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RIESZ SPACE VALUED SUBMEASURES AND APPLICATION TO GROUP-VALUED FINITELY ADDITIVE MEASURES

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As a consequence of a general Domination Theorem given for a subadditive measure with values in a Riesz space, we prove the arcwise connectedness of the range of a L.C.V.T.S.-valued and of a group-valued finitely additive measure.

1. Introduction.

In 1973 Landers [7] proved that a group-valued measure dominated by a semiconvex real valued countably subadditive set function has arcwise connected range; this very property was later extended in [9] to the case of finitely additive measures (f.a.m.'s) dominated by semiconvex real-valued submeasures (i.e. subadditive set functions).

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The key idea of this paper is that a large class of group-valued f.a.m.'s admits a dominating submeasure ranging on a more general Riesz space, namely submeasures obtained by means of seminorms: therefore we can get the arcwise connectedness of the range of such f.a.m.'s mimicking the proof of the Domination Theorem of [9].

Most of the connectedness or of the arcwise connectedness results for the range of group-valued measures or f.a.m.'s base upon some properties of the ranges of their restriction over any measurable subset (briefly subranges); more precisely this kind of results is obtained under the assumption of compactness of subranges (as in [6], [9], [8]), or of their closedness ([9]) or their metrizability ([8]). We also quote in connection to this topic the results in [12] and ([11] - section 2). To obtain the existence of a vector-valued dominating submeasure we present here a property of subranges generalizing the idea of positive f.a.m.'s in the scalar case: it is shown, by means of an example , that this property, called quasi-monotonicity, is crucial to get the result even in a locally convex topological vector space.

Given the preliminary definitions in Section 2, in Section 3 we prove a general Domination Theorem; as an application in Section 4 we first obtain the arcwise connectedness of the range for a f.a.m. with values in a locally convex topological vector space (L.C.T.V.S.); furtherly, rearranging an idea due to Musial [10], this last result is extended to the case of f.a.m. ranging on a locally compact group.

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2. Preliminaries.

Let $(G, +, \tau)$ be any abelian topological group. By $U_G(0)$ we denote the family of the neighbourhoods of the neutral element $0 \in G$.

DEFINITION 2.1. *Given a measurable space* $(\Omega, \mathcal{A})$, *a set function* $m : \mathcal{A} \to G$ *is called a finitely additive measure (f.a.m.) if*

1) $m(\emptyset) = 0$ *(where* 0 *denotes the neutral element in* G*)*

2) for every $A, B \in \mathcal{A}$ *with* $A \cap B = \emptyset$, $m(A \cup B) = m(A) + m(B)$.

DEFINITION 2.2. *We will say that a f.a.m. is semiconvex provided for each* $A \in \mathcal{A}$ *there exists an* $\mathcal{A}$*-measurable subset of* A, *denoted by* $A_{1/2}$, *such that*

3) $m(A_{1/2}) + m(A_{1/2}) = m(A)$,

while m *will be said to be continuous if for every neighbourhood* $U \in U_G(0)$ *there exists a finite decomposition of* Ω, *say* $\{A_1, \ldots, A_n\}$ *such that* $m(A_i \cap E) \in U$, $i = 1, \ldots, n$ $E \in \mathcal{A}$.

DEFINITION 2.3. *A set function* $m : \mathcal{A} \to G$ *(not necessarily a finitely additive one) is said to be s-bounded iff any sequence* $\{A_n\}_n$ *in* A *of pairwise disjoint sets is such that* $\lim_{n\to\infty} m(A_n) = 0$.

It is known that semiconvexity and continuity are not equivalent (see [9] Examples 4.3, 4.4); nevertheless, when the group G does not have second order cyclic elements, any semiconvex s-bounded f.a.m. is necessarily continuous ([4]).

DEFINITION 2.4. *A set function* $m : \mathcal{A} \to G$ *will be said to be quasi-monotone if for each neighbourhood* $U \in U_G(0)$ *and for every set* $A \in \mathcal{A}$ *such that* $m(A) \in U$ *it follows* $m(B) \in U$ *for every* $B \subset A$, $B \in \mathcal{A}$.

