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CONSTRUCTIONS OVER LOCALIZATIONS OF RINGS

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In this paper we construct a category of effective noetherian rings in which linear equations can be «solved». This category is closed with respet to some important constructions like trascendental extensions, quotientations, finite products and localizations with respect to a large class of multiplicatively closed systems. Hence it gives a definition of «constructive» rings.

1. Introduction.

Motivated by the success of the concept of Gröbner bases to attack computational problems in the ideal theory for polynomial rings over a field, several authors have recently suggested different notions of effectivity over a ring powerful enough:

1) to guarantee ideal theoretic computations;

2) to be preserved from a ring A to its polynomial extensions $A[X_1, X_2, \dots, X_n]$;

3) to allow for a generalization of Buchberger algorithm for Gröbner

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bases to polynomial rings with coefficients in a ring satisfying such a notion.

One of the most general of these notions, however quite powerful to produce feasible algorithms in many situations, is the one proposed by Zacharias ([15]; cf. also [4] and [8]).

We recall that if A is an effective noetherian ring, A satisfies Zacharias' conditions if, roughly speaking, in A we can solve linear systems. In this paper we first consider the category Z whose objects are all effective noetherian rings which satisfy such conditions. The category Z results closed with respect to some important constructions like: trascendental extensions, quotientations, finite products, and it is possible to compute the first n-modules of a free resolution of an ideal of an object of Z.

A class of rings, which are significant in algebraic geometry applications, is obtained by localizing rings (usually quotients of polynomial rings over a field) at multiplicatively closed systems (usually either finitely generated systems, or complements of a prime ideal [12]). Clearly it is of interest to have effective ideal theory on such rings. However only partial results are known, generally without explicetely given proofs (cf. [14], [8], [4]).

We propose here a class of multiplicatively closed subsets of rings which seems to be sufficiently large to contain all the interesting examples known but which is sufficiently specific to guarantee that Zacharias' conditions are preserved under localizations. In such a way we can construct a subcategory A of Z which contains all the most common rings usually considered and which is closed with respect to localizations (as well as the previously mentioned operations). Hence it probably gives a good example of definition of «constructive» rings.

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2. Preliminary results.

Throughout the paper, a ring A is assumed to be Noetherian, commutative, with identity and «explicitly given», meaning that:

- we can represent (with a finite expression) every object of A;
- operations: $+, -, \cdot$ are constructive;
- given an element in A it is possible to decide whether it is 0 or not (and so, given two representations we can say if they give the same element).

We will say that an ideal I of A is «given» if we are given a finite set of generators of I.

DEFINITION 2.1. *We will say that A satisfies Zacharias' conditions if the following hold:*

i) given $a, a_1, a_2, \dots, a_m \in A$ it is possible to decide whether a is in the ideal $(a_1, a_2, \dots, a_m)$ and if so, find $b_1, b_2, \dots, b_m$ such that $a = \sum b_i a_i$; (that is A is detachable);

ii) given $a_1, a_2, \dots, a_m \in A$ it is possible to find a finite set of generators for the A-module $\{(b_1, b_2, \dots, b_m) \in A^m | a = \sum b_i a_i\}$; (that is we can determine the first module of syzygies of an ideal (Cf. [15]).

PROPOSITION 2.2. *The following conditions on A are equivalent:*

i) A satisfies Zacharias' conditions;

ii) If the following linear equation:

$$a_1x_1 + a_2x_2 + \dots + a_mx_m = a \quad with \quad a, a_1, a_2, \dots, a_m \in A$$

is given, then it is possible to determine all its solutions (or it is possible to see that there are none);

iii) A satisfies condition i) with $a = 1$ and condition ii) of definition 2.1.

Proof. i)$\Rightarrow$ii) by definition; i)$\Rightarrow$iii) obvious; iii)$\Rightarrow$i): let $a \in A$, let $B_1 := (\beta_{11}, \beta_{12}, \ldots, \beta_{1m}, \beta_{1m+1})$, $B_2 := (\beta_{21}, \beta_{22}, \ldots, \beta_{2m}, \beta_{2m+1}), \ldots, B_r := (\beta_{r1}, \beta_{r2}, \ldots, \beta_{rm}, \beta_{rm+1})$, a finite set of generators for the first module of *syzygies* of the ideal $(a_1, a_2, \ldots, a_m, -a)$. Then $a \in (a_1, a_2, \ldots, a_m)$ iff $a_1b_1 + a_2b_2 + \ldots + a_mb_m + (-a)1 = 0$ that is iff $(b_1, b_2, \ldots, b_m, 1) \in (B_1, B_2, \ldots, B_r)$, hence iff $1 \in (\beta_{1m+1}, \beta_{2m+1}, \ldots, \beta_{rm+1})$. So if we can determine $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum \lambda_i \beta_{im+1} = 1$, then we can determine $b_1, b_2, \ldots, b_m$ such that $a = a_1b_1 + a_2b_2 + \ldots + a_mb_m$.

We call Z the category whose objects A are all the commutative, Noetherian, explicitly given rings with identity, that satisfy Zacharias' conditions.

Remark 2.3. 1) If $A \in Z$ then we can define the concept of Gröbner basis for any ideal of $A[X_1, X_2, \ldots, X_n]$ (where $X_1, X_2, \ldots, X_n$ are indeterminates) and we can define an algorithm which constructs a Gröbner basis of any given ideal as it is shown for instance in [14], [15], [10].

2) If A is a field, it is shown in [3] and [11] that Gröbner basis for a module $U \subseteq A[X_1, X_2, \ldots, X_n]^r$ can also be defined. It is not difficult to see that in the same way we can give the concept of Gröbner basis for any module $U \subseteq A[X_1, X_2, \ldots, X_n]^r$ where A is in Z. It is also possible to construct an algorithm which gives a Gröbner basis from any system of generators of U. The details are very easy but cumbersome, so we omit them.

