«LE MATEMATICHE»
Vol. XLII (1987) - Fasc. I-II, pp. 151-162

SIMPLE WAVE SOLUTIONS FOR A RADIATION DOMINATED MAGNETOFLUID

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Simple waves are studied for a radiating magnetofluid in special relativity. For some of the Riemann invariants explicit expressions are given. The behaviour of these solutions is also studied numerically.

0. Introduction.

Relativistic Magnetofluiddynamics (R.M.F.D.) is of relevance for several areas of astrophysics (e.g. gravitational collapse and supernova explosions [1], neutron stars and pulsars [2]) and plasma physics (e.g. ionizing strong shocks [3], intense charged particle beams [4]).

Non linear waves in R.M.F.D. represent a fundamental chapter of the theory. In this article a particular class of non linear waves is investigated, i.e. that of simple waves, which, for quasilinear hyperbolic systems, are the nonlinear analog of plane waves.

We shall restrict the analysis to that of a radiation dominated magnetofluid, which is of considerable interest in astrophysics: the

(*) Entrato in Redazione il 16 gennaio 1989

case of a monoatomic relativistic gas (obeying Synge's equation of state) has already been treated in [5].

In sec. 1 we briefly review the equations of R.M.F.D.; in sec. 2 we introduce the basic quasilinear system describing R.M.F.D. in special relativity. In sec. 3 magnetoacoustic simple waves are treated and their behaviour is investigated in sec. 4. Also we notice how by using a suitable conformal transformation [6] we obtain from the above solutions, other solutions which represent waves in a spatially flat Robertson-Walker universe.

1. Test Relativistic Magnetofluiddynamics.

When we neglect the gravitational field of the fluid in comparison with the background gravitational field we deal with a «test fluid»: the equations describing an elettromagnetically interacting fluid are then [7]:

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{1.1}$$

$$\nabla_\alpha (\rho u^\alpha) = 0 \tag{1.2}$$

$$\nabla_\alpha F^{\alpha\beta} = J^\beta \tag{1.3}$$

$$\nabla_{[\gamma} F_{\alpha\beta]} = 0 \tag{1.4}$$

where $T^{\alpha\beta}$ is the total fluid and elettromagnetic field energy-momentum tensor, u^α the fluid 4-velocity, ρ is the rest mass density, $F^{\alpha\beta}$ is the electromagnetic field tensor, J^α is the current four-vector; the metric $g_{\alpha\beta}$ has signature +2 and the units have been chosen such that $c = 1$.

To equations (1.1), (1.2), (1.3), (1.4) we must add Ohm's law:

$$J^\alpha = \varepsilon u^\alpha + \sigma e^\alpha \tag{1.5}$$

where σ is the conductivity and ε the charge density of the fluid.

The electric and magnetic field with respect to a comoving observer are:

$$E_\alpha = F_{\alpha\beta} u^\beta \tag{1.6}$$

$$b_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} u^\beta F^{\gamma\delta} \tag{1.7}$$

with $b_\alpha u^\alpha = 0$, $E_\alpha u^\alpha = 0$.

In the magnetofluiddynamical approach $\sigma \to \infty$ (that happens for hot and dense plasmas [14]), and we have necessarily $E^\alpha = 0$ so that the elettromagnetic field reduces to the magnetic field with respect to the fluid; equation (1.3) defines J^α, and equation (1.4) yields [7]:

$$\nabla_\alpha (u^\alpha b^\beta - u^\beta b^\alpha) = 0. \tag{1.8}$$

For a perfect fluid (non dissipative), the total energy momentum tensor writes:

$$T^{\alpha\beta} = (e + p + |b|^2) u^\alpha u^\beta + (p + |b|^2/2) g^{\alpha\beta} - b^\alpha b^\beta$$

with e the total energy density, p the pressure and $|b|^2 = b^\alpha b_\alpha > 0$.

The fluid quantities ρ, p, e (all measured in the local rest frame) are related by the first law of thermodynamics:

$$\vartheta ds = d\left(\frac{e}{\rho}\right) + pd\left(\frac{1}{\rho}\right) \tag{1.9}$$

where s is the specific entropy and ϑ is the absolute temperature.

