«LE MATEMATICHE»
Vol. XLII (1987) - Fasc. I-II, pp. 67-86

ON THE CONVERGENCE OF VARIATIONAL PROBLEMS WITH STOCHASTIC UNILATERAL OBSTACLES

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This paper presents a study about the Γ-convergence of a sequence of variational stochastic obstacle problems to a deterministic one.

The aim of this paper is to compute explicitely the $\Gamma(L^2(Q)^-)$ limit of the sequence of variational functionals defined by

$$J_h^\omega(u, A) = ||u||^2_{H^1(Q)} + \Phi_h^\omega(u, A)$$

where Φ_h^ω is a stochastic unilateral obstacle functional

$$\Phi_h^\omega(u, A) = \begin{cases} 0 & \text{if } \tilde{u}(x) \geq \varphi_h^\omega(x) \text{ cap a.e. on } A \\ +\infty & \text{otherwise} \end{cases}$$

Here

$Q =]0, 1[\times]0, 1[$; by various amount of technicalities it could be taken as any open subset of $\mathbb{R}^n$, even unbounded (see e.g. [2]); ω

(*) Entrato in Redazione il 16 novembre 1988

runs over some probabilized space Ω, with probability P; A is an open subset of Q; u runs over $H^1(Q)$; $\tilde{u}(x)$ is any quasi-continuous representative of u, e.g. (see [2])

$$\tilde{u}(x) = \liminf_{\rho \to 0}[\text{meas}\{y \in Q : |y-x| < \rho\}]^{-1} \int_{\{y \in Q : |y-x| < \rho\}} u(y)dy;$$

the capacity is defined, for any Borel subset of Q by

$$\text{cap } B = \inf\{||w||^2_{H^1(Q)}, w \in H^1(Q), w \text{ l.s.c.}, \ w \geq 1 \text{ on } B\};$$

a property is said to hold cap. almost everywhere (cap. a.e.) if the capacity of the set where it does not hold is zero. Finally $\varphi_h^\omega : Q \to \bar{\mathbb{R}}$, $\forall h \in \mathbb{N}$ $\forall \omega \in \Omega$, is the obstacle function, that will be specified later.

It is known (see [2]) that $J_h^\omega(u, A)$ has a unique minimum point on $H^1(Q)$, for any fixed $h \in \mathbb{N}$ and $\omega \in \Omega$, provided it is not indentically $+\infty$.

In the following we study the asymptotic behaviour of the minima of $J_h^\omega(u, A)$ for a sequence of stochastic obstacles that is closely related to the sequence studied in [1].

More precisely, denoting by u_h^ω the minimum point of $J_h^\omega(u, A)$ in $H^1(Q)$, it will be shown that there exists a «deterministic» $u \in H^1(A)$, i.e. independent of ω, such that

$$u_h^{\bar{\omega}} \to u \text{ in } L^2(Q)$$

for almost all fixed $\bar{\omega} \in \Omega$. Moreover u is the minimum point of a deterministic «limit» functional, in the sense of Γ-convergence.

The main theorem.

Let $Q =]0, 1[$. For each fixed $h \in \mathbb{N}$ divide Q into h^2 equal squares Q_h^i, $i = 1, 2, \dots, h^2$. Let B_h^i and x_h^i be the circle inscribed in Q_h^i and its center, respectively. Let S_h^i be the circle centered at x_h^i with radius e^{-h^2}. Set

$$\varphi_h^\omega(x) = \begin{cases} \alpha_h^i(\omega) & \text{if } x \in S_h^i \\ -\infty & \text{otherwise} \end{cases}$$

where $\alpha_h^i(\omega)$, $h \in \mathbb{N}$ $i = 1, 2, \ldots, h^2$, are independent random variables, having the same distribution function of some fixed random variable $\alpha(\omega)$, i.e.

$$P\{\omega \in \Omega : \alpha_h^\omega(\omega) > t\} = P\{\omega \in \Omega : \alpha(\omega) > t\}$$

for any $h \in \mathbb{N}$, any $i = 1, 2, \ldots, h^2$ and any $t \in \mathbb{R}$.

Finally let $a_+ = \frac{1}{2}(a + |a|)$ and $a_+^2 = (a_+)^2$ $\forall a \in \mathbb{R}$.

THEOREM. *Let* $(\varphi_h^\omega)_{h \in \mathbb{N}}$, $\omega \in \Omega$, *be defined as above and suppose that* $\int_\Omega \alpha^4(\omega) dP < +\infty$.

Then

$$\Gamma(L^2(Q)^- \lim_{\substack{h\to\infty \\ w\to u}} [||w||^2_{H^1(Q)} + \Phi_h^\omega(w, A)] = ||u||^2_{H^1(Q)} + 2\pi \int_A \int_\Omega (\alpha(\omega) - u(x))^2_+ dP dx$$

for any $u \in H^1(Q)$, *any open set* $A \subseteq Q$ *such that* $meas(\partial A) = 0$ *and almost all* $\omega \in \Omega$.

As it will be seen in Proposition 0.2, this result implies the stated convergence of minima.

The proof of this theorem requires some lemmas and preliminaries contained in the following section.

It must be noted that the stochastic Γ-convergence studied in this paper, i.e. the almost-everywhere-in-Ω-convergence, is not the only resonable one. For another relevant kind of stochastic convergence and its applications see, e.g., [3].

