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BETTI NUMBERS OF POWERS OF IDEALS

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Let $A = K[x_1,\dots,x_n]$ be a standard graded polynomial ring over a field K, let $\mathscr{M} = (x_1,\dots,x_n)$ be the graded maximal ideal and I a graded ideal of A. For each i the Betti numbers $\beta_i(I^k)$ of I^k are polynomial functions for $k \gg 0$. We show that if I is $\mathscr{M}$-primar, then these polynomial functions have the same degree for all i .

1. Introduction

Let $A = K[x_1,\dots,x_n]$ be a standard graded polynomial ring over a field K and $I \subset A$ be a graded ideal. Many authors studied the resolution of the graded ideal $I^k, k > 0$. More precisely they are interested to the total Betti numbers, the graded Betti numbers and the regularity of I^k as a function of k, see ([3]) for a survey on these results. To study these invariants one considers the Rees algebra $\mathscr{R}(I)$ of I, since I^k is its kth graded component. Let $S = K[x_1,\dots,x_n,y_1,\dots,y_m]$ be a bigraded polynomial ring over the field K with $\deg x_i = (1,0)$ for $i = 1,\dots,n$ and $\deg y_j = (d_j,1)$ for $j = 1,\dots,m$. The Rees algebra $\deg\mathscr{R}(I) = \bigoplus_{k\geq 0} I^k t^k \subset S[t]$ is a finitely generated bigraded module over S, and $\mathscr{R}(I)_{(*,k)} = \bigoplus_j \mathscr{R}(I)_{(j,k)} = I^k$. In our paper, we are interested in the asymptotic behavior of the total Betti numbers of I^k. Kodiyalam [2] proved that there are polynomials $P_i(t)$ with $P_i(k) = \beta_i(I^k)$ for $k \gg 0$, and Singla [3] showed $\deg P_{i+1}(t) \leq \deg P_i(t)$

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for any $i \geq 0$. Now we prove that, if I is an $\mathcal{M}$-primary graded ideal of A where $\mathcal{M} = (x_1, \ldots, x_n)$ is the graded maximal ideal of A, then $\deg P_{i+1}(t) = \deg P_i(t)$ for all $i \geq 0$.

2. Notation and Results

Let $A = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field K, and let $\mathcal{M} = (x_1, \ldots, x_n) \subset A$ be the graded maximal ideal of A. Let $I \subset A$ be a graded ideal, minimally generated by the homogeneous elements $f_1, f_2, \ldots, f_s$ with $\deg f_i = d_i$ for $i = 1, \ldots, s$, and let $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k t^k \subset S[t]$ be the Rees algebra of I.

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a bigraded polynomial ring over the field K with $\deg x_i = (1,0)$ for $i = 1, \ldots, n$ and $\deg y_j = (d_j, 1)$ for $j = 1, \ldots, m$. Then the K-algebra homomorphism $S \to \mathcal{R}(I)$ induced by $x_i \mapsto x_i$ and $y_j \mapsto f_j t$ is a surjective homomorphism of bigraded K-algebras (provided we assign to an element $ft^k \in \mathcal{R}(I)$ the natural bidegree ($\deg f$, k)). Thus $\mathcal{R}(I)$ may be viewed as bigraded S-module.

Let W by a graded A-module. The numbers $\beta_{i,j}(W) = \dim_K \operatorname{Tor}_i(K,W)_j$ are called the *graded Betti numbers* of the module W, and the numbers

$$\beta_i(W) = \sum_j \beta_{ij}(W) = \dim_K \operatorname{Tor}_i(K,W)$$

are called the *total Betti number* of W.

Now let N be any finitely generated bigraded S-module. We set

$$N^{(k)} = \bigoplus_i N_{(i,k)}.$$

Then for each k, $N^{(k)}$ is a graded A-module. In the special case m$N = \mathcal{R}(I)$, we have $\mathcal{R}(I)^{(k)} = I^k$.

We quote the following results from [2] and [3, Theorem 2.2.4]:

Theorem 2.1. *With the assumptions and notation introduced one has:*

(a) (Kodiyalam) *There exist polynomials P_i^N such that $P_i^N(k) = \beta_i(N^{(k)})$ for all $k \gg 0$.*

(b) (Singla) $\deg P_{i+1}^N \leq \deg P_i^N$ *for all $i \geq 0$.*

In this note we show

Theorem 2.2. *Let $I \subset A = K[x_1, \ldots, x_n]$ be a $\mathcal{M}$-primary ideal. Then*

$$\deg P_0^I = \deg P_1^I = \cdots = \deg P_{n-1}^I = n-1.$$

For the proof of the theorem we need the following simple

Lemma 2.3. *Let $(R,\mathcal{N})$ be a Noetherian local domain of dimension 1, and let $I \subset \mathcal{N}$ be a nonzero ideal. Then $R:I \neq R$.*