To the system (1.1), (1.2), (1.8) we must add an appropriate state equation: in the following we shall limit ourselves to the case of a radiation dominated gas, for which the state equation is

$$p = e/3. \tag{1.10}$$

2. Quasilinear systems and Simple wave solutions.

The equations of test R.M.F.D. (1.1), (1.2), (1.8) can be written in the form of a quasilinear system [8]:

$$\mathcal{A}_B^{\alpha A}(U^c)\nabla_\alpha U^B = 0 \quad {\scriptstyle \alpha=0,1,2,3 \atop A,B=0,1,\dots,9} \tag{2.1}$$

where the unknown field vector is:

$$U^A = (u^\alpha, b^\alpha, p, s)^T$$

and the matrix $\mathcal{A}_B^{\alpha A}$ is:

$$\mathcal{A}_B^{\alpha A} = \begin{vmatrix} Eu^\alpha \delta_\nu^\mu & , & -b^\alpha\delta_\nu^\mu + p^{\mu\alpha}b_\nu & , & m^{\nu\alpha} & , & 0^\mu \\ b^\alpha\delta_\nu^\mu & , & -u^\alpha\delta_\nu^\mu & , & f^{\alpha\mu} & , & 0^{\alpha\mu} \\ \eta\delta_\nu^\alpha & , & 0_\nu^\alpha & , & e_p' u^\alpha & , & 0^\alpha \\ 0_\nu^\alpha & , & 0_\nu^\alpha & , & 0^\alpha & , & u^\alpha \end{vmatrix}$$

where $0^{\mu\alpha}$, 0^α, indicate tensors and vectors with vanishing components respectively, $\eta = e + p$, $E = \eta + |b|^2$, $p^{\mu\alpha} = h^{\mu\alpha} + u^\mu u^\alpha$, $h^{\mu\alpha} = g^{\mu\alpha} + u^\mu u^\alpha$ $m^{\mu\alpha} = (\eta h^{\mu\alpha} - e_p'|b|^2 u^\mu u^\alpha + b^\mu b^\alpha)/\eta$, $f^{\mu\alpha} = (u^\mu b^\alpha e_p' - u^\alpha b^\mu)/\eta$, $e_p' = (\partial e/\partial)_s$, and the thermal gas sound speed (without elettromagnetic effects) is $c_s^2 = 1/e_p' \leq 1$.

We shall consider a 1-dimensional flow in the space-time of special relativity: then $g^{\alpha\beta}$ =diag$(-1, 1, 1, 1)$ in Minkowski coordinates. From Maxwell's equations we obtain the first integral of the motion

$$J_1 = u^0 b^1 - u^1 b^0 \tag{2.2}$$

which, in the non relativistic limit, coincides with a well known classical integral of motion [9].

For one-dimensional motion from the system (2.1) one obtains:

$$\mathcal{A}_B^{0A}\partial_t U^B + \mathcal{A}_B^{1A}\partial_x U^B = 0. \tag{2.3}$$

A class of relevant solutions for the system (2.3) is that of simple waves [10], which depend on only one independent variable such that:

$$U^A = U^A(\phi) \tag{2.4}$$

$$\phi = x - \lambda(U^A)t \tag{2.5}$$

Then from (2.1) it follows:

$$(\mathcal{A}_B^{1A} - \lambda \mathcal{A}_B^{0A})\frac{dU^B}{d\phi} = 0$$

and in order to have not trivial solutions

$$\begin{cases} \det(\mathcal{A}_B^{1A} - \lambda \mathcal{A}_B^{0A}) = 0 & (2.6) \\ \dfrac{dU^A}{d\phi} = \Pi^I(\phi)R_I^A & (2.7) \end{cases}$$

where $\{R_I\}$ is a basis of eigenvectors associated with λ and $\Pi^I(\phi)$ are proportionality factors. With this ansatz we calculate from (2.7), numerically or by an integration (in this case the solution is equivalent to determining $N-1$ first integrals called Riemann invariants), $U^A = U^A(\phi)$ and then (2.6) defines implicitly $\phi = \phi(x,t)$.

We observe that, at striking variance of the linear case, λ in general is not a constant, then the initial profile changes its shape while propagates: the simple wave regular solution develops into a shock wave.

In our problem equation (2.6) writes:

$$Ea^2A^2N_4 = 0$$

where $a = u^\alpha\phi_\alpha$, $\phi_\alpha = \partial_\alpha\phi$, $G = \phi^\alpha\phi_\alpha$, $B = b^\alpha\phi_\alpha$, $A = Ea^2 - B^2$ $N_4 = \eta(e'_p - 1)a^4 - (\eta + e'_p|b|^2)a^2G + B^2G$.

The solutions of $N_4 = 0$ correspond to magnetoacoustic waves, slow and fast; $A = 0$ correspond to Alfvén waves and $a = 0$ to material waves. The associated eigenvector are given in [8].