The author wishes to thank prof E. De Giorgi for his suggestions and comments.

0. Preliminaries.

In the following, the class of the open subsets of Q is denoted by $\mathcal{A}(Q)$. Furthermore the norms $||\cdot||_{H^1(A)}$ and $||\cdot||_{L^2(A)}$ will be denoted by $||\cdot||_{1,A}$ and $||\cdot||_{0,A}$.

Abstract Γ-convergence.

In separable metric spaces, Γ-convergence is characterized by the following proposition.

PROPOSITION 0.1. *Let X be a separable metric space. Let f_h, f be functionals from X to $\bar{R}$.*

Then

$$f(u) = \Gamma(X^-) \limsup_{\substack{h\to\infty \\ w\to u}} f_h(w)$$

if and only if

1) for every sequence (w_h) of X, such that $w_h \to u$ in X,

$$f(u) \le \limsup_{h\to\infty} f_h(w_h)$$

2) there exists a sequence (u_h) of X, such that

$$u_h \to u \text{ in } X \text{ and } f(u) = \limsup_{h\to\infty} f_h(u_h).$$

The characterization of the $\Gamma(X^-)\liminf\limits_{h\to\infty} f_h(w)$ obtained from the above one by replacing limsup by liminf.

Finally, if

$$\Gamma(X^-) \liminf_{\substack{h\to\infty \\ w\to u}} f_h(w) = \Gamma(X^-) \limsup_{\substack{h\to\infty \\ w\to u}} f_h(w)$$

their cammon value is denoted by $\Gamma(X^-) \lim\limits_{\substack{h\to\infty \\ w\to u}} f_h(w)$.

In this paper only $\Gamma(L^2(Q)^-)$ convergence will be used. Thus the symbols $\Gamma\lim$, $\Gamma\limsup$, $\Gamma\liminf$ will denote the corresponding limits in the $L^2(Q)$ topology.

There is an extensive literature regarding theory and applications of Γ-convergence. Here only the necessary definitions and properties will be quoted. For more information on this subject see [4].

Perhaps the most relevant property of Γ-convergence is its relationship with the convergence of minima and minimum points.

PROPOSITION 0.2. *Let* $f(u) = \Gamma(X^-) \lim_{\substack{h\to\infty \\ w\to u}} f_h(w)$. *Suppose also that* f_h *has a minimum point* u_h *in* X *and that there exists a fixed compact subset* K *of* X *such that* $u_h \in K\ \forall h \in \mathbb{N}$.

Then f *has a minimum points* $u, u_h \to u$ *in* X *and* $f_h(u_h) \to f(u)$.

A useful tool in computing Γ-limits is the following formula.

PROPOSITION 0.3. *Let* $X = L^2(Q)$. *Let* (f_h) *be a sequence of l.s.c. convex functionals. Suppose that* f_h *are equicoercive, i.e.*

$$\forall t \in \mathbb{R}\ \exists \mathcal{K}_t \subset X\ \mathcal{K}_t \text{ compact: } \{w \in X : f_h(w) < t\} \subseteq K_t\ \forall h \in \mathbb{N}.$$

Thus

$$\Gamma(X^-) \limsup_{\substack{h\to\infty \\ w\to u}} f_h(w) = \sup_{\lambda\in\mathbb{N}} \limsup_{h\to\infty} \min_{w\in X}[f_h(w) + \lambda||w-u||_{0,Q}].$$

Analogous formulae hold for Γ lim inf *and* Γ lim, *by replacing* lim sup *by* lim inf *and* lim, *respectively.*

It is not hard to prove that the last proposition remains valid if one replaces $||w-u||_{0,Q}$ by $||w-u||^2_{0,Q}$, or $\sup_{\lambda\in\mathbb{N}}$ by $\lim_{\lambda\to\infty}$.

Obstacles.

The definition of capacity already introduced is different from but equivalent, for our purpose, to the classical one. In fact the Borel subset of capacity zero are the same for both definitions.

The definition of obstacle given in the previous section gives rise to an $H^1(Q)$-l.s.c. functional, without any assumption of regularity on φ. Therefore the variational problem

$$\min_{w\in H^1(Q)} [||w||^2_{1,Q} + \Phi(w, A)]$$

has a unique solution for each given $A \in \mathcal{A}(Q)$. More information about obstacles is in [2].

A positive Borel measure μ will be said to belong to $H^{-1}(Q)$ if the linear functional $\varphi \to \int_Q \varphi d\mu$ is continuous in $C_0^\infty(Q)$ with respect to the $H^1(Q)$-topology. The basic property of these measures is that they are null on each set of capacity zero (see [2]).

Finally a class $\mathcal{B}$ of Borel subsets of Q will be said to be *rich* if for any family of Borel subsets of Q $(B_t)_{t\in\mathbb{R}}$, such that $t < s$ implies $\bar{B}_t \subseteq \overset{\circ}{B}_s$, $B_t \in \mathcal{B}$ holds for all $t \in \mathbb{R}$, except a set at most denumerable. As an example of a rich class may be considered the class of the Borel sets B such that $\mu(\partial\mathcal{B}) = 0$, where μ is a fixed Radon measure. Remark that denumerable intersection of rich classes is rich.