LEMMA 2.4. *Let $A \in Z$. If $I \subseteq A[X_1, X_2, \ldots, X_n]$ is a given ideal, where $X_1, X_2, \ldots, X_n$ are indeterminates, then it is possible to determine a system of generators for the ideal $I \cap A$.*

Proof. It is easy to see that if G is a Gröbner basis of I then $G \cap I$ is a set of generators of $I \cap A$ (See [4]; prop. 3.3).

PROPOSITION 2.5. *$A \in Z$, $f \in A$, and $I, J \subseteq A$ are given ideals, then we can determine a finite set of generators for the following ideals:*

i) $I \cap J$;

ii) $I + J$;

iii) $(0) : f$;

iv) $I : f$;

v) $I : J$;

and it is possible to decide if $I \subseteq J$ (See [4], corollary 3.2).

Proof. i) We have: $I \cap J = (tI, (1-t)J)A[t] \cap A$, where t is an indeterminate and so a system of generators for $I \cap J$ can be computed by lemma 2.4. An other proof is as follows: let $I := (f_1, f_2, \dots, f_m)$ and $J := (g_1, g_2, \dots, g_k)$. By hypothesis we can compute a system of generators of the module $\{(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_k) \in A^{m+k} | \sum u_i f_i + \sum v_i(-g_i) = 0\}$. Let's call it: $(c_{i1}, c_{i2}, \dots, c_{im}, d_{i1}, d_{i2}, \dots, d_{ik})$, $i = 1, \dots, s$. Then it is easy to see that $I \cap J$ is generated by $\sum_j c_{ij} g_j$, $i = 1, \dots, s$.

ii) trivial.

iii) $(0) : f = \{a \in A | af = 0\}$ and a set of generators for this ideal can be determined because of ii) of def. 2.1.

iv) $I : f = \{a \in A | af \in I\}$. Let $\{u_1, u_2, \dots, u_m\}$ be a set of generators of $I \cap (f)$, (it can be computed because of i)), then $u_i = f v_i$, for suitable $v_i \in A$ (v_i can be computed because of i) of def. 2.1). Then $I : f = (0) : f + (v_1, v_2, \dots, v_m)$.

v) If $J = (f_1, f_2, \dots, f_m)$, then $I : J = \bigcap_{i=1}^{m} (I : f_i)$.

LEMMA 2.6. *Let $A \in Z$, $r \in \mathbb{N}$, $M, N \subseteq A^r$ A-modules (given via a finite set of generators); then it is possible to compute a finite set of generators for the A-module $M \cap N$.*

Proof. We have: $M \cap N = [(tM, (1-t)N)A[t]^r] \cap A^r$, where t is an indeterminate. As we have remarked previously, we can compute a Gröbner basis of the module $(tM, (1-t)N)A[t]^r$. Then (as in prop. 2.5 i)) we can see that the elements of this Gröbner basis that are in A^r are a set of generators of $M \cap N$.

PROPOSITION 2.7. *Let $A \in Z$, $n \in \mathbb{N}$ then:*

i) given B, $B_1, B_2, \dots, B_m \in A^n$, it is possible to decide wether B is in the A-module $(B_1, B_2, \dots, B_m) \subseteq A^n$ and if so, find $c_1, c_2, \dots, c_m$ such that $B = \sum c_i B_i$;

ii) given $B_1, B_2, \dots, B_m \in A^n$, (with $B_i := (b_{i1}, b_{i2}, \dots, b_{in})$, $(i = 1, \dots, m)$, it is possible to find a finite set of generators for the A-module $M := \{(c_1, c_2, \dots, c_m) \in A^m | \sum c_i B_i = 0\}$.

Proof. ii) is a consequence of lemma 2.6 in fact:

$$M = \{C := (c_1, c_2, \dots, c_m) \in A^m | \sum c_i B_i = 0\} =$$

$$\{C = (c_1, c_2, \dots, c_m) \in A^m | (C|\beta_q) = 0, q = 1, \dots, n\} = \bigcap_{q=1}^{n} \{C \in A^m | (C|\beta_q) = 0\},$$

where

$$\beta_q := (b_{1q}, b_{2q}, \dots, b_{mq}).$$

i) is as follows:

$$B = c_1 B_1 + c_2 B_2 + \dots + c_m B_m \quad \text{iff} \quad c_1 B_1 + c_2 B_2 + \dots + c_m B_m + 1(-B) = 0.$$

If $D_i := (d_{i1}, d_{i2}, \dots, d_{im+1}) \in A^{m+1}$, $i = 1, \dots, r$ is a set of generators for the A-module $\{(d_1, d_2, \dots, d_{m+1}) \in A^{m+1} | d_1 B_1 + d_2 B_2 + \dots + d_m B_m + d_{m+1}(-B) = 0\}$ (which can be computed by ii) of this proposition) then $B \in (B_1, B_2, \dots, B_m)$ iff 1 is in the ideal $(d_{1m+1}; d_{2m+1}, \dots, d_{rm+1})$. So if we can determine $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\sum \lambda_i d_{im+1} = 1$, then we can determine $c_1, c_2, \dots, c_m$ such that $B = c_1 B_1 + c_2 B_2 + \dots + c_m B_m$.

COROLLARY 2.8. *Let $A \in Z$, $I \subseteq A$ a given ideal and $n \in \mathbb{N}$. Then it is possible to compute the first n modules of a free resolution of I.*

Proof. It is an obvious cosequence of ii) of the previous proposition.