3. Magnetoacoustic simple waves.

In the following we shall limit ourselves to the magnetoacoustic case (e.g. we shall consider the solutions of $N_4 = 0$) and then we must exclude the case in which a material wave or an Alfvén wave coincide with a magnetoacoustic one: this implies

$$\left\{ \begin{array}{l} B \neq 0 \\ \\ \Lambda = \eta a^2 - G|b|^2 \neq 0 \end{array} \right. \tag{3.1}$$

These conditions will be imposed during the integration. Under the hypothesis (3.1) it is shown in [8] that $N_4 = 0$ admits 4 real and distinct roots for λ (with $|\lambda| \leq 1$): the corresponding eigenvectors are:

$$R^A = \left| \begin{array}{c} d^\alpha \\ d^{\alpha+3} \\ Ea^2 A \\ 0 \end{array} \right|$$

$$d^\alpha = Ea^2(Bf^\alpha - am^\alpha) + Ea(B^2 - e'_p|b|^2a^2)(\phi^\alpha + 2au^\alpha)/\eta$$

$$d^{\alpha+3} = d^\alpha B/a + EAaf^\alpha$$

$$f^\alpha = f^{\alpha\mu}\phi_\mu, \; m^\alpha = m^{\alpha\mu}\phi_\mu$$

If we take as independent variable $\phi = p$, then equations (2.7) write explicitly:

$$\frac{d(\Gamma v_x)}{dp} = \alpha_1 \Gamma v_x + \alpha_1/a + \alpha_2 b_x \tag{3.2)a}$$

$$\frac{d(\Gamma v_y)}{dp} = \alpha_1 \Gamma v_y + \alpha_2 b_y \tag{3.2)b}$$

$$\frac{d(\Gamma v_z)}{dp} = \alpha_1 \Gamma v_z + \alpha_2 b_z \tag{3.2)c}$$

(3.2)d
$$\frac{db_x}{dp} = \beta_1 \Gamma v_x + \beta_2 b_x - (e'_p - 1) B a^2 / GA$$

(3.2)e
$$\frac{db_y}{dp} = \beta_1 \Gamma v_y + \beta_2 b_y$$

(3.2)f
$$\frac{db_z}{dp} = \beta_1 \Gamma v_z + \beta_2 b_z$$

(3.2)g
$$\frac{ds}{dp} = 0$$

where

$$\alpha_1 = -a^4 (e'_p - 1)/AG, \ \alpha_2 = aB(e'_p - 1)\eta A$$
$$\beta_1 = B(\alpha_1 - 1/\eta)/a, \ \beta_2 = e'_p/\eta + B\alpha_2/a$$

then we have the obvious integral

(3.3)
$$J_0 = s = \text{const.}$$

Let us obtain some useful integrals: from equations (3.2) follow

(3.4)
$$\frac{d}{dp}[\Gamma(v_x b_z - v_z b_y)] = (e'_p - 1)\frac{a^2}{A}[\Gamma(v_x b_z - v_z b_y)]$$

Now we suppose that such waves will propagate into a constant state: then at the stagnation point p_0

$$[\Gamma(v_x b_z - v_z b_y)]_{p_0} = 0$$

hence, by the uniqueness theorem for ordinary differential equation, (3.4) gives the following invariant:

(3.5)
$$J_2 = v_x b_z - v_z b_y = 0$$

Similarly, it can be shown that

$$\frac{d}{dp}\frac{by}{bz} = \frac{\beta_1 \Gamma}{bz}(vy - byvz/bz)$$

whence another invariant

$$J_s = by/bz \tag{3.6}$$

Notwithstanding the knowledge of these invariants, the equations (3.2) can be integrated analytically only in particular cases [11] and, in general, a numerical integration is needed.

4. Numerical Integration and qualitative behaviour.

Now we have performed a numerical integration of the system (3.2) in the case:

$$v_y = b_y = 0, \ p = e/3$$

It is convenient to write our system in dimensionless form by introducing the variables:

$$\bar{p} = \frac{p}{p_0}, \ \bar{b}x = \frac{bx}{J_1}, \ \bar{b}z = \frac{bz}{J_1}, \ \bar{B} = \frac{B}{J_1}, \ \bar{E} = \frac{E}{p_0}, \ \bar{A} = \bar{E}a^2 - L\bar{B}^2$$

where p_0 is the initial pressure and $L = J_1^2/p_0$ is a non dimensional parameter. Therefore the results presented in this way are completely independent of the scale and have a wide range of applications.