Moreover, a rich class $\mathcal{B}$ is dense, i.e. for any B_1, B_2, Borel subset of Q, such that $\bar{B}_1 \subseteq \overset{\circ}{B}_2$, there exists $B \in \mathcal{B}$ such that $\bar{B}_1 \subseteq \overset{\circ}{B} \subseteq \bar{B} \subseteq \overset{\circ}{B}_2$.

More information about rich classes and increasing set functions can be found in [2] and in the bibliography given there.

Two important results about Γlimit of obstacles are the following theorems (see [2]).

THEOREM 0.4. *Let (φ_h) be any sequence of obstacles.*

Let

$$F_+(u, A) = \Gamma \limsup_{\substack{h\to\infty \\ w\to 0}} [||w||^2_{1,Q} + \Phi_h(u + w, A)]$$

and

$$F_-(u, A) = \Gamma \liminf_{\substack{h\to\infty \\ w\to 0}} [||w||^2_{1,Q} + \Phi_h(u + w, A)].$$

Then

$$\Gamma \limsup_{\substack{h\to\infty \\ w\to u}} [||w||^2_{1,Q} + \Phi_h(w, A)] = ||u||^2_{1,Q} + F_+(u, A)$$

$$\Gamma \liminf_{\substack{h\to\infty \\ w\to u}} [||w||^2_{1,Q} + \Phi_h(w, A)] = ||u||^2_{1,Q} + F_-(u, A)$$

THEOREM 0.5. *(Integral representation of Γlimit).*

Let (φ_h) F_+, F_- *be as above. Moreover suppose that there exists* $M \in \mathbb{R}$ *such that*

$$(*) \qquad \min_{w \in H^1(Q)} [||w||^2_{1,Q} + \Phi_h(w, Q)] < M \quad \forall h \in \mathbb{N}.$$

Then, there exist

- *two positive Radon measures* μ *and* ν *such that* $\mu \in H^{-1}(Q)$
- *a Borel function* $f : Q \times \mathbb{R} \to \bar{R}$ *such that* $f(x, \cdot)$ *is l.s.c., positive, decreasing, convex, for* μ*-almost all* $x \in Q$.
- *a rich family* $\mathcal{A} \subseteq \mathcal{A}(Q)$

such that

$$F_+(u, A) = F_-(u, A) = \int_A f(x, \tilde{u}(x)) d\mu + \nu(A)$$

for any $u \in H^1(Q)$ *and any* $A \in \mathcal{A}$.

For the proof see [2]. For a complete characterization of Γ-limits of obstacles see [6].

By this theorem, computations done for constant u extend easily to any $u \in H^1(Q)$.

Finally the following two lemmas are needed.

LEMMA 0.6. (see [5]). *Let* $B \subset B'$ *be two concentric circles and let* v *be the solution of*

$$\begin{cases} -\Delta v + \lambda v = 0 & \text{on } B' \setminus B \\ v = 1 & \text{on } B \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial B' \end{cases}$$

Then

- $0 \leq v \leq 1$ *on* B';
- v *has spherical symmetry;*

- *decreases when the distance from the center increases.*

Remark that v, defined as above, is the minimum point in $H^1(B')$ of the functional $||u||^2_{1,B'} + \lambda||u||^2_{0,B'} + \Phi(u)$, where

$$\Phi(u) = \begin{cases} 0 & \text{if } u(x) \geq 1 \text{ a.e. on } B \\ +\infty & \text{otherwise} \end{cases}$$

Probability.

Recall that a probabilized space is defined by a set Ω and a measure P in Ω such that $P(\Omega) = 1$; P is defined on some σ-algebra of subsets of Ω, whose elements are called events.

A property will be said to hold almost surely (a.s.) if it holds for all $\omega \in \Omega$, except a set of probability zero.

A random variable is just a measurable function from Ω to $\mathbb{R}$. Given a random variable the expectation $E(\alpha)$ and the variance $V(\alpha)$ are defined by

$$E(\alpha) = \int_\Omega \alpha(\omega) dP$$

and

$$V(\alpha) = E((\alpha - E(\alpha))^2) = \int_\Omega (\alpha - E(\alpha))^2 dP.$$

Remark that the variance is finite if and only if $\alpha \in L^2_P(\Omega)$, i.e. $\int_\Omega \alpha^2 dP < +\infty$.

The following Lemma is known as Borel-Cantelli's Lemma; it provides a useful tool in avoiding «exceptional» situations, at least almost surely.

LEMMA 0.7. (Borel-Cantelli). *Let (A_n) be a sequence of events such that $\sum_{n=1}^{\infty} P(A_n) < +\infty$.*

Then, for P-almost all $\omega \in \Omega$, there are only finitely many $\bar{n} \in \mathbb{N}$ such that $\omega \in \bar{A}_n$.

In other words, for every «history» ω, except a set of probability zero, only a finite number of the events (A_n) actually occur.

The application of Borel-Cantelli's Lemma needs often an estimate of $P\{\omega \in \Omega : |\alpha(\omega) - E(\alpha)| > t\}$. It is supplied by the following inequalities.

LEMMA 0.8. *Let α be a random variables and suppose that $E(\alpha) < +\infty$.*

Then

$$P\{\omega \in \Omega : |\alpha| > t\} \leq \frac{E(\alpha)}{t}$$

moreover, if $V(\alpha) < +\infty$ then

$$P\{\omega \in \Omega : |\alpha(\omega) - E(\alpha)| > t\} \leq \frac{V(\alpha)}{t^2}.$$

Some book on probability theory refers to the former estimate as to «Markov's inequality» and to the latter as to «Čebyšev's inequality».