Remark 2.9. a) Prop. 2.7 is enounced in [8], theorem 1.1.

b) Conditions i) and ii) of prop. 2.7 are considered in [9].

It is clear that Zacharias' conditions and the two conditions of prop. 2.7 are equivalent.

3. Properties of Z.

THEOREM 3.1. *If $A \in Z$ and if $X_1, X_2, \ldots, X_n$ are indeterminates, then $A[X_1, X_2, \ldots, X_n] \in Z$.*

Proof. See [14] or [15].

PROPOSITION 3.2. *Z is closed under quotientations, that is, if $A \in Z$, and $I \subseteq A$ is a given ideal, then $A/I \in Z$.*

Proof. If $A \in Z$, every element of A/I can be represented if we give an element of its equivalence class, so it is clear that $+, -, \cdot$ are constructive in A/I. If an element $[a] \in A/I$ is given, $[a] = 0$ in A/I iff $a \in I$, so we can also decide if an element of A/I is 0.

If $[a], [a_1], [a_2], \ldots, [a_s] \in A/I$, $[a] \in ([a_1], [a_2], \ldots, [a_s])$ in A/I iff $a \in (a_1, a_2, \ldots, a_s, f_1, f_2, \ldots, f_r)$ in A (where $I := (f_1, f_2, \ldots, f_r)$) so i) of def. 2.1 is satisfied.

Let $[a_1], [a_2], \ldots, [a_s] \in A/I$, and let $J := (a_1, a_2, \ldots, a_s) \subseteq A$. In the following commutative diagram we define the maps as follows:

$\alpha_1([u_1], [u_2], \ldots, [u_s]) := [a_1u_1 + a_2u_2 + \ldots + a_su_s]$;

$\alpha_2(c_1, c_2, \ldots, c_s, d_1, d_2, \ldots, d_r) := a_1c_1 + a_2c_2 + \ldots + a_sc_s + d_1f_1 + d_2f_2 + \ldots + d_rf_r$;

$\alpha_3(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_r) := a_1u_1 + a_2u_2 + \ldots + a_su_s + v_1f_1 + v_2f_2 + \ldots + v_rf_r$;

β_1 is the canonical projection;

$\beta_2(c_1, c_2, \ldots, c_s, d_1, d_2, \ldots, d_r) := ([c_1], [c_2], \ldots, [c_s])$;

γ_1 and γ_2 are the canonical immersion;

β_3 and γ_3 are defined by the universal property of the kernel (and so are the restriction of β_2 and γ_2 respectively).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \ker\alpha_1 & \to & (A/I)^s & \xrightarrow{\alpha_1} & (I+J)/I & \to & 0 \\
 & & \uparrow \beta_3 & & \uparrow \beta_2 & & \uparrow \beta_1 & & \\
0 & \to & \ker\alpha_2 & \to & A^s \oplus A^r & \xrightarrow{\alpha_2} & I+J & \to & 0 \\
 & & \uparrow \gamma_3 & & \uparrow \gamma_2 & & \uparrow \gamma_1 & & \\
0 & \to & \ker\alpha_3 & \to & I^s \oplus A^r & \xrightarrow{\alpha_3} & I & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In this diagram every row is exact and also the last two columns, and so, by 3×3 lemma ([13], 6.16), the first column is exact. Since β_3 is surjective, the image under β_3 of a system of generators of ker α_2 can be computed since $A \in Z$ and β_3, being the restriction of β_2, is computable. We can then obtain explicitly a system of generators of ker α_1 as an A-module and hence as an A/I-module, and so the second condition of def. 2.1 is satisfied.

Remark 3.3. From corollary 2.8 and from prop. 3.2 it follows that if $A \in Z$ and $I \subseteq A$ is a given ideal then it is possible to compute a free resolution for an ideal of A/I. We observe however that there is a faster way to compute such resolution, using the commutativity of the following diagram (which is a straightforward generalization of the previous one):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \ker\alpha_1 & \to & (A/I)^s & \xrightarrow{\alpha_1} & M & \to & 0 \\
 & & \uparrow \beta_3 & & \uparrow \beta_2 & & \uparrow \beta_1 & & \\
0 & \to & \ker\alpha_2 & \to & A^s \oplus A^{rt} & \xrightarrow{\alpha_2} & M' & \to & 0 \\
 & & \uparrow \gamma_3 & & \uparrow \gamma_2 & & \uparrow \gamma_1 & & \\
0 & \to & \ker\alpha_3 & \to & I^s \oplus A^{rt} & \xrightarrow{\alpha_3} & I^t & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$M := ([m_1], [m_2], \dots, [m_t]) \subseteq (A/I)^t$

($[m_i]$ indicates $([m_{i1}], [m_{i2}], \dots, [m_{it}) \in (A/I)^t$);

$M' := (m_1, m_2, ..., m_s, (f_1, 0, ..., 0), (f_2, 0, ..., 0), ..., (f_r, 0, ..., 0), (0, f_1, ..., 0),$
$(0, f_2, ..., 0), ..., (0, f_r, \dots, 0), \dots, (0, 0, \dots, f_1), \dots, (0, 0, \dots, f_r))$

(m_i indicates $(m_{i1}, m_{i2}, \dots, m_{it})$).