Throughout this paper $(E, \mathbf{C}, \tau)$ will denote a topological Riesz space with cone $\mathbf{C}$.

DEFINITION 2.5. *A set function* $\nu : \mathcal{A} \to \mathbf{C}$ *will be called a submeasure if it satisfies (1) and:*

4) for every, $A, B \in \mathcal{A}$ *with* $A \cap B = \emptyset$, $\nu(A \cup B) \leq \nu(A) + \nu(B)$;

5) for every $A \in \mathcal{A}$, *and for every* $B \in \mathcal{A} \cap A$, $\nu(B) \leq \nu(A)$.

DEFINITION 2.6. *If G_1, G_2 are two topological groups, with neutral elements 0_1 and 0_2 respectively, we will say that a set function $\nu : \mathcal{A} \to G_2$ dominates a set function $m : \mathcal{A} \to G_1$, and we will write $m \ll \nu$, iff whenever $\nu(A_n) \to 0_2$ it follows $m(A_n) \to 0_1$.*

For any locally compact group G, we will denote by $G^\wedge$ the character group, and by φ any element of $G^\wedge$. $\mathbf{T}$ will denote the unitary thorus, and by $*$ and 1 we will denote the group multiplication and the neutral element of $\mathbf{T}$.

3. The Domination Theorem.

LEMMA 3.1. *If $\nu : \mathcal{A} \to \mathbf{C}$ is a submeasure, and if $U_E(0)$ is a neighbourhood basis for 0 in E then the family $U_V(A) = \{B \in \mathcal{A} : \nu(B\Delta A) \in V\}$, $V \in U_E(0)$ forms a neighbourhood basis of $A \in \mathcal{A}$.*

Proof. It is obvious that $\{U_V, V \in U_E(0)\}$ is non empty, that $A \in U_V(A)$ for every $A \in \mathcal{A}$ and $V \in U_E(0)$ and that for every $U_{V_1}(A)$, $U_{V_2}(A)$ there exists $U_W(A) \subset U_{V_1}(A) \cap U_{V_2}(A)$. We now want to show that for every $U_V(A)$ and for every $B \in U_V(A)$ there exists $U_W(B)$ such that $U_W(B) \subset U_V(A)$.

Set $v = \nu(B\Delta A)$; let $W \in U_E(0)$ be such that $v + W \subset V$.

Take $C \in U_W(B)$; then since

$$A\Delta C \subset (A\Delta B) \cup (B\Delta C) \quad \text{one as}$$

$$\nu(A\Delta C) \leq \nu(A\Delta B) + \nu(B\Delta C) \quad \text{and thus}$$

$$\nu(A\Delta C) \leq v + z \quad \text{where} \quad z \in W, \quad \text{i.e.}$$

$$\nu(A\Delta C) \in v + W \subset V$$

whence $C \in U_V(A)$ for every $C \in U_W(B)$.

From now on we will denote by τ_ν the topology induced on $\mathcal{A}$ from the quasi-monotone submeasure ν.

LEMMA 3.2. *Let* $\nu : \mathcal{A} \to \mathbf{C}$ *be a semiconvex submeasure; then for every* $A \in \mathcal{A}$ *there exists a family of* $\mathcal{A}$*-measurable subsets of* A, $\{A_t\}_{t\in[0,1]}$, *such that*

i) $A_0 = \emptyset,\ A_1 = A$;

ii) $\nu(A_t) = t\nu(A)$;

iii) if $t' < t$ *then* $A_{t'} \subset A_t$ *and* $\nu(A_t - A_{t'}) = (t - t')\nu(A)$.