In order to state, in lemma 0.11, a suitable form of the so called «strong large number law», we give an important definition.

DEFINITION 0.9. *Two events $A, B \subseteq \Omega$ are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.*

Two random variables α, β are said to be independent if the events

$$\{\omega : s \leq \alpha(\omega) \leq t\} \text{ and } \{\omega : s' \leq \beta(\omega) \leq t'\}$$

are independent for every $s, t, s', t' \in \mathbb{R}$.

If α and β are independent it is not hard to verify that each event belonging to the σ-algebra generated by $\{\omega : s \leq \alpha(\omega) \leq t\}_{s,t \in \mathbb{R}}$ is independent from each other one belonging to the σ-algebra generated by $\{\omega : s \leq \beta(\omega) \leq t\}_{s,t \in \mathbb{R}}$.

One of the most important property of the independent random variables is the following formula.

PROPOSITION 0.10. *If α, β are independent random variables then*

$$V(\alpha+\beta) = V(\alpha) + V(\beta)$$

The following Lemma is an application of the previous results. It will be essential in the following.

LEMMA 0.11. *(Strong large number law). Suppose that, for any $h \in \mathbb{N}$, J_h is a subset of $\{1, 2, 3, \dots, h^2\}$ such that the number of its elements $\#J_h$ verifies $\lim_h h^{-2}(\#J_h) > 0$.*

Suppose also that $\alpha_h^i(\omega)$, $h \in \mathbb{N}$, $i = 1, 2, \dots, h^2$, are independent random variables having the same expectation $\bar{E}$ and the same variance $\bar{V} < +\infty$.

Then

$$\lim_{h\to\infty} (\#J_h)^{-1} \sum_{i \in J_h} \alpha_h^i(\omega) = \bar{E} \quad a.s.$$

Proof. Set $S_h(\omega) = \sum_{i \in J_n} \alpha_h^i(\omega)$ and fix $\varepsilon > 0$. By Cebysev inequality it follows that

$$P\left\{\omega : |\frac{S_h}{\#J_h} - \bar{E}| > \varepsilon\right\} \le \frac{\bar{V}(S_h)}{\varepsilon^2(\#J_h)^2}.$$

Since α_h^i are independent, by 0.10 $V(S_h) = (\#J_h)\bar{V}$ and therefore

$$P\left\{\omega : |\frac{S_h}{\#J_h} - \bar{E}| > \varepsilon\right\} \le \frac{\bar{V}}{\varepsilon^2}(\#J_h)^{-1}.$$

Since $\lim_{h\to\infty} h^{-2}(\#J_h) > 0$, it is possible to apply Borel-Cantelli's Lemma to the events $\left\{\omega : |\frac{S_h}{\#J_h} - \bar{E}| > \varepsilon\right\}$ and Lemma follows immediately.

Without any information about the variance of α_h^i, a weaker result holds.

LEMMA 0.12. *(Weak large number law). Let* (α_h^i) $h \in \mathbb{N}$, $i = 1, 2, \dots, h^2$, *a sequence of indipendent random variables, having the same distribution function of a fixed random variables* α.

Assume also that $E(\alpha) < +\infty$.

Then, for any $t \in \mathbb{R}^+$,

$$\lim_h P \left\{ \omega : |h^{-2} \sum_{i=1}^{h^2} (\alpha_h^i(\omega) - E(\alpha))| > t \right\} = 0$$

The proof can be obtained in the same way of the classical weak large number law (see, for instance, W. Feller: «An introduction to Probability Theory and its Applications» J. Wiley, New York, pag. 195).

1. Proof of the main theorem.

It will be used a modification of an argument already used in [5], in a different context.

It starts by proving that, for any $t \in \mathbb{R}$ and $A \in \mathcal{A}(Q)$ such that meas $(\partial A) = 0$

$$F_+^\omega(t, A) = F_-^\omega(t, A) = 2\pi \int_A \int_\Omega (\alpha(\omega) - t)_+^2 dP dx \quad \text{a.s.} \tag{1}$$

For, fix $t \in \mathbb{R}$ and $A \in \mathcal{A}(Q)$; let $\mathcal{J}_h$ denote the set of the indices i such that $Q_h^i \cap A \neq \emptyset$.

Assume, at first, the additional hypothesis

$$\lim_h h^{-2} \sum_{i=1}^{h^2} (\alpha_h^i(\omega) - t)_+^4 = E((\alpha(\omega) - t)_+^4) \quad \text{a.s.}$$

Moreover, let v_h be the solution of the problem

$$\begin{cases} \Delta w = 0 & \text{on } B_h \backslash S_h \\ w = 0 & \text{on } \partial B_h \\ w = 1 & \text{on } \bar{S}_h \end{cases}$$

where B_h and S_h are the open circles centered at the origin with radius $(2h)^{-1}$ and e^{-h^2}, respectively.

It is easy to determine explicitely v_h by using the logaritmic potential.