We now define the maps as follows:

$$\alpha_1([u_1], [u_2], \dots, [u_s]) := [u_1 m_1 + u_2 m_2 + \dots + u_s m_s];$$

$$\alpha_2(c_1, c_2, \dots, c_s, d_{11}, d_{12}, \dots, d_{1r}, \dots, d_{t1}, d_{t2}, \dots, d_{tr}) :=$$
$$= \sum_i c_i m_i + \sum_j d_{1j}(f_j, 0, \dots, 0) + \dots + \sum_j d_{tj}(0, \dots, 0, f_j);$$

$$\alpha_3(c_1, c_2, \dots, c_s, d_{11}, d_{12}, \dots, d_{1r}, \dots, d_{t1}, d_{t2}, \dots, d_{tr}) :=$$
$$= \sum_i c_i m_i + \sum_j d_{1j}(f_j, 0, \dots, 0) + \dots + \sum_j d_{tj}(0, \dots, 0, f_j).$$

$$\beta_1(m_i) := [m_i], i = 1, \dots, s; \beta_1(f_i, 0, \dots, 0) =$$
$$\dots = \beta_1(0, \dots, 0, f_i) = 0 \ i = 1, \dots, r;$$

$$\beta_2(c_1, c_2, \dots, c_s, d_{11}, d_{12}, \dots, d_{1r}, \dots, d_{t1}, d_{t2}, \dots, d_{tr}) := ([c_1], [c_2], \dots, [c_s]);$$

γ_1 and γ_2 inclusions.

As in the previous proposition, we can compute a system of generators of $\ker \alpha_2$ which gives via β_3 (or β_2) a system of generators of $\ker \alpha_1$.

PROPOSITION 3.4. *The category Z is closed for finite products.*

Proof. If $A, B \in Z$, then it is clear that $+, -, \cdot$ are constructive in $A \times B$ and that it is possible to decide if $(a, b) \in A \times B$ is 0.

i) of def 2.1 is trivial:

$$(a, b) \in ((a_1, b_1), \dots, (a_m, b_m)) \Leftrightarrow a \in (a_1, \dots, a_m), b \in (b_1, \dots, b_m).$$

ii) is as follows: if $(a_1, b_1), \dots, (a_m, b_m) \in A \times B$ then

$$M := \{((t_1, \tau_1), \dots, (t_m, \tau_m)) | \sum_{i=1}^{m} (t_i, \tau_i)(a_i, b_i) = 0\} =$$

$$= \{((t_1, \tau_1), \dots, (t_m, \tau_m)) | \sum_{i=1}^{m} t_i a_i = 0, \sum_{i=1}^{m} \tau_i b_i = 0\}.$$

Clearly the $A \times B$-module M is isomorph to the $A \times B$-module:

$$M' := \{(t_1, \dots, t_m, \tau_1, \dots, \tau_m) | \sum_{i=1}^{m} t_i a_i = 0, \sum_{i=1}^{m} \tau_i b_i = 0\}.$$

If $(t_{1q}, \dots, t_{mq})$, $q = 1, \dots, p$ is a system of generators for the A-module $\left\{ (t_1, \dots, t_m) | \sum_{i=1}^{m} t_i a_i = 0 \right\}$ and if $(\tau_{1k}, \dots, \tau_{mk})$ $k = 1, \dots, j$ is a system of generators for the B-module $\left\{ (\tau_1, \dots, \tau_m) | \sum_{i=1}^{m} \tau_i b_i = 0 \right\}$ then $(t_{11}, \dots, t_{m1}, 0, \dots, 0), \dots, (t_{1p}, \dots, t_{mp}, 0, \dots, 0)$, $(0, \dots, 0, \tau_{11}, \dots, \tau_{m1}), \dots,$ $(0, \dots, 0, \tau_{1j}, \dots, \tau_{mj})$ is a system of generators of M'.

We remark that by an effective PID we mean a ring A which is a PID and, moreover, such that given $a, b \in A$ we can:

- compute $d := MCD(a, b)$;
- compute $u, v \in A$ such that $d = au + bv$;
- if $c \in A$ decide if c is a multiple of d and in this case determine $k \in A$ such that $c = dk$.

PROPOSITION 3.5. *If A is an effective PID then $A \in Z$.*

Proof. (cf. [2], [4], [10]). Given $a, a_1, a_2, \dots, a_r$ we compute $b := GCD(a_1, a_2, \dots, a_r)$ and $b_1, b_2, \dots, b_r$ s.t. $\sum b_i a_i = b$. $a \in (a_1, a_2, \dots, a_r)$ iff $a = cb$, in which case $a = \sum c b_i a_i$.

By [10], prop. 3.6, it is possible to compute a basis $u_1, u_2, \dots, u_s$ of $(a_1, a_2, \dots, a_{r-1}) : a_r$; $u_i a_r \in (a_1, a_2, \dots, a_{r-1})$ and it is possible to compute b_{ij} s.t. $u_i a_r = \sum b_{ij} a_j$.

$\{(a_{i1}, a_{i2}, \dots, a_{ir-1}, u_i) | i = 1, \dots, s\}$ is then clearly a basis for the *syzygies* of $(a_1, a_2, \dots, a_r)$.

PROPOSITION 3.6. *Let $A \in Z$, $S \subseteq A$ a multiplicatively closed subset of A. Then if $I \subseteq A_S$ is a given ideal, we can compute the first n modules of a free resolution of I for every $n \in \mathbb{N}$.*

Proof. The only thing to verify is the following: given $u_1, u_2, ..., u_k \in (A_S)^r$, we can determine a system of generators for the A_S-module $M := \{(b_1, b_2, \ldots, b_k) \in (A_S)^k | \sum_{i=1}^{k} b_i u_i = 0\}$.

Let U be the matrix whose rows are the vectors $u_1, u_2, \ldots, u_k$ and let's call $U_1, U_2, \ldots, U_r$ its columns.