Proof. The proof is exactly the same as that of Lemma 2.1 in [5] once observed that in any Riesz space if $(q_n)_n$ is a decreasing sequence of positive numbers and $z \in \mathbf{C}$ then

$$\inf\{q_n \cdot z, n \in \mathbb{N}\} = \{\inf_n q_n\} \cdot z.$$

LEMMA 3.3. *Let* $\nu : \mathcal{A} \to \mathbf{C}$ *be a submeasure such that for every* $A \in \mathcal{A}$ *there exists a family of* $\mathcal{A}$*-measurable subsets of* A, $\{A_t\}_{t\in[0,1]}$ *satisfying i, ii), iii). Then* $\mathcal{A}$ *equipped with the* τ_ν*-topology is arcwise connected.*

Proof. We will show that, for every $A, B \in \mathcal{A}$ fixed, the function

$$f : [0,1] \to (\mathcal{A}, \tau_\nu)$$

defined by

$$f(t) = (A - B)_{(1-t)} \cup (A \cap B) \cup (B - A)_t$$

is a continuous arc joining A and B, namely $\tau_\nu\text{-}\lim_{t\to t'} f(t) = f(t')$. Assume that $t > t'$. Then

$$f(t) - f(t') = [(A - B)_{(1-t)} \cup (A \cap B) \cup (B - A)_t] -$$

$$- [(A - B)_{(1-t')} \cup (A \cap B) \cup (B - A)_{t'}];$$

set

$$R = (A - B)_{(1-t)}, \quad T = (B - A)_t,$$

$$R' = (A - B)_{(1-t')}, \quad T' = (B - A)_{t'}, \quad S = (A \cap B);$$

it follows $R \subset R'$ and $T' \subset T$ and therefore

$$f(t) - f(t') = [R \cup S \cup T] - [R' \cup S \cup T'] = (R - R') \cup (T - T') = (T - T')$$

because $R - R' = \emptyset$. Hence

$$f(t) - f(t') = (B - A)_t - (B - A)_{t'}.$$

Analogously one finds

$$f(t') - f(t) = (A - B)_{(1-t)} - (A - B)_{(1-t')},$$

whence

$$\begin{aligned} \nu[f(t)\Delta f(t')] &\leq \nu[(B - A)_t - (B - A)_{t'}]+ \\ &+ \nu[(A - B)_{(1-t')} - (A - B)_{(1-t)}] = \\ &= (t - t')\nu(B - A) + (t - t')\nu(A - B). \end{aligned}$$

So for $t \to t'$ we have $\nu[f(t)\Delta f(t')] \to 0$ i.e. $f(t) \xrightarrow{\tau_\nu} f(t')$.

COROLLARY 3.4. *If $\nu : \mathcal{A} \to \mathbf{C}$ is a semiconvex submeasure then $(\mathcal{A}, \tau_\nu)$ is arcwise connected.*

THEOREM 3.5 (Domination Theorem) *Let $m : \mathcal{A} \to G$ be a f.a.m. and let $\nu : \mathcal{A} \to \mathbf{C}$ a semiconvex submeasure dominating m. Then $R(m)$ is arcwise connected.*

Proof. By means of Corollary 3.4 we just have to show that m is a τ_ν continuous map. Indeed if $A_n \xrightarrow{\tau_\nu} A$ then $\nu(A_n \Delta A) \to 0$ and as

$$\nu(A_n \Delta A) \geq \frac{1}{2}[\nu(A - A_n) + \nu(A_n - A)]$$

it follows that

$$\nu(A - A_n) \to 0 \quad \text{and} \quad \nu(A_n - A) \to 0.$$

As ν dominates m it is also true that

$$m(A - A_n) \to 0, \ m(A_n - A) \to 0.$$

From

$$m(A_n) - m(A) = m(A - A_n) + m(A_n - A)$$

we find

$$\lim_{n\to\infty} m(A_n) = m(A).$$

4. Application: arcwise connectedness of the range of vector and group-valued f.a.m.'s.

We are now going to examine a large class of f.a.m.'s which always admit a dominating submeasure.

