QUASICONFORMAL MAPPINGS
AND DEGENERATE ELLIPTIC
AND PARABOLIC EQUATIONS

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In this paper two Harnack inequalities are proved concerning a
degenerate elliptic and a degenerate parabolic equation. In both cases
the weight giving the degeneracy is a power of the jacobian of a
quasiconformal mapping.

Introduction.

The purpose of this paper is to prove Harnack inequalities for
some elliptic and parabolic degenerate equations. The degeneracy is
in both cases given by a power of the Jacobian of a quasiconformal
(Q.C.) mapping. This kind of weights are of interest because they
allow some «large» degeneracies (e.g. one can take as a weight $|x|^\alpha$,
$\alpha > -1$). In the elliptic case the result is not new. It was proved by
E. Fabes, C. Kenig and R. Serapioni in their paper [7], obtaining

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appropriate weighted Sobolev estimates and then following the usual J. Moser's pattern.

Our proof however shows that in the elliptic case the Harnack inequality can be obtained using only a Q.C. change of variable and exploiting only one fundamental property of those mappings: the \textit{Distortion Theorem} (Theorem 1.2).

The same approach is not available for degenerate parabolic equations because the form of the equation is not preserved under such a change of variable.

Then, to prove our result in the parabolic case, we followed the usual technique using an easy consequence of a weighted Sobolev imbedding proved in [7].

Finally let us remark that we considered a parabolic equation with weights in both sides (as previously considered with a different kind of degeneracy in [2], [5]) because for the usual parabolic operator (as in [4], [6]) no local regularity is to be hoped without requiring some high integrability to the inverse of the weight (for some examples and comments see [3], [4]).

1. The elliptic equation.

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) bijective, \( n \geq 2 \). Assume that the components \( f, f_i \ (i = 1, \ldots, n) \), have distributional derivatives which belong to \( L^p_{\text{loc}}(\mathbb{R}^n) \). We denote by \( f'(x) \) the Jacobian matrix of \( f \) and by \( |f'| \) its determinat.

\( f \) is \textit{quasiconformal} if

\[
\left[ \sum_{i,j=1}^{n} (f_i f_j^*)^2(x) \right]^{1/2} \leq k |f'(x)|^{1/n}
\]

for some positive constant \( k \) (the \textit{dilation constant} of \( f \)). The following theorems are well known.
THEOREM 1.1. (see e.g. [1]) If $f$ is Q.C. with constant $k$ then $f^{-1}$

is Q.C. with dilation constant $k^{n-1}$.

THEOREM 1.2. (Distortion Theorem, see e.g. the proof of Lemma 4

in [11]). Let $S_r(f(x_0))$ a quasicircle in $R^n$, i.e. $S_r(f(x_0)) = f(B(x_0, r))$\(^{(1)}\). 

Then there exist $\lambda = \lambda(n) > 1$ and positive numbers $r', s'$ such that if $y_0 = f(x_0)$ and $S_r(f(x_0)) = S_r(y_0)$ we have

$$S_r(y_0) \subseteq B(y_0, r') \subseteq B(y_0, s') \subseteq S_r(\lambda(\sqrt{n+1})k(y_0)).$$

Moreover: $\frac{r'}{s'} \leq \frac{1}{\sqrt{n}}$.

Let $\Omega$ a bounded domain in $R^n$, $a_{ij} : \Omega \to R$ $(i, j = 1, \ldots, n)$ measurable functions in $\Omega$ satisfying

\begin{equation}
vw(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq v^{-1}w(x)|\xi|^2 \quad \forall \xi \in R^n, \text{ a.e. in } \Omega
\end{equation}

where we set $w(x) = |f'(x)|^{1-\frac{2}{n}}$ for some Q.C. mapping $f$ and $v > 0$.

Following [7] we denote by $H^1(\Omega; w)$ and $H^1_0(\Omega; w)$ the completions of $C^\infty(\overline{\Omega})$ and of $C_0^\infty(\Omega)$ with respect to the norms

$$||u||_{H^1(\Omega; w)} = \left[ \int_\Omega u^2(x)w(x)dx + \int_\Omega |Du|^2w(x)dx \right]^{1/2}$$

$$||u||_{H^1_0(\Omega; w)} = \left[ \int_\Omega |Du|^2w(x)dx \right]^{1/2}$$

respectively. We notice that as it is shown in [7] because of the

properties of the weight a vector valued function, denoted by $Du$, is

uniquely associated to any function $u$ in $H^1(\Omega; w)$.