Then $M = \{(b_1, b_2, \ldots, b_k) \in (A_S)^k | (B|U_j) = 0, j = 1, \ldots, r\}$, where, as usual, $(..|..)$ is the scalar product. If $B \in (A_S)^k$ then we can compute $t_B \in S : t_B B \in A^k$ and $s_j \in S : s_j U_j \in A^k$, $j = 1, \ldots, r$. But $(B|U_j) = 0$ in A_S iff $(t_B B|s_j U_j) = 0$ in A_S iff there exists $v \in S$ such that $(v t_B B|s_j U_j) = 0$ in A.

Let's consider in A^k the A-module:

$$N := \{C := (c_1, c_2, \ldots, c_k) \in A^k | (C|s_j U_j) = 0, j = 1, \ldots, r\}.$$

If $C_1, C_2, \ldots, C_p$ is a system of generators for N (we can compute it by prop. 2.7), then $C_1, C_2, \ldots, C_p$ is a system of generators of M as an A_S-module: $(C_i|U_j) = \frac{1}{s_j}(C_i|s_j U_j) = 0$ hence $C_i \in M$; and if $B \in M$, then $t_B B \in A^k$ and $(t_B B|s_j U_j) = t_B s_j (B|U_j) = 0$ then $t_B B \in N$, so $t_B B$ is a linear combination, with coefficents in A of $C_1, C_2, \ldots, C_p$ therefore B is a linear combination, with coefficients in A_S of $C_1, C_2, \ldots, C_p$.

COROLLARY 3.7. *If A and S are as in prop. 3.6, then A_S satisfies condition ii) of def. 2.1.*

DEFINITION 3.8. *Let A be a ring and $S \subseteq A$ a multiplicatively closed subset. We wil call it admissible (a.m.c.s.) if*

$$S := S_1 + \alpha$$

where $\alpha \subseteq A$ is a given ideal and where S_1 is a multiplicatively closed subset of the following kinds:

– $S_1 := \langle s_1, s_2, \dots, s_m \rangle$ *(that is S_1 is finitely generated by $s_1, s_2, \dots, s_m \in A$).*

or

– $S_1 := C\bigcup_{i=1}^{n} \mathcal{P}_i$ *with $\mathcal{P}_i \subseteq A$ given prime ideals.*

Examples of a.m.c.s.: $S := 1 + \alpha$, $S := C\mathcal{P}$, $S :=$finitely generated.

PROPOSITION 3.9. *Let $A \in Z$, $S \subseteq A$ an a.m.c.s., then A_S is detachable (i.e. it satisfies i) of def. 2.1).*

Proof. We have $\frac{a}{s} \in \left(\frac{a_1}{s_1}, \frac{a_2}{s_2}, \dots, \frac{a_m}{s_m}\right)$ (where $a, a_1, a_2, \dots, a_m \in A$, $s_1, s_2, \dots, s_m \in S$) iff $a \in S(a_1, a_2, \dots, a_m)$ where $S(a_1, a_2, \dots, a_m) := \{a \in A |$ there exists $t \in S : ta \in (a_1, a_2, \dots, a_m)\}$ is the saturation of $(a_1, a_2, \dots, a_m)$ in A w.r.t. S.

Moreover, in case $\frac{a}{s} \in \left(\frac{a_1}{s_1}, \frac{a_2}{s_2}, \dots, \frac{a_m}{s_m}\right)$, it is clear that we can give a representation of $\frac{a}{s}$ of the king required in def. 2.1 i) if we can construct an element $t \in S$ such that $ta \in (a_1, a_2, \dots, a_m)$.

Case 1. Let $S := \langle s_1, s_2, \dots, s_m \rangle$, and let $I := (a_1, a_2, \dots, a_m)$ be a given ideal of A. In this case we have:

$$S(I) = (I, s_1T_1 - 1, \dots, s_mT_m - 1)A[T_1, T_2, \dots, T_m] \cap A$$

where $T_1, T_2, \dots, T_m$ are indeterminates. We can compute a Gröbner basis for the ideal $(I, s_1T_1 - 1, \dots, s_mT_m - 1)A[T_1, T_2, \dots, T_m]$ and from lemma 2.4 we can compute a system of generators of $S(I)$.

Moreover, if $a \in S(I)$ then:

$$a = vf(T_1, \dots, T_m) + (s_1T_1 - 1)g_1(T_1, \dots, T_m) + \dots$$

$$+(s_mT_m - 1)g_m(T_1, \dots, T_m), (v \in I)$$

and $a \cdot s_1^{q_1} \cdot s_2^{q_2} \cdot \dots \cdot s_m^{q_m} \in I$ where q_i are the maximum of the degrees of T_i in the polynomials $f, g_1, g_2, \dots, g_m$.

Case 2. let $S := C\bigcup_{i=1}^{n} \mathcal{P}_i$. Then if $a \in A$, $a \in S(I)$ iff there exists $s \notin \mathcal{P}_i$ for every i such that $sa \in I$ iff $(I : a) \not\subset \mathcal{P}_i$ for every i and this last condition can be verified using prop. 2.5. Moreover, if such an s exists, then it can be constructed by induction as follows: if $n = 1$, then we have only to find one of the generators of (I:a) that is not in $\mathcal{P}_1$.

If $n > 1$ and $(I : a) \not\subset \mathcal{P}_i$ for every i, then, by induction, we can construct, for every j, x_j s.t. $x_j \notin \mathcal{P}_i$, for $i \neq j$. If there exists j such that $x_j \notin \mathcal{P}_j$, then $s := x_j$ is the desired element.

Otherwise we put $s := \sum_i x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_n$ (cf. [G], prop. 1.11).