THEOREM 4.1. *Let V be a L.C.T.V.S. and let $m : \mathcal{A} \to V$ be a semiconvex quasi-monotone f.a.m.. Then $R(m)$ is arcwise connected.*

Proof. Denote by $\mathcal{P}$ the family of admissible seminorms of V. The space $E = \mathbb{R}^{\mathcal{P}}$ with cone $\mathbf{C} = \{f \in E : f(p) \geq 0 \ \forall p \in \mathcal{P}\}$ and with the pointwise convergence topology is an order-complete topological Riesz space. Let $\sigma : \mathcal{A} \to \mathbf{C}$ be defined – for every $A \in \mathcal{A}$ – by $\sigma(A) : \mathcal{P} \to \mathbb{R}_0^+$, with $\sigma(A)(p) = (p \circ m)(A)$; we will show that σ is a submeasure dominating m. Indeed $(p \circ m)(\emptyset) = 0$ for each $p \in \mathcal{P}$, i.e. $\sigma(\emptyset) = 0$. Let $A, B \in \mathcal{A}$, $A \cap B = \emptyset$; then for all $p \in \mathcal{P}$

$$p[m(A \cup B)] = p[m(A) + m(B)] \leq (p \circ m)(A) + (p \circ m)(B)$$

whence

$$\sigma(A \cup B) \leq \sigma(A) + \sigma(B).$$

We are now going to prove the monotonicity of σ. Let A, $B \in \mathcal{A}$, with $A \subset B$ and let $p \in \mathcal{P}$; two possible cases may happen: either $(p \circ m)(B) = 0$ or $(p \circ m)(B) > 0$. In the first case we find

$$m(B) \in U_{p,\varepsilon} = \{x \in V : p(x) \leq \varepsilon\} \quad \text{for every} \quad \varepsilon > 0$$

and by quasi-monotonicity $m(A) \in U_{p,\varepsilon}$ for every $\varepsilon > 0$, i.e. $(p o m)(A) \leq \varepsilon$ for every $\varepsilon > 0$ whence $(p \circ m)(A) = 0$. In the second case let $\varepsilon_p = (p \circ m)(B)$. By the same quasi-monotonicity argument one finds

$$m(A) \in U_{p,\varepsilon_p}, \quad \text{i.e.} \quad (p \circ m)(A) \leq \varepsilon_p = (p \circ m)(B).$$

Thus for each $p \in \mathcal{P}$ we proved that $(p \circ m)(A) \leq (p \circ m)(B)$, and so $\sigma(A) \leq \sigma(B)$. Furthermore from the semiconvexity of m and from the properties of seminorms in L.C.T.V.S. we have for every $p \in \mathcal{P}$ and $A \in \mathcal{A}$

$$(p \circ m)(A_{1/2}) = (p \circ m)(A - A_{1/2}) = p\left(\frac{1}{2}m(A)\right) = \frac{1}{2}(p \circ m)(A)$$

i.e. $\sigma(A_{1/2}) = \sigma(A - A_{1/2}) = \frac{1}{2}\sigma(A)$. Finally σ dominates m. Indeed if $(A_n)_n$ is a sequence in $\mathcal{A}$ with $\sigma(A_n) \to 0$, the topology of E being the pointwise convergence one, we have

$$\sigma(A_n)(p) = (p \circ m)(A_n) \to 0 \quad \text{for all} \quad p \in \mathcal{P}.$$

As V has the weak topology induced by the seminorms it turns out that $m(A_n) \to 0$. Thus σ satisfies all the assumption of Theorem 3.5 and the assertion follows.

We now show that the quasi-monotonicity assumption is crucial in Theorem 4.1; namely we give an example of a semiconvex f.a.m. which is not quasi-monotone and such that the range is not arcwise connected.

EXAMPLE. Let $\Omega = [0, 1]$, $\mathcal{B}$ be the σ-algebra of Borel subset of Ω, $\lambda : \mathcal{B} \to [0, 1]$ be the usual Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{R}$ be any non continuous additive function, and take $m : \mathcal{B} \to \mathbb{R}^2$ as $m(B) = (\lambda(B), f(\lambda(B)))$. Then m is semiconvex: indeed, for each $A \in \mathcal{B}$ choose $B \subset A$, $B \in \mathcal{B}$ such that $\lambda(B) = \frac{1}{2}\lambda(A)$, i.e. $2\lambda(B) = \lambda(A)$; from the additivity of f one finds

$$2f(\lambda(B)) = f(\lambda(B)) + f(\lambda(B)) = f(\lambda(B) + \lambda(B)) = f(\lambda(A))$$

and so $m(B) = \frac{1}{2}m(A)$.