Proof. The assumptions imply that R/I is a local ring of dimension 0. Therefore, there exists an integer $k>0$ such that $(\mathcal{N}/I)^k=(0)$. This implies that $\mathcal{N}^k \subset I$. Hence $R:\mathcal{N} \subset R:\mathcal{N}^k \subset R:I$, and it is enough to prove that R is a proper subset of $R:\mathcal{N}$. In order to see this, consider the exact sequence:

$$0 \to \mathcal{N} \to R \to R/\mathcal{N} \to 0.$$

This sequence yields the exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathcal{N},R) \to \operatorname{Hom}_R(R,R) \to \operatorname{Hom}_R(\mathcal{N},R) \to \operatorname{Ext}^1_R(R/\mathcal{N},R) \to 0.$$

Since $\operatorname{Hom}_R(R/\mathcal{N},R)=0$ and $\operatorname{Hom}_R(R,R)=R$, it follows that $(R:\mathcal{N})/R \cong \operatorname{Ext}^1(R/\mathcal{N},R)$, and since $\operatorname{Ext}^1(R/\mathcal{N},R)\neq 0$, we conclude that $R \neq R:\mathcal{N}$, as desired. □

Now we are ready to prove our main result.

Proof of 2.2. We observe that $\beta_0(I^k)=\mu(I^k)=\dim(I^k/\mathcal{M}I^k)$, where $\mu(I^k)$ is the minimal number of generators of I^k. Thus $\beta_0(I^k)$ is the Hilbert function of $\overline{\mathcal{R}(I)}=\mathcal{R}(I)/\mathcal{M}\mathcal{R}(I)$, and hence $\beta_0(I^k)$ is a polynomial function for $k \gg 0$ (which we denoted by P^I_0), whose degree is $\dim\overline{\mathcal{R}(I)}-1$. Since I is $\mathcal{M}$-primary, one has, according to [1, 4.6.13], that $\dim\overline{\mathcal{R}(I)}=n$, so that $\deg P^I_0=n-1$.

By Theorem 2.1, $\deg P^I_0 \geq \deg P^I_1 \geq \cdots \geq \deg P^I_{n-1}$. Thus it remains to prove that $\deg P^I_{n-1}=n-1$.

Note that

$$\begin{aligned}\beta_{n-1}(I^k) &= \dim_K \operatorname{Tor}_{n-1}(x_1,\ldots,x_n;I^k)=\dim_K \operatorname{Tor}_n(x_1,\ldots,x_n;A/I^k)\\ &= \dim_K H_n(x_1,\ldots,x_n;A/I^k)=\dim_K(I^k:\mathcal{M})/I^k).\end{aligned}$$

Thus P^I_{n-1} is the Hilbert polynomial of the graded $\overline{\mathcal{R}(I)}$-module

$$(\mathcal{R}(I):\mathcal{M}\mathcal{R}(I))/\mathcal{R}(I),$$

and hence $\deg P^I_{n-1}=d-1$, where $d=\dim(\mathcal{R}(I):\mathcal{M}\mathcal{R}(I))/\mathcal{R}(I)$.

In order to complete the proof of the theorem, we have to show that $d=n$. It is clear that $d\leq n$. Suppose $d<n$; then there exists a prime ideal $\mathcal{Q}$ of $\overline{\mathcal{R}(I)}$ with $\dim\overline{\mathcal{R}(I)}/\mathcal{Q}=\dim\overline{\mathcal{R}(I)}=n$, and such that $(\mathcal{R}(I):\mathcal{M}\mathcal{R}(I))/\mathcal{R}(I))_{\mathcal{Q}}=0$. Let $\mathcal{P}\in\operatorname{Spec}(\mathcal{R}(I))$ be the preimage of $\mathcal{Q}$ under the canonical epimorphism $\mathcal{R}(I)\to\overline{\mathcal{R}(I)}$. Then $\mathcal{P}$ is a minimal prime ideal of $\mathcal{M}\mathcal{R}(I)$ of height 1, since

$$\operatorname{height}\mathcal{P}=\dim\mathcal{R}(I)-\dim\mathcal{R}(I)/\mathcal{P}=$$

$$= (n+1) - \dim \overline{\mathscr{R}(I)}/\mathscr{Q} = (n+1) - n = 1.$$

It follows that $\dim \mathscr{R}(I)_{\mathscr{P}} = 1$, and $(\mathscr{R}(I) : \mathscr{M}\mathscr{R}(I))/\mathscr{R}(I))_{\mathscr{Q}} = 0$ implies that $\mathscr{R}(I)_{\mathscr{P}} : \mathscr{M}\mathscr{R}(I)_{\mathscr{P}} = \mathscr{R}(I)_{\mathscr{P}}$, contradicting Lemma 2.3. □

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