Case 3. $S := S_1 + \alpha$ then $a \in S(I)$ iff there exists $s_1 \in S_1$ and there exists $r \in \alpha$ such that $(s_1 + r)a \in I$ iff $s_1 a \in I + \alpha a$ iff $a \in S_1(I + \alpha a)$ and so we have reduced the problem to one of the two problems considered above. Suppose $\alpha := (u_1, u_2, \ldots, u_k)$; once we have constructed $s_1 \in S_1$ such that $s_1 a \in I + \alpha a$ then we can write $s_1 a = \sum b_j a_j + \sum c_j a u_j$, that is: $(s_1 - \sum c_j u_j)a \in I$ and $s_1 - \sum c_j u_j \in S$.

As an immediate consequence of prop. 3.9 and prop. 3.6 we have:

COROLLARY 3.10. *If $A \in Z$ and if $S \subseteq A$ is an a.m.c.s. then $A_S \in Z$.*

4. The category A.

Denote by $\mathbf{A}_0$ a class of domains which are contained in Z, and denote by $\mathbf{A}$ the smallest category that contains $\mathbf{A}_0$ and closed for the following operations:

1) trascendental extensions;

2) finite products;

3) quotientations;

4) localisations w.r.t. an a.m.c.s..

Therefore **A** satisfies:

1) $A \in \mathbf{A} \Rightarrow A[X] \in \mathbf{A}$ (X is an indeterminate);

2) $A \in \mathbf{A}$, $I \subseteq A$ a given ideal $\Rightarrow A/I \in \mathbf{A}$;

3) $A, B \in \mathbf{A} \Rightarrow A \times B \in \mathbf{A}$;

4) $A \in \mathbf{A}$, $S \subseteq A$ is an a.m.c.s. $\Rightarrow A_S \in \mathbf{A}$.

PROPOSITION 4.1. *The category* $\mathbf{A}$ *is a subcategory of* Z.

Proof. It follows immediately from teor. 3.1, prop. 3.4, prop. 3.2 and cor. 3.10.

Any object of **A** can be constructed from one (or more) objects of $\mathbf{A}_0$ using a finite number of times 1), 2), 3), and 4).

The following propositions show that some of the operations 1), 2), 3), and 4) commutes.

LEMMA 4.2. *Let* A, B *be rings,* $\alpha \subseteq A \times B$ *an ideal,* $\mathcal{P} \subseteq A \times B$ *a prime ideal,* $p : A \times B \to A$ *and* $q : A \times B \to B$ *the two projections. Then we have:*

1) $\alpha = p(\alpha) \times q(\alpha)$*;*

2) $\mathcal{P} = \mathcal{P}_1 \times B$ *(or* $\mathcal{P} = A \times \mathcal{P}_2$*) where* $\mathcal{P}_1$ *is a prime ideal of* A *(respectively* $\mathcal{P}_2$ *is a prime ideal of* B*).*

Proof. 1) is obvious. 2) $\mathcal{P} = p(\mathcal{P}) \times q(\mathcal{P})$ and $(0,0) = (1,0)(0,1) \in \mathcal{P}$ therefore $(1,0) \in \mathcal{P}$, for instance.

PROPOSITION 4.3. A, B *be rings,* $S \subseteq A \times B$ *an a.m.c.s.. Then we can construct an a.m.c.s.* $T \subseteq A$ *and an a.m.c.s.* $U \subseteq B$ *such that:*

$$(A \times B)_S \cong A_T \times B_U .$$

Proof. *Case* 1. $S = \langle s_1, s_2, \ldots, s_m \rangle$. Then $(A \times B)_S \cong A_T \times B_U$ where $T := \langle ps_1, ps_2, \ldots, ps_m \rangle$ and $U := \langle qs_1, qs_2, \ldots, qs_m \rangle$.

Case 2. $S = C \bigcup_{i=1}^{n} \mathcal{P}_i$, $\mathcal{P}_i \subseteq A \times B$ given prime ideals. From lemma 4.2 we know that $\mathcal{P}_i = \mathcal{R}_i \times B$ or $\mathcal{P}_i = A \times \mathcal{Q}_i$ ($\mathcal{R}_i$ and $\mathcal{Q}_i$ prime ideals). We can suppose that $\mathcal{P}_i = A \times \mathcal{Q}_i$ for $i = 1, \ldots, m$ and $\mathcal{P}_i = \mathcal{R}_i \times B$ for $i = m+1, \ldots, n$. Then we have: $C \bigcup_{i=1}^{n} \mathcal{P}_i = \left(C \bigcup_{i=m+1}^{n} \mathcal{R}_i \right) \times \left(C \bigcup_{i=1}^{n} \mathcal{Q}_i \right)$, $(A \times B)_S \cong A_T \times B_U$ where $T := C \bigcup_{i=m+1}^{n} \mathcal{R}_i$, $U := C \bigcup_{i=1}^{n} \mathcal{Q}_i$.

Case 3. $S = 1 + \alpha$, $\alpha \subseteq A \times B$ a given ideal. Then from lemma 4.2 we obtain: $S = (1 + p\alpha) \times (1 + q\alpha)$ and so the thesis follows with $T := 1 + p\alpha$ and $U := 1 + q\alpha$.

Case 4. $S = S_1 + \alpha$ with S_1 finitely generated or $S_1 = C \bigcup_{i=1}^{n} \mathcal{P}_i$. If we call β the extension of α in $(A \times B)_{S_1}$, we have:

$$(A \times B)_S \cong ((A \times B)_{S_1})_{1+\beta}.$$

PROPOSITION 4.4. *Let A be a ring, $I \subseteq A$ an ideal, $S \subseteq A/I$ an a.m.c.s. and $p : A \to A/I$ the quotient map. If we define $T := p^{-1}S$ we have:*

1) T is an a.m.c.s. of A;

2) $(A/I)_S \cong A_T/I_T$.