In [3] it is proved that for every $x \in \mathbb{R}$ f is unbounded on any neighbourhood of x; therefore m isn't quasi-monotone. As we have $R(m) = graph(f|_{[0,1]})$, and f is not Lebesgue-measurable, (see [3]), $R(m)$ cannot be arcwise connected.

LEMMA 4.2. *Let A and B be two topological groups and let $f: A \to B$ be a continuous morphism. Then $f(U_A(0)) = \{f(U), U \in U_A(0)\}$ is a neighbourhood basis for the neutral element $f(0) \in B$.*

The proof is straightforward.

LEMMA 4.3. *Let G be a locally compact group. If $m : \mathcal{A} \to G$ is a semiconvex quasi-monotone f.a.m. and φ is any element in the character group then $(\varphi \circ m) : \mathcal{A} \to \mathbf{T}$ is a semiconvex quasi-monotone f.a.m.. Thus the f.a.m. $\lambda : \mathcal{A} \to \mathbf{T}^{G^\wedge}$ defined – for every $A \in \mathcal{A}$ – by $\lambda(A) : G^\wedge \to \mathbf{T}$ with $\lambda(A)(\varphi) = (\varphi \circ m)(A)$ is semiconvex and quasi-monotone with respect to the pointwise convergence topology.*

Proof. Let $\varphi \in G^\wedge$ be fixed. Observe first that $(\varphi \circ m)(\emptyset) = 1$ and for $A, B \in \mathcal{A}$, $A \cap B = \emptyset$

$$(\varphi \circ m)(A \cup B) = \varphi[m(A) + m(B)] = (\varphi \circ m)(A) * (\varphi \circ m)(B)$$

by means of the linearity of φ. Furthermore, as m is quasi-monotone if $A \in \mathcal{A}$, for any $B \in A \cap \mathcal{A}$ it follows from Lemma 4.2 that $(\varphi \circ m)(B)$ lies in each neighbourhood of 1 in $\mathbf{T}$ containing $(\varphi \circ m)(A)$, i.e. $(\varphi \circ m)$ is quasi-monotone. From the semiconvexity of m for each $A \in \mathcal{A}$, $A_{1/2} \in A \cap \mathcal{A}$ exists such that

$$(\varphi \circ m)(A_{1/2}) * (\varphi \circ m)(A_{1/2}) = (\varphi \circ m)(A)$$

and so

$$(\varphi \circ m)(A_{1/2}) = [(\varphi \circ m)(A)]^{1/2}.$$

This shows that each $(\varphi \circ m)$ is semiconvex and that for any $A \in \mathcal{A}$ the «halving» subset is the same for every $\varphi \in G^\wedge$: thus λ is semiconvex itself. The other properties of λ being trivial the assertion follows.

THEOREM 4.4. *Let G be a locally compact group without second order cyclic elements. If $m : \mathcal{A} \to G$ is a semiconvex quasi-monotone, s-bounded f.a.m., then $R(m)$ is arcwise connected.*

Proof. Let $I = \{z \in \mathbf{T} : \mathrm{Re}(z) \in [0,1]\}$; for every $\varphi \in G^{\wedge}$ there exists $U^{(\varphi)} \in U_G(0)$ such that $\varphi(U^{(\varphi)}) \subset I$. From the semiconvexity and the s-boundedness the continuity of m follows ([4]). Hence there exists a finite decomposition $\{A_1, \dots, A_n\}$ of Ω such that $m(A_i \cap \mathcal{A}) \subset U^{(\varphi)}$ $i = 1, \dots, n$. Then $(\varphi \circ m)(A_i \cap \mathcal{A}) \subset I$, $i = 1, \dots, n$.