Then the system (3.2) reduces (we omit the bar) to:

$$\frac{dv_x}{dp} = -\frac{2a^3}{AG\Gamma}(1 - \lambda v_x) + \frac{aB}{2Ap}\frac{2}{\Gamma^2} \tag{4.1)a}$$

$$\frac{dv_z}{dp} = \frac{2a^3\lambda v_z}{AG\Gamma} + \frac{aBL}{2p\Gamma A}(b_z - b^0 v_z) \tag{4.1)b}$$

$$\frac{db_z}{dp} = -\frac{B\Gamma v_z}{a}\left(\frac{2a^4}{AG} + \frac{1}{4p}\right) + \frac{b_z}{4p}\left(1 + \frac{2Ea^2}{A}\right) \tag{4.1)c}$$

together with an algebraic equation for λ:

$$N_4 = \alpha_4\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0 \tag{4.2}$$

where the coefficient α_i depend on

$$\alpha_i = \alpha_i(p, v_x, v_z, b_z).$$

Moreover the system (4.1), (4.2) must satisfy during the integration the conditions (3.1).

We start our integration assigning the initial data

$$v_x(p_0) = v_z(p_0) = 0, \quad b_z(p_0) = b_{z_0}.$$

In this case, initially, $N_4 = 0$ reduces to a biquadratic equation which admits progressive ($\lambda > 0$) and regressive ($\lambda < 0$) fast and slow magnetoacoustic waves.

Then we choose one root to be followed during the integration: a suitable algorithm ensures that we are always following the same root. In figures 1,2,3 we plot the results obtained for the progressive fast wave ($b_z(p_0) = 0.2$), for various values of the parameter L. We observe that the component v_z of the velocity is a non monotone functions of p, at striking variance with the non relativistic results.

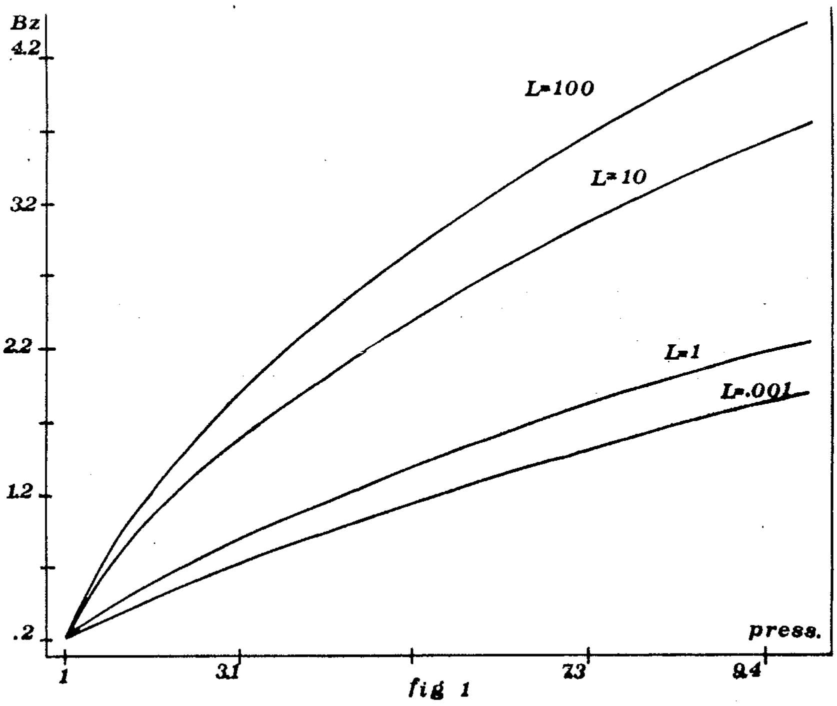

Fig. 1 - Pressure versus b_z for several values of L.