Proof. Let $S := S_1 + \alpha$ with $\alpha \subseteq A/I$ ideal then $p^{-1}S = p^{-1}S_1 + p^{-1}\alpha$ and if S_1 is finitely generated with $[s_1], [s_2], \ldots, [s_m]$, $s_i \in A$, the $p^{-1}S_1 = \langle s_1, s_2, \ldots, s_m \rangle + I$; if $S_1 = C \bigcup_{i=1}^{n} \mathcal{P}_i$ then $p^{-1}S_1 = C \bigcup_{i=1}^{n} p^{-1}\mathcal{P}_i$ and $p^{-1}\mathcal{P}_i$ are prime ideals of A.

Observe that the localizations w.r.t. an a.m.c.s. do not commute, in general, with the trascendental extensions, as it is showed by the following example:

EXAMPLE 4.5. Let $\mathcal{P} \subseteq \mathbb{Z}$, $\mathcal{P} := (0)$; $S := C\mathcal{P}$ is an a.m.c.s.. We will prove that $\mathbb{Z}_S[X]$ is not isomorphic to $\mathbb{Z}[X]_T$ for every T a.m.c.s. of $\mathbb{Z}[X]$.

Since $\mathbb{Z}_S = \mathbb{Q}$, it sufficies to prove that $\mathbb{Q}[X]$ is not isomorphic to $\mathbb{Z}[X]_T$ for every T.

Let's assume on the contrary that $\phi : \mathbb{Z}[X]_T \to \mathbb{Q}[X]$ is a ring isomorphism. It is clear that $\phi(z) = z$ for every $z \in \mathbb{Z}$. To get the contradiction it is enough to find a non costant polynomial $g \in \mathbb{Z}[X]$ invertible in $\mathbb{Z}[X]_T$. In fact in this case $\phi(g) = \dfrac{a}{b}(a, b \in \mathbb{Z})$, hence $\phi(bg) = a = \phi(a)$, therefore $bg = a$, a contradiction.

Case 1. Let $T := \langle s_1, s_2, \ldots, s_m \rangle + \alpha$ with $s_i \in \mathbb{Z}[X]$, $\alpha \subseteq \mathbb{Z}[X]$.

If there exists $s_i \notin \mathbb{Z}$, let $g := s_i$; if $s_i \in \mathbb{Z}$ for every $i = 1, \ldots, m$, but $\alpha \neq (0)$, let be $f \in \alpha$ non constant. In this case it is enough, for example, to put $g := f + s_1$. If $\alpha = (0)$ $\mathbb{Z}[X]_T = \mathbb{Z}_T[X]$, i.e. $\mathbb{Z}_T[X] \cong \mathbb{Q}[X]$ therefore $\mathbb{Z}_T \cong \mathbb{Q}$, a contradiction.

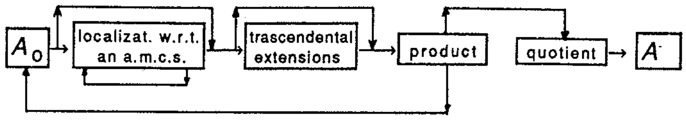
Case 2. $T := C\bigcup\limits_{i=1}^{n} \mathcal{P}_i + \alpha$ and $\alpha \neq (0)$. If $f \in \alpha$ is not constant, let $g := 1 + f \in T$.

Finally we have to consider the case $T := \bigcup\limits_{i=1}^{n} \mathcal{P}_i$. In this case either $X + r \in T$ for some $r \in \mathbb{Z}$, and then we define $g := X + r$, or $X + r \in \mathcal{P}_{i_r}$ for every r. In this last case there exist an $i \in \{1, 2, \ldots, n\}$ and $r, s \in \{1, 2, \ldots, n+1\}$ with $r \neq s$, such that $X + r$, $X + s \in \mathcal{P}_i$. Hence $r - s \in \mathcal{P}_i \cap \mathbb{Z}$. But $r - s$ must be invertible in $\mathbb{Z}[X]_T$ since $\phi(r - s)$ is such in $\mathbb{Q}[X]$, therefore there exist $a \in \mathbb{Z}[X]$, $b \in T$ such that $(r - s)a = b$, hence $(r - s)a \notin \mathcal{P}_i$, a contradiction.

As a consequence of the previous propositions and of the (obvious) fact that, if U, V, W are rings, $I \subseteq U$ is an ideal, and X is an indeterminate, then it is possible to construct the following isomorphisms:

$$(V \times W)[X] \cong V[X] \times W[X] \text{ and } U/I[X] \cong U[X]/IU[X],$$

we get that every object of $\mathbf{A}$ can be constructed as a quotient of a finite product of rings A_i, where every ring A_i is obtained starting from an object of $\mathbf{A}_0$ localizing w.r.t. an a.m.c.s. and/or adding indeterminates for a finite number of times as is showed by the following scheme:



Remark 4.6: 1) It is possible to take several different subclasses $\mathbf{A}_0$ of rings: for instance all the effective PID (considering in such a way every effective field).

2) If the ring of integers $\mathbb{Z}$ is in $\mathbf{A}_0$, then every finite field is in $\mathbf{A}$.

3) If $A \in \mathbf{A}$ (or, more generally, $A \in Z$) and if $f_1, f_2, \ldots, f_m \in A[X_1, X_2, \ldots, X_n]$ where $X_1, X_2, \ldots, X_n$ are indeterminates, then $A[f_1, f_2, \ldots, f_m] \in \mathbf{A}$ (or $A[f_1, f_2, \ldots, f_m] \in Z$). This is an immediate consequence of the following isomorphism:

$A[f_1, f_2, \ldots, f_m] \cong A[Y_1, Y_2, \ldots, Y_m]/J$ where $Y_1, Y_2, \ldots, Y_m$ are indeterminates, and $J := (Y_1 - f_1, \ldots, Y_m - f_m)A[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m] \cap A[Y_1, Y_2, \ldots, Y_m]$ (Cfr. [4], corollary 3.2).