Set now $\mu_k(B) = (\varphi \circ m)(B \cap A_k)$, $B \in \mathcal{A}$; then $\mu_k(B) = e^{i\nu_k(B)}$ with $|\nu_k| \leq \dfrac{\pi}{2}$. We show now that each ν_k is a quasi-monotone, semiconvex f.a.m.. In fact $\nu_k(\emptyset) = 0$ as $(\varphi \circ m)(\emptyset \cap A_k) = 1$. Moreover if $B, C \in \mathcal{A}$, $B \cap C = \emptyset$ then from

$$e^{i\nu_k(B \cup C)} = \mu_k(B \cup C) = (\varphi \circ m)[(B \cup C) \cap A_k] =$$

$$(\varphi \circ m)[(B \cap A_k) \cup (C \cap A_k)] = (\varphi \circ m)(B \cap A_k) * (\varphi \circ m)(C \cap A_k) =$$

$$e^{i\nu_k(B)} \cdot e^{i\nu_k(C)} = e^{i[\nu_k(B)+\nu_k(C)]}$$

we get $\nu_k(B \cup C) = \nu_k(B) + \nu_k(C)\cdot$ ν_k is semiconvex because, for fixed $A \in \mathcal{A}$; from Lemma 4.3 there exists $A_{1/2} \subset A$, $A_{1/2} \in \mathcal{A}$ such that

$$(\varphi \circ m)(A_{1/2} \cap A_k) * (\varphi \circ m)(A_{1/2} \cap A_k) = (\varphi \circ m)(A \cap A_k),$$

$$\text{i.e.} \quad \mu_k(A_{1/2}) * \mu_k(A_{1/2}) = \mu_k(A) \quad \text{and so}$$

$$e^{2i\nu_k(A_{1/2})} = e^{i\nu_k(A)}, \quad \text{whence} \quad \nu_k(A_{1/2}) = \frac{1}{2}\nu_k(A).$$

The quasi monotonicity of ν_k follows immediately from that of $(\varphi \circ m)$ established in Lemma 4.3. Set now $\nu_\varphi = \sum_{k=1}^{n} \nu_k$: it is obvious that ν_φ is again a quasi-monotone semiconvex f.a.m.; furthermore as the A_k's are pairwise disjoint $(\varphi \circ m) = e^{i\nu_\varphi}$ and $\mathcal{N}(\varphi \circ m) = \mathcal{N}(\nu)$ (where $\mathcal{N}(\cdot)$ denotes the ideal of $\mathcal{A}$-measurable subset having $(\cdot)$

measure equal to neutral element of the group.) Consider now $E = \mathbb{R}^{G^\wedge}$ equipped with the pointwise order induced by the cone $\mathbf{C} = \{f : G^\wedge \to \mathbb{R}_0^+\}$ and with the pointwise convergence topology. Then it is possible to define the following f.a.m.: $\nu : \mathcal{A} \to E$, i.e. $\nu(A) : G^\wedge \to \mathbb{R}$ defined by $\nu(A)(\varphi) = \nu_\varphi(A)$. It is obvious that for the topology of E and from the same properties for ν_φ, ν is quasi monotone. Moreover, the «halving» sets for ν_φ being independent from the choice of $\varphi \in G^\wedge$, ν is semiconvex. We now prove that ν dominates m. Indeed if $(A_n)_n$ is a sequence in $\mathcal{A}$ such that $\nu(A_n) \to 0$ then for each $\varphi \in G^\wedge$ $(\varphi \circ m)(A_n) \to 1$. As G has the weak topology induced by $G^\wedge$ ([1]) it follows $m(A_n) \to 0$. We now consider the family $\mathcal{P}$ of admissible seminorms on the L.C.T.V.S. E; set $F = \mathbb{R}^{\mathcal{P}}$ and endow F with the pointwise order $\mathbf{K} = \{f : \mathcal{P} \to \mathbf{R}_0^+\}$ and with the pointwise convergence topology. Taking $\sigma : \mathcal{A} \to \mathbf{C}$ as in the proof of Theorem 4.1 we get a semiconvex submeasure dominating ν; then $m \ll \nu$, $\nu \ll \sigma$, implies $m \ll \sigma$, and therefore from Theorem 3.5 the assertion follows.

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