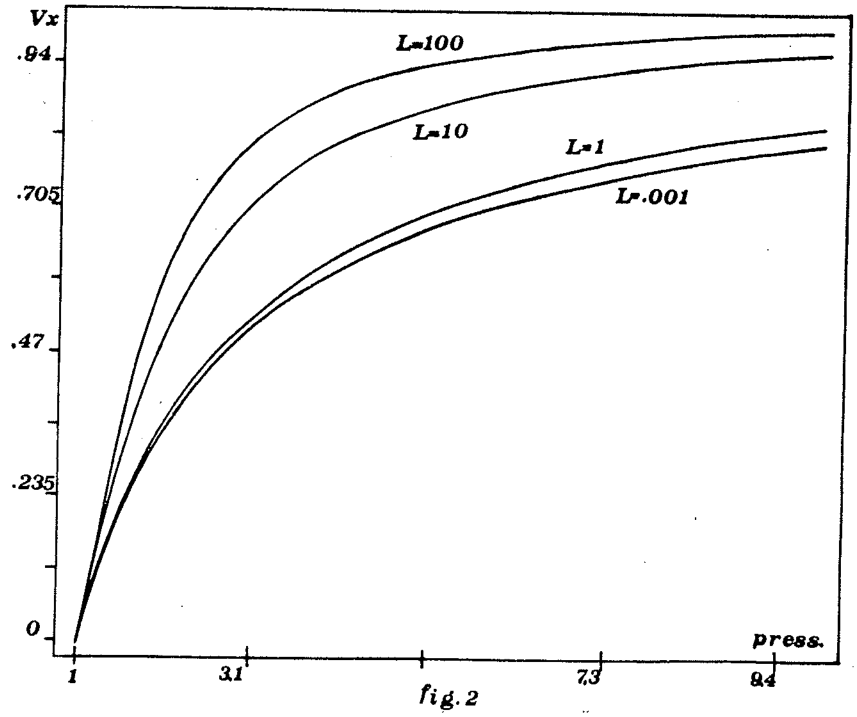

Fig. 2 - Pressure versus v_x for several values of L.

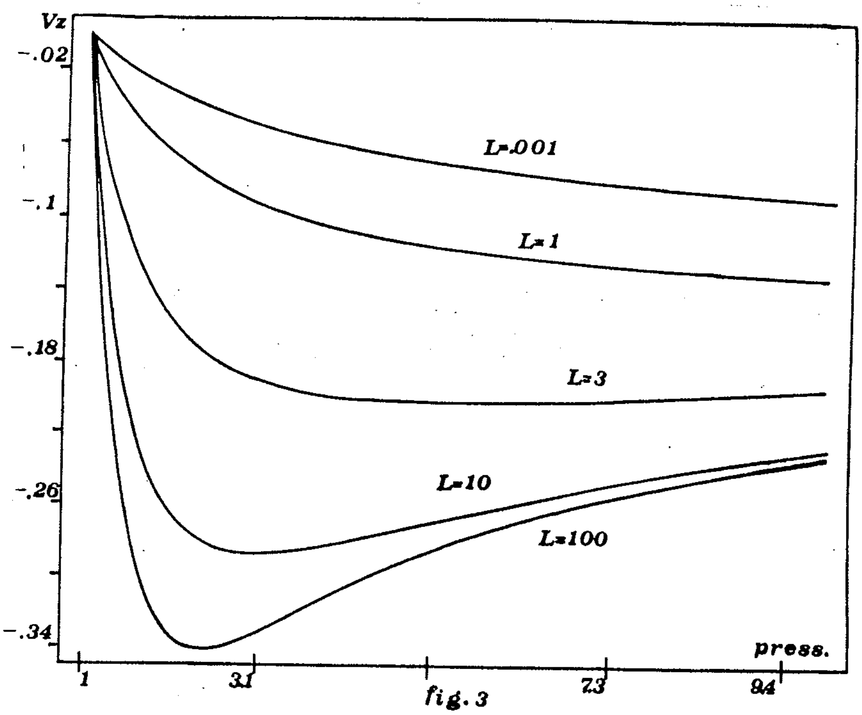

Fig. 3 - Pressure versus v_z for several values of L.

This phenomenon can be explained as follows: in a relativistic framework one has the limitations:

$$v_x^2 + v_z^2 \leq 1$$

therefore if v_x increases, the other component cannot be a monotone function of p.

This result could have interesting consequences for the solution of the relativistic magnetic piston problem [12], where the solution is constructed by patching together simple waves [13].

We notice that the above solutions could be extended to R.M.F.D. in a curved background, in the case of conformally flat spacetimes [6].

In particular, for the case of a spatially flat Robertson - Walker universe, with the metric:

$$ds^2 = a^2(t)(-dt^2 + dx^2 + dy^2 + dz^2) \tag{4.3}$$

with the trasformation (u^α, b^α, p are the new variables in the metric (4.3))

$$u^\alpha = u^\alpha/a, \; b^\alpha = b^\alpha/a^3, \; \rho = p/a^4$$

one obtains solutions representing «simple waves» propagating in this spacetime.

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