Set

$$w_h^\omega(x) = \begin{cases} [v_h(x - x_h^i)] \cdot (\alpha_h^i(\omega) - t)_+ & \text{if } i \in \mathcal{J}_h,\ x \in B_h^i \\ 0 & \text{otherwise} \end{cases}$$

and remark that

$$\Phi_h^\omega(t + w_h^\omega, A) = 0 \quad \forall h \in \mathbb{N} \quad \forall \omega \in \Omega.$$

In order to obtain estimates for $F_+^\omega(t, A)$, we verify that $w_h^\omega \to 0$ in $L^2(Q)$ a.s.. In fact

$$\begin{aligned} \|w_h^\omega\|_{0,Q} &= \sum_{i\in\mathcal{J}_h} \|v_h(x - x_h^i)\|_{0,B_h^i} \cdot (\alpha_h^i(\omega) - t)_+ = \\ &= \|v_h\|_{0,B_h} \sum_{i\in\mathcal{J}_h} (\alpha_h^i(\omega) - t)_+ \leq \\ &\leq \|v_h\|_{0,B_h} \cdot h^2 \left(h^{-2} \sum_{i=1}^{h^2} (\alpha_h^i(\omega) - t)_+ \right). \end{aligned}$$

A direct computation shows that $\lim_h \|v_h\|_{0,B_h} \cdot h^2 = 0$.

Therefore it will be enough to verify that

$$\lim_{h\to 0} h^{-2} \sum_{i=1}^{h^2} (\alpha_h^i(\omega) - t)_+ = E((\alpha(\omega) - t)_+) \quad \text{a.s.} \tag{2}$$

and

$$E((\alpha(\omega) - t)_+) < +\infty. \tag{3}$$

Infact, since (α_h^i), $h \in \mathbb{N}$ and $1, 2, \ldots, h^2$, are independent and $\tau \to (\tau - t)_+$ is a Borel function, it follows that $((\alpha_h^i(\omega) - t)_+)$, $h \in \mathbb{N}$ and $i = 1, 2, \ldots, h^2$, are independent.

Moreover, α_h^i have the same distribution of the fixed random variable α. Therefore, for any $s \geq 0$

$$P\{\omega : (\alpha_h^i(\omega) - t)_+ > s\} = P\{\omega : \alpha_h^i(\omega) > t + s\} =$$

$$= P\{\omega : \alpha(\omega) > t + s\} = P\{\omega : (\alpha(\omega) - t)_+ > s\}$$

and if $s < 0$ one has

$$P\{\omega : (\alpha_h^i(\omega) - t)_+ > s\} = 1 = P\{\omega : (\alpha(\omega) - t)_+ > s\}.$$

Hence the random variables $(\alpha_h^i(\omega)-t)_+$ have the same distribution function, and, a fortiori, the same expectation and the same variance of $(\alpha(\omega) - t)_+$.

By Lemma 0.11 it follows (2), provided $V((\alpha(\omega) - t)_+) < +\infty$. Remark that this condition, by Schwartz inequality, implies also (3). It is an immediate consequence of $V(\alpha) < +\infty$, and thus (2) and (3) are verified and $w_h^\omega \to 0$ in $L^2(Q)$ almost surely.

Now remark that $\lim_{h\to 0} h^{-2}||\nabla v_h||_{0,B_h} = 2\pi$. It can be proved by direct computation. Thus, by Proposition 0.1 it follows that

$$F_+^\omega(t, A) \leq \limsup_{h\to\infty} ||\nabla w_h^\omega||_{0,Q}^2 = \limsup_{h\to\infty} \left[||\nabla v_h||_{0,B_h}^2 \cdot \sum_{i\in\mathcal{J}_h} (\alpha_h^i(\omega) - t)_+^2 \right] \leq$$

$$\leq 2\pi \cdot \limsup_{h\to\infty} (\#\mathcal{J}_h) h^{-2} \cdot \limsup_{h\to\infty} (\#\mathcal{J}_h)^{-1} \sum_{i\in\mathcal{J}_h} (\alpha_h^i(\omega) - t)_+^2$$

almost surely.

Since $\lim_{h\to\infty} h^{-2}(\#\mathcal{J}_h) = m(\bar{A})$, by Lemma 0.11 $(\#\mathcal{J}_h)^{-1} \sum_{i\in\mathcal{J}_h} (\alpha_h^i(\omega) - t)_+^2)$ converges a.s to $E((\alpha(w) - t)_+^2)$, and finally one obtains

$$F_+^\omega(t, A) \leq 2\pi E(\alpha(\omega) - t)_+^2) \cdot m(\bar{A}). \tag{4}$$

The next step is to prove that

$$F_-^\omega(t, A) \geq 2\pi E(\alpha(\omega) - t)_+^2) \cdot m(A).$$

These inequalities are actually equivalent to (1), being $F_+^\omega \geq F_-^\omega$.

