpha(\chi_i(x)) \subset \bigcap_{i=1}^{p} \chi_i(x) + \frac{\varepsilon}{8} \cdot \mathbb{B} \subset f(x) \cdot \mathbb{B} + \frac{\varepsilon}{2} \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B}. \tag{2.12}$$

The condition (2.9) implies, for each $\alpha \in I$, $x \in X$ and $i = 1, \dots, m$, $\mathcal{J}_\alpha\left(\left(f + \frac{\varepsilon}{8}\right) \cdot \mathbb{B}\right)(x) \subset \mathcal{J}_\alpha(\chi_i)(x)$, that is $\mathcal{J}_\alpha(f \cdot \mathbb{B})(x) + \frac{\varepsilon}{8} \cdot \mathcal{J}_\alpha(\mathbb{B})(x) \subset \mathcal{J}_\alpha(\chi_i)(x)$ and therefore

$$\mathcal{J}_\alpha(f \cdot \mathbb{B})(x) + \frac{\varepsilon}{8} \cdot \mathcal{J}_\alpha(\mathbb{B})(x) \subset \bigcap_{i=1}^{p} \mathcal{J}_\alpha(\chi_i)(x); \tag{2.13}$$

by (2.6), for each $\alpha \geq \iota_2$, $\mathcal{J}_\alpha(\mathbb{B})(x) \subset \mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B}$, and hence (2.13) becomes

$$\mathcal{J}_\alpha(f \cdot \mathbb{B})(x) \subset \bigcap_{i=1}^{p} \mathcal{J}_\alpha(\chi_i)(x) + \frac{\varepsilon}{8}\left(\mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B}\right). \tag{2.14}$$

By putting together (2.12) and (2.14), we obtain, for each $x \in U_2$ and $\alpha \in I$ with $\alpha \geq \iota_2$, $\alpha \geq \iota_3$,

$$\begin{aligned}\mathcal{J}_\alpha(f \cdot \mathbb{B})(x) &\subset f(x) \cdot \mathbb{B} + \frac{\varepsilon}{2} \cdot \mathbb{B} + \frac{\varepsilon}{8} \cdot \mathbb{B} + \frac{\varepsilon}{8}\left(\mathbb{B} + \frac{\varepsilon}{4} \cdot \mathbb{B}\right) \\ &\subset f(x) \cdot \mathbb{B} + \varepsilon \cdot \mathbb{B}.\end{aligned} \tag{2.15}$$

Finally, put $U = U_1 \cap U_2$ and let $\iota \in I$ such that $\iota \geq \iota_1$, $\iota \geq \iota_2$, $\iota \geq \iota_3$; for each $\alpha \geq \iota$ and $x \in U$, we have (cf. (2.7) and (2.15))

$$d(\mathcal{J}_\alpha(f \cdot \mathbb{B})(x), f(x) \cdot \mathbb{B}) \leq \varepsilon.$$

Now, it is enough to apply a standard covering argument on the compact space X to establish the existence of $\lambda \in I$ such that, for each $\alpha \geq \lambda$, $d(\mathcal{J}_\alpha(f \cdot \mathbb{B}), f \cdot \mathbb{B}) \leq \varepsilon$; since $\varepsilon \in \mathbb{R}_+^*$ is arbitrary, this shows that the net $(\mathcal{J}_\alpha(f \cdot \mathbb{B}))_{\alpha \in I}$ converges to $f \cdot \mathbb{B}$.

Remark 2.3. We esplicitly observe that Theorem 2.2 generalizes Theorem 2.1; in fact, if we take H and $\mathcal{G}$ as in Theorem 2.1 then, for each $f \in H$, the function $(f + \varepsilon) \cdot \mathbb{B}$ belongs to $\mathcal{G}$ and consequently the conditions (2.1) and (2.2) are trivially statisfied for each $x_0 \in X$ and $\varepsilon \in \mathbb{R}_+^*$.

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