We now consider in $\Omega$ the operator

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_i} \right)$$

\(^{(1)}\) $\forall x_0 \in R^n$, $r \in ]0, +\infty[\text{ we set } B(x_0, r) = \{ x \in R^n : |x - x_0| < r \}$. 

and local solutions of the equation $Lu = 0$ (see [F.K.S.]) Our first result is the following

**THEOREM 1.3.** (*Harnack inequality, see [F.K.S.] p. 97.*) It exists $c' > 1$, $c' = c'(n, k)$, such that if $B(x_0, c'r)$ is contained in $\Omega$ and $u$ is any local solution of $Lu = 0$ positive in $B(x_0, c'r)$, we have

$$\max_{B(x_0, r)} u \leq \gamma \min_{B(x_0, r)} u$$

for some positive constant $\gamma$ independent of $u$, $x_0$, $r$.

Before proving Theorem 1.3 we make some easy remarks.

Let $y = f(x)$, $x = f^{-1}(y) \equiv g(y)$, $\Omega' = f(\Omega)$. We have $|g'(y)| = \frac{1}{|f'(x)|}$ and then if $A = (a_{ij})$ setting $\tilde{A}(y) = |g'(y)|(g'(y)^{-1})^t A(g(y))g'(y)^{-1}$ we have

$$(1.2) \quad \frac{1}{\nu} |\xi|^2 \leq \tilde{A}(y)\xi \cdot \xi \leq \alpha \nu^{-1} |\xi|^2.$$  

(Obviously above $M^{-1}$ and $M^t$ denote the inverse and the transpose matrix of a matrix $M$).

(1.2) is a straightforward consequence of the quasiconformality of $g$. Indeed because Q.C. if $\lambda_i(y)$ are the eigenvalues of $g'(y)^t g'(y)$ there exists $\alpha > 0$ such that

$$\frac{1}{\alpha} \leq \frac{\lambda_i(y)}{\lambda_j(y)} \leq \alpha \quad \forall i, j = 1, \ldots, n.$$ 

We conclude that the original degenerate equation in $\Omega$ has been transformed through the performed Q.C. change of variables in the non degenerate equation in $\Omega$

$$(1.3) \quad \text{div}(\tilde{A}(y)Dv(y)) = 0,$$

where $v(y) = u(g(y))$.

Concerning solutions of (1.3) we have
THEOREM 1.4. (Harnack inequality on Quasi Circles). Let \( S_r(y_0) \) a quasicircle in \( \Omega' \), such that \( S_{\tau'(\lambda/\sqrt{n+1})^2}(y_0) \equiv S_{\tilde{c}\tau} \subseteq \Omega' \) and suppose \( v \) is a local solution of (1.3) \( v \geq 0 \) in \( S_{\tilde{c}\tau} \). Then
\[
\max_{S_r} v \leq \gamma' \min_{S_r} v
\]
for some positive constant \( \gamma' \) independent of \( \tau, \nu \) and \( x_0 \).

Proof. With the notation of Theorem 1.2 being \( v \geq 0 \) in \( B_{\tau'} \) we have
\[
(1.4) \quad \max_{B_{\tau'}} v \leq \gamma' \min_{B_{\tau'}} v
\]
by the usual Moser's Harnack Theorem for non degenerate equations (\( \gamma' \) independent of \( \tau', \lambda_0 \) and \( v \)). Then
\[
\max_{S_r} v \leq \gamma' \min_{S_r} v.
\]
We are now ready for the Proof of Theorem 1.3.

Let \( u \) be a local solution of \( Lu = 0 \) in \( \Omega \). Then \( v(y) = u(g(y)) \) is a local solution of (1.3) in \( \Omega' \). Let \( c' = \tilde{c} \), \( \tilde{\omega} \) as in Theorem 1.4. Because \( u \geq 0 \) in \( B(x_0, c'r) \) we have \( v \geq 0 \) in \( S_{\tilde{c}\tau} \) and then, because Theorem 1.4
\[
\max_{S_r} v \leq \gamma' \min_{S_r} v
\]
which means, going back to \( u \)
\[
\max_{B(x_0,r)} u \leq \gamma' \min_{B(x_0,r)} u.
\]

2. The parabolic equation.

Let \( \Omega \) a bounded open set in \( \mathbb{R}^n \), \( T > 0 \) and \( Q = \Omega \times ]0, T[ \).