5. A particular case.

Let $A := k[X_1, X_2, \ldots, X_n]$ and let $\mathcal{M} \subseteq A$ be a maximal ideal. Let's fix a Gröbner basis of $\mathcal{M}$ and let $r : A \to A$ be the reduction, that is $r(f)$ is the reduced of f w.r.t. the given Gröbner basis of $\mathcal{M}$.

Let $v_1, v_2, \ldots, v_m$ be a basis of $r(A)$ as a k-vector space (and so a basis of $A/\mathcal{M}$ as a k-vector space).

Let $S := C\mathcal{M}$ and let $\alpha \subseteq A$ be a given ideal ($\alpha \subseteq \mathcal{M}$ so to avoid trivial cases). We want to see how to determine if $a \in S(\alpha)$ in this particular case. This will allow us to find in some specific cases an algorithm in general faster then the one given in proposition 3.9.

PROPOSITION 5.1. *If $a \in A$, $a \in S(\alpha)$ iff there exists $s \in S$ such that $r(s)a \in (a\mathcal{M} + \alpha)$.*

Proof. If $a \in S(\alpha)$ then $sa \in \alpha$ with $s \in S$. $s - r(s) \in \mathcal{M}$ and so $r(s)a \in (a\mathcal{M} + \alpha)$.

Conversely: $r(s)a \in (a\mathcal{M} + \alpha)$ then $r(s)a = am + b$ with $m \in \mathcal{M}$, $b \in \alpha$, so $(r(s) - m)a \in \alpha$ and therefore $a \in S(\alpha)$, since obviously $r(s) - m \in S$.

Moreover we have: $r(s) = f_1v_1 + f_2v_2 + \ldots + f_mv_m$ with $f_i \in k$, hence $r(s)a \in (a\mathcal{M} + \alpha)$ iff there exist $f_1, f_2, \ldots, f_m \in k$ such that $(f_1v_1+f_2v_2+\ldots+f_mv_m)a \in (a\mathcal{M}+\alpha)$. We fix a Gröbner basis $g_1, g_2, \ldots, g_k$ of $a\mathcal{M}+\alpha$, and we reduce $r(s)a = (f_1v_1+f_2v_2+\ldots+f_mv_m)a$ w.r.t. this Gröbner basis; what we obtain is a polynomial $F(X_1, X_2, \ldots, X_n, f_1, f_2, \ldots, f_m)$ where $f_1, f_2, \ldots, f_m$ are all of degree at most 1. Then $r(s)a \in (a\mathcal{M}+\alpha)$ iff there exist $f_1, f_2, \ldots, f_m \in k$ such that $F(X_1, X_2, \ldots, X_n, f_1, f_2, \ldots, f_m)$ is the zero polynomial in $X_1, X_2, \ldots, X_n$. Hence we obtain a homogeneous linear system in $f_1, f_2, \ldots, f_m$ with coefficents in k.

Therefore $r(s)a \in (a\mathcal{M} + \alpha)$ iff this linear system has a non-trivial solution.

Let now $\mathcal{P} \subseteq A$ be a given prime ideal of dimension d. It is well known that there exist d indeterminates $X_{i_1}, X_{i_2}, \ldots, X_{i_d}$ such that $\mathcal{P} \cap k[X_{i_1}, X_{i_2}, \ldots, X_{i_d}] = (0)$ and $\mathcal{P} \cap k[X_{i_1}, X_{i_2}, \ldots, X_{i_d}, X_{i_j}] \neq (0)$ for $j > d$ (cfr. [6], Cap II, 1 satz 1). In [7] and [5] it is shown that to determine $X_{i_1}, X_{i_2}, \ldots, X_{i_d}$ we need only to compute the Gröbner basis of $\mathcal{P}$ with respect to the lexicographical order on the monomials.

Let $\mathcal{M} := \mathcal{P}B$ (where $B := k(X_{i_1}, X_{i_2}, \ldots, X_{i_d})\ [X_1, \ldots, X_{i_1-1}, X_{i_1+1}, \ldots, X_n]$).

Let $S := C\mathcal{P}$, and let $\alpha \subseteq A$ be a given ideal (contained in $\mathcal{P}$).

PROPOSITION 5.2. *if $a \in A$, then $a \in S(\alpha)$ iff $a \in S'(\beta)$ where $\beta := \alpha B$, $S'(\beta)$ is the saturated ideal of β w.r.t. $S' := B \backslash \mathcal{M}$, and a is considered as an element of B.*

Proof. $a \in S(\alpha)$ iff $\frac{a}{1} \in \alpha A_{\mathcal{P}}$. But $A_{\mathcal{P}} = B_{\mathcal{M}}$ $(\subseteq Q(A))$ then

$\frac{a}{1} \in \alpha A\mathcal{P}$ iff $\frac{a}{1} \in \beta B_{\mathcal{M}}$ iff $a \in S'(\beta)$.

From this proposition and the previous consideration we have:

PROPOSITION 5.3. *If* $a \in A$, $\mathcal{P} \subseteq A$ *is a prime ideal, and* $\alpha(\subseteq \mathcal{P})$ *is a given ideal, then we can establish if* $a \in S(\alpha)$ ($S := A\backslash\mathcal{P}$) *solving a suitable linear system in* $K := k(X_{i_1}, X_{i_2}, \ldots, X_{i_d})$.

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