By Proposition 0.3, there exists a function $h(\lambda)$, $h : \mathbb{N} \to \mathbb{N}$, such that $h(\lambda) \geq \lambda$ and

$$F_-^\omega(t, A) = \lim_{\lambda\to 0} \min_{w\in H^1(Q)} [||\nabla w||_{0,Q}^2 + \lambda||w||_{0,Q}^2 + \Phi_{h(\lambda)}^\omega(t+w, A)].$$

Now, set $J_h = \{i \in \{1, 2, \ldots, h^2\} : Q_h^i \subseteq A\}$ and remark that, for any $\lambda, h \in \mathbb{N}$,

$$\min_{w\in H^1(Q)} [||\nabla w||_{0,Q}^2 + \lambda||w||_{0,Q}^2 + \Phi_h^\omega(t+w, A)] \geq$$

$$\geq \min_{w\in H^1\left(\bigcup_{i\in J_h} B_h^i\right)} \left[||\nabla w||^2_{0,\bigcup_{i\in J_h} B_h^i} + \lambda||w||^2_{0,\bigcup_{i\in J_h} B_h^i} + \Phi_h^\omega\left(t+w, \bigcup_{i\in J_h} B_h^i\right)\right].$$

Being B_h^i pairwise disjoint the minimum problem on $\bigcup_{i\in J_h} B_h^i$ can be splitted into the corresponding problems on each circle B_h^i, $i \in J_h$.

Then

$$F_-^\omega(t, A) \geq \lim_{\lambda\to\infty} \sum_{i\in J_{h(\lambda)}} \min_{H^1(B^i_{h(\lambda)})} [||\nabla w||^2_{0,B^i_{h(\lambda)}} + \lambda||w||^2_{0,B^i_{h(\lambda)}} + \Phi^\omega_{h(\lambda)}(t+w, B^i_{h(\lambda)})]. \tag{5}$$

Denote by M_i^λ and μ_i^λ the (stochastic) minimum and minimum point of the functional in the right hand side.

Remark that

$$\lambda||u_i^\lambda||^2_{0,B^i_{h(\lambda)}} \leq M_i^\lambda.$$

Since, by (4)

$$F_-^\omega(t, A) \leq F_+^\omega(t, A) < +\infty \quad \text{a.s.}$$

it follows by (5) that $\sum_{i\in J_{h(\lambda)}} M_i^\lambda$ is bounded from above, as λ tends to infinity and then

(6) $$\lim_{\lambda\to\infty} \sum_{i\in J_{h(\lambda)}} ||u_i^\lambda||^2_{0,B^i_{h(\lambda)}} = 0 \text{ a.s.}$$

Now, fix $\varepsilon > 0$ and let A^ε_λ be the union of the cubes $Q^i_{h(\lambda)}$ contained in A such that

$$||u_i^\lambda||^2_{0,B^i_{h(\lambda)}} \leq \varepsilon \text{ meas } (B^i_{h(\lambda)}).$$

By (6) it follows that

(7) $$\lim_{\lambda\to\infty} \text{meas } (A\backslash A^\varepsilon_\lambda) = 0. \quad \text{a.s.}$$

In order to obtain an estimate from below of the right hand side of (5), Dirichlet's principle will be used.

Set $c_i^\lambda = \inf_{x\in B^i_{h(\lambda)}} u_i^\lambda$ and remark that, by Lemma 0.6, c_i^λ is equal to the (constant) value of the trace of u_i^λ on $\partial B^i_{h(\lambda)}$. Thus

$$v_i^\lambda(x) \equiv c_i^\lambda + [(\alpha^i_{h(\lambda)}(\omega) - t)_+ - c_i^\lambda] v_{h(\lambda)}(x - x^i_{h(\lambda)})$$

is the harmonic function that agrees with u_i^λ on $\partial S^i_{h(\lambda)}$ and $\partial B^i_{h(\lambda)}$. Therefore, by Dirichlet's principle one has

$$||\nabla u_i^\lambda||^2_{0,B^i_{h(\lambda)}} \geq ||\nabla v_i^\lambda||^2_{0,B^i_{h(\lambda)}} \geq (\alpha^i_{h(\lambda)}(\omega) - t)^2_+ ||\nabla v_{h(\lambda)}||^2_{0,B^i_{h(\lambda)}} -$$
$$- 2c_i^\lambda (\alpha^i_{h(\lambda)}(\omega) - t)_+ ||\nabla v_{h(\lambda)}||^2_{0,B^i_{h(\lambda)}}.$$

By (5) and $\lim_{h\to\infty} h^2 ||\nabla v_h||^2_{0,B_h} = 2\pi$, one obtains

(8) $$F^\omega_-(t, A) \geq 2\pi \lim_\lambda \left[[h(\lambda)]^{-2} \cdot \sum_{i\in J_{h(\lambda)}} (\alpha^i_{h(\lambda)}(\omega) - t)^2_+ \right] - $$
$$- 4\pi \lim_\lambda \left[[h(\lambda)]^{-2} \cdot \sum_{i\in J_{h(\lambda)}} (\alpha^i_{h(\lambda)}(\omega) - t)_+ \right].$$

Now

$$\lim_{\lambda\to\infty}\left[[h(\lambda)]^{-2}\cdot\sum_{i\in J_{h(\lambda)}}(\alpha^i_{h(\lambda)}(\omega)-t)^2_+\right]=E((\alpha(\omega)-t)^2_+)\cdot\text{meas }(A).\text{ a.s.}$$

The last equality follows by Lemma 0.11 and by

$$\lim_{\lambda\to\infty}[h(\lambda)]^{-2}\cdot(\#J_{h(\lambda)})=\text{ meas}(A).$$

Thus, (1) follows immediately, provided the second term in the right hand side of (8) is zero. Let $L=[h(\lambda)]^{-2}\cdot\sum_{i\in J_h}c^\lambda_i(\alpha^i_{h(\lambda)}(\omega)-t)_+$.