In the cylinder \( Q \) we will study the degenerate parabolic operator
\[
Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) - \frac{\partial}{\partial t} (w(x)u)
\]
with the same assumptions as in Section 1 on $a_{ij}$ and $w$.

We will consider weak solutions $u$ of $Lu = 0$ in $Q$ where $u \in W$,

$$W = \{ u \in L^2(0,T; H_0^1(\Omega; w)) : \frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega; w)) \}.$$ 

The meaning of weak solution is the usual (see eg. [2], [5]).

We need the following imbedding theorem proved in [7] under our assumptions on the weight.

**THEOREM 2.1.** It exists $k > 1$ such that for any ball $B(x_0,r) \subseteq \Omega$ and any $u \in H_0^1(B,w)$

$$\left( \frac{1}{w(B)} \int_B |u|^{2k} w \, dx \right)^{1/2k} \leq cr \left( \frac{1}{w(b)} \int_B |Du|^2 w \, dx \right)^{1/2},$$

where $w(B) = \Omega \int_B w \, dx$ and $c > 0$ depends on $n$ and $w$ only.

From Theorem 2.1 it is immediately deduced

**THEOREM 2.2.** It exists $\tilde{k} > 1$ such that for any $a, b \in R$, $a < b$, $B = B(x_0,r)$ and for any $u \in L^\infty(a,b; L^2(B; w)) \cap L^2(a,b; H_0^1(B; w))$ we have

$$\left( \frac{1}{b-a} \int_a^b \frac{1}{w(B)} \int_B |u|^{2\tilde{k}} w \, dx \, dt \right)^{1/2\tilde{k}} \leq$$

$$\leq (cr)^{1/k} \left( \sup_{[a,b]} \left( \frac{1}{w(B)} \int_B u^2 w \, dx \right) \right)^{1/2} 1^{-1/\tilde{k}}$$

$$\left( \frac{1}{b-a} \int_a^b \frac{1}{w(B)} \int_B |Du|^2 w \, dx \, dt \right)^{1/\tilde{k}},$$

where $c > 0$ depends on $n$ and $w$ only.
For any \((x_0, t_0) \in Q, \rho > 0\) we set

\[D_{x_0, t_0}(\rho) = \{(x, t) \in Q : |t - t_0| < \rho^2, |x - x_0| < 2\rho\}\]

\[D^+_{x_0, t_0}(\rho) = \{(x, t) \in Q : t_0 + \frac{3}{4}\rho^2 < t < t_0 + \rho^2, |x - x_0| < \rho/2\}\]

\[D^-_{x_0, t_0}(\rho) = \{(x, t) \in Q : t_0 - \frac{3}{4}\rho^2 < t < t_0 - \frac{1}{4}\rho^2, |x - x_0| < \rho/2\}\].

We then have the following parabolic Harnack inequality.

**THEOREM 2.3.** Let \(u(x, t)\) be a positive solution of \(Lu = 0\) in \(D_{x_0, t_0}(\rho)\). Then

\[
(2.1) \quad \sup_{D^+_{x_0, t_0}(\rho)} u(x, t) \leq \gamma \inf_{D^-_{x_0, t_0}(\rho)} u(x, t)
\]

\(\forall \rho > 0\) such that \(D_{x_0, t_0}(\rho) \subseteq Q\). \(\gamma\) in (2.1) is a positive constant independent of \(x_0, t_0, \rho\) and \(u\).

Because we have the parabolic Sobolev embedding given in Theorem 2.2 all we need to prove (2.1) following the Moser’s technique (see e.g. [10]) is an appropriate weighted local energy estimate with constant independent of \(\rho\).

This is guaranteed, exactly as in [5], by the special form of the equation in which the same weight appears in both the sides. After these remarks the proof follows by the same steps as in [5].

**REFERENCES**


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