Remark that

$$|L|\le\left|[h(\lambda)]^{-2}\cdot\sum_{i:B^i_{h(\lambda)}\subseteq A^\varepsilon_\lambda}c^\lambda_i(\alpha^i_{h(\lambda)}(\omega)-t)_+\right|+$$
$$+\left|[h(\lambda)]^{-2}\cdot\sum_{i:B^i_{h(\lambda)}\subseteq A\setminus A^\varepsilon_\lambda}k^\lambda_i(\alpha^i_{h(\lambda)}(\omega)-t)^2_+\right|$$

where $k^\lambda_i=c^\lambda_i/(\alpha^i_{h(\lambda)}(\omega)-t)_+$.

By Lemma 0.6, $0\le c^\lambda_i\le(\alpha^i_{h(\lambda)}(\omega)-t)_+$ and therefore $0\le k^\lambda_i\le 1$. Moreover being $c^\lambda_i=\inf_{B^i_{h(\lambda)}}u^\lambda_i$, from $B^i_{h(\lambda)}\subseteq A^\varepsilon_\lambda$ it follows that $0\le c^\lambda_i\le\varepsilon^{1/2}$. Thus

$$|L|\le[h(\lambda)]^{-2}\cdot\left[\varepsilon^{1/2}\sum_{i\in J_{h(\lambda)}}(\alpha^i_{h(\lambda)}(\omega)-t)_++\sum_{i:B^i_{h(\lambda)}\subseteq A\setminus A^\varepsilon_\lambda}(\alpha^i_{h(\lambda)}-t)^2_+\right]$$

Since $\lim_{\lambda\to\infty}h(\lambda)=+\infty$ it follows by Lemma 0.11 that

$$\lim_\lambda[h(\lambda)]^{-2}\cdot\sum_{i\in J_{h(\lambda)}}(\alpha^i_{h(\lambda)}(\omega)-t)_+=E((\alpha(\omega)-t)_+)\text{ a.s.}$$

Set $\chi^i_{h(\lambda)}(\omega) = 1$ if $B^i_{h(\lambda)} \subseteq A\backslash A^\varepsilon_\lambda$ and $\chi^i_{h(\lambda)} = 0$ otherwise. By Schwartz inequality in $\mathbb{R}^{h^2(\lambda)}$ it follows that

$$\left|[h(\lambda)]^{-2} \cdot \sum_{i:B^i_{h(\lambda)}\subseteq A\backslash A^\varepsilon_\lambda} (\alpha^i_{h(\lambda)}(\omega) - t)^2_+\right| =$$

$$= \left|[h(\lambda)]^{-2} \sum_{i\in J_{h(\lambda)}} \chi^i_{h(\lambda)}(\omega)(\alpha^i_{h(\lambda)}(\omega) - t)^2_+\right| \leq$$

$$\leq \left([h(\lambda)]^{-2} \sum_{i=1}^{h^2(\lambda)} (\alpha^i_{h(\lambda)}(\omega) - t)^4_+\right)^{1/2} \cdot ([h(\lambda)]^{-2}\#\{i : B^i_{h(\lambda)} \subseteq A\backslash A^\varepsilon_\lambda\})^{1/2}.$$

By (7) one has $\lim_\lambda [h(\lambda)]^{-2}\#\{i : B^i_{h(\lambda)} \subseteq A^\varepsilon_\lambda\} = 0$ a.s. and by the additional hypothesis $\lim_h \frac{1}{h^2} \sum_{i=1}^{h^2} (\alpha^i_h(\omega) - t)^4_+ < \infty$ a.s.

Hence it follows that

$$\lim_\lambda |L| \leq \varepsilon^{1/2} E(\alpha(\omega) - t)_+).$$

Being ε arbitrary, (1) is verified.

Now, in order to drop the additional hypothesis

$$(H) \qquad \lim_h h^{-2} \sum_{i=1}^{h^2} (\alpha^i_h(\omega) - t)^4_+ = E((\alpha(\omega) - t)^4_+) \text{ a.s.}$$

remark that any subsequence of the original sequence of obstacles has in turn a subsequence that verifies (H).

In fact, since $V(\alpha^2) < +\infty$, Lemma 0.12 implies that (H) holds in probability; hence it holds also almost surely, by taking a suitable subsequence.

Thus, the previous argument shows that any subsequence of the given sequence of functionals admits a subsequence Γ-converging to some limit, independing of the choice of the initial subsequence.

Since Γ-convergence in the class of the functionals greater than $||w||^2_{H^1}$ (more generally, in the class of $L^2(Q)$-equicoercive functionals) is induced by a metric, the whole sequence Γ-converges to the limit, and the theorem follows, at least for constant u.

Now, it is possible to accomplish the proof by using a suitable form of a standard argument using the integral representation theorem.

In fact, fix $\omega \in \Omega$ such that $F^\omega_+(0, Q) < +\infty$. Since the hypothesis (*) in theorem 0.5 is fullfilled there exist f^ω and μ^ω, ν^ω, $\mathcal{A}^\omega$ such that

$$F^\omega_+(u, A) = \int_A f^\omega(x, \tilde{u}(x))d\mu^\omega + \nu^\omega(A) \quad \forall u \in H^1(Q), \forall A \in \mathcal{A}^\omega, \ \mathcal{A}^\omega \text{ rich.}$$

By (1) it follows that

$$2\pi \int_A E((\alpha(\omega) - t)^2_+)dx = \int_A f^\omega(x, t)d\mu^\omega + \nu^\omega(A) \tag{9}$$

for every $t \in \mathbb{R}$, $A \in \mathcal{A}^\omega \cap \{B \subseteq Q : B$ Borel set, meas $(\partial B) = 0\}$.

Now, for each $t \in \mathbb{R}$ both sides of the last equality are finite Radon measures, with respect to the set variable A.

They coincide on a rich subset of $\mathcal{A}(\Omega)$. Since a rich subset is dense they actually coincide everywhere in $\mathcal{A}(\Omega)$.

Let $\mu^\omega = \bar{\mu}^\omega + \bar{\mu}^\omega_S$ and $\nu^\omega = \bar{\nu}^\omega + \bar{\nu}^\omega_S$ the decomposition into the absolutely continuous part and the singular part with respect to the Lebesgue measure of μ^ω and ν^ω, respectively. Moreover set

$$g^\omega(x) = \frac{d\bar{\mu}^\omega}{dx}, \ h^\omega(x) = \frac{d\bar{\nu}^\omega}{dx}.$$

Remark that from (9) it follows immediately that

$$\int_A f^\omega(x, t)d\bar{\mu}^\omega_S = \bar{\nu}^\omega_S(A) = 0 \text{ for all } A \in \mathcal{A}(\Omega)$$

and thus one obtains

$$f^\omega(x, t) = 0 \quad \bar{\mu}^\omega_S - \text{a.e. on supp } \bar{\mu}^\omega_S \tag{10}$$

and

$$2\pi \int_A E((\alpha(\omega) - t)_+^2)dx = \int_A [f^\omega(x,t)g^\omega(x) + h^\omega(x)]dx$$

for all $t \in \mathbb{R}$, for all $A \in \mathcal{A}(\Omega)$.

Therefore

$f^\omega(x,t)g^\omega(x) + h^\omega(x) = 2\pi E((\alpha(\omega) - t)_+^2)$, a.e. in Q, for any $t \in \mathbb{R}$.

Moreover, by (10), it follows that

$$\int_A f^\omega((x, u(\tilde{x}))d\bar{\mu}_S^\omega + \bar{\nu}_S^\omega(A) = 0 \quad \forall A \in \mathcal{A}(Q) \quad \forall u \in H^1(Q).$$

Then, since

$$F^\omega(u,A) = \int_A [F^\omega(x,u(x))g^\omega(x) + h^\omega(x)]dx + \int_A f^\omega(x,\tilde{u}(x))d\mu_s^\omega + \nu_S^\omega(A)$$

one obtains

$$F^\omega(u,A) = 2\pi \int_A \int_\Omega (\alpha(\omega) - u(x))_+^2 dP dx \tag{11}$$

for each $u \in H^1(Q)$ and each $A \in \bar{A}^\omega$, where

$$\bar{\mathcal{A}}^\omega = \mathcal{A}^\omega \cap \{A \subseteq Q,\ A \text{ open, meas } (\partial A) = 0\}.$$

Finally we prove (11) for all $A \subseteq Q$, A open and verifing meas $(\partial A) = 0$.

For, remark that if $A_1, A_2 \in \mathcal{A}(\Omega)$ and $A_1 \subseteq A_2$, the following inequalities hold

$$F_-^\omega(\cdot, A_1) \le F_-^\omega(\cdot, A_2) \tag{12}$$

$$F_+^\omega(\cdot, A_1) \le F_+^\omega(\cdot, A_2).$$

Now, fix any $A \in \mathcal{A}(Q)$ such that meas $(\partial A) = 0$. Let K, A' be a compact and an open subset of Q, respectively, such that

$$K \subseteq A \subseteq \bar{A} \subseteq A' \quad \text{meas } (A'/K) < \varepsilon.$$

Since $\bar{\mathscr{A}}^\omega$ is rich, it is also dense. Then there exist $A_1, A_2 \in \bar{\mathscr{A}}^\omega$ such that

$$K \subseteq A_1 \subseteq \bar{A}_1 \subseteq A \subseteq \bar{A} \subseteq A_2 \subseteq \bar{A}_2 \subseteq A'.$$

Obviously meas $(A_2 \backslash A_1) < \varepsilon$. By (12) one obtains

$$-2\pi \int_{A/A_1} \int_\Omega (\alpha - u)_+^2 dP dx \leq F_-^\omega(u, A) - 2\pi \int_A \int_\Omega (\alpha - u)_+^2 dP dx \leq$$
$$\leq F_+^\omega(u, A) - 2\pi \int_A \int_\Omega (\alpha - u)_+^2 dP dx \leq \int_{A_2/A} \int_\Omega (\alpha - u)_+^2 dP dx$$

for any $u \in H^1(Q)$.

By the absolute continuity of the integral, being $A \backslash A_1$ and $A_2 \backslash A$ subsets of $A_2 \backslash A_1$, it follows that

$$F_-^\omega(u, A) = 2\pi \int_A \int_\Omega (\alpha - u)_+^2 dP dx = F_+^\omega(u, A)$$

since ω is arbitrary in Ω, except for the set of probability 0 where $F_+^\omega(0, Q) = +\infty$, the proof is accomplished.

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