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QUASICONFORMAL MAPPINGS AND DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS

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In this paper two Harnack inequalities are proved concerning a degenerate elliptic and a degenerate parabolic equation. In both cases the weight giving the degeneracy is a power of the jacobian of a quasiconformal mapping.

Introduction.

The purpose of this paper is to prove Harnack inequalities for some elliptic and parabolic degenerate equations. The degeneracy is in both cases given by a power of the Jacobian of a quasiconformal (Q.C.) mapping. This kind of weights are of interest because they allow some «large» degeneracies (e.g. one can take as a weight $|x|^\alpha$, $\alpha > -1$). In the elliptic case the result is not new. It was proved by E. Fabes, C. Kenig and R. Serapioni in their paper [7], obtaining

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appropriate weighted Sobolev estimates and then following the usual J. Moser's pattern.

Our proof however shows that in the elliptic case the Harnack inequality can be obtained using only a Q.C. change of variable and exploiting only one fundamental property of those mappings: the *Distortion Theorem* (Theorem 1.2).

The same approach is not available for degenerate parabolic equations because the form of the equation is not preserved under such a change of variable.

Then, to prove our result in the parabolic case, we followed the usual technique using an easy consequence of a weighted Sobolev imbedding proved in [7].

Finally let us remark that we considered a parabolic equation with weights in both sides (as previously considered with a different kind of degeneracy in [2], [5]) because for the usual parabolic operator (as in [4], [6]) no local regularity is to be hoped without requiring some high integrability to the inverse of the weight (for some examples and comments see [3], [4]).

1. The elliptic equation.

Let $f : R^n \to R^n$ bijective, $n \geq 2$. Assume that the components of f, f_i $(i = 1, \dots, n)$, have distributional derivatives which belong to $L^n_{\text{loc}}(R^n)$. We denote by $f'(x)$ the Jacobian matrix of f and by $|f'|$ its determinat.

f is *quasiconformal* if

$$\left[\sum_{i,j=1}^{n} (f_i)^2_{x_j}(x)\right]^{1/2} \leq k|f'(x)|^{1/n}$$

for some positive constant k (the *dilation constant* of f). The following theorems are well known

THEOREM 1.1. (see e.g. [1]) *If f is Q.C. with constant k then f^{-1} is Q.C. with dilation constant k^{n-1}.*

THEOREM 1.2. (Distortion Theorem, see e.g. the proof of Lemma 4 in [11]). *Let $S_r(f(x_0))$ a quasicircle in R^n, i.e. $S_r(f(x_0)) = f(B(x_0, r))$(1). Then there exist $\lambda = \lambda(n) > 1$ and positive numbers r', s' such that if $y_0 = f(x_0)$ and $S_r(f(x_0)) = S_r(y_0)$ we have*

$$S_r(y_0) \subseteq B(y_0, r') \subset B(y_0, s') \subseteq S_{r(\lambda(\sqrt{n}+1))^k}(y_0).$$

Moreover: $\dfrac{r'}{s'} \leq \dfrac{1}{\sqrt{n}}$.

Let Ω a bounded domain in R^n, $a_{ij} : \Omega \to R$ $(i, j = 1, \ldots, n)$ measurable functions in Ω satisfying

$$\nu w(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}w(x)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. in} \quad \Omega \tag{1.1}$$

where we set $w(x) = |f'(x)|^{1-\frac{2}{n}}$ for some Q.C. mapping f and $\nu > 0$. Following [7] we denote by $H^1(\Omega; w)$ and $H^1_0(\Omega; w)$ the completions of $C^\infty(\bar{\Omega})$ and of $C^\infty_0(\Omega)$ with respect to the norms

$$||u||_{H^1(\Omega;w)} = \left[\int_\Omega u^2(x)w(x)dx + \int_\Omega |Du|^2 w(x)dx\right]^{1/2}$$

$$||u||_{H^1_0(\Omega;w)} = \left[\int_\Omega |Du|^2 w(x)dx\right]^{1/2}$$

respectively. We notice that as it is shown in [7] because of the properties of the weight a vector valued function, denoted by Du, is uniquely associated to any function u in $H^1(\Omega; w)$.

We now consider in Ω the operator

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)$$

(1) $\forall x_0 \in R^n, r \in]0, +\infty[$ we set $B(x_0, r) = \{x \in R^n : |x - x_0| < r\}$.

and local solutions of the equation $Lu = 0$ (see [F.K.S.]) Our first result is the following

THEOREM 1.3. *(Harnack inequality, see [F.K.S.] p. 97). It exists $c' > 1$, $c' = c'(n,k)$, such that if $B(x_0, c'r)$ is contained in Ω and u is any local solution of $Lu = 0$ positive in $B(x_0, c'r)$, we have*

$$\max_{B(x_0,r)} u \leq \gamma \min_{B(x_0,r)} u$$

for some positive constant γ independent of u, x_0, r.

Before proving Theorem 1.3 we make some easy remarks.

Let $y = f(x)$, $x = f^{-1}(y) \equiv g(y)$, $\Omega' = f(\Omega)$. We have $|g'(y)| = \dfrac{1}{|f'(x)|}$ and then if $A = (a_{ij})$ setting $\tilde{A}(y) = |g'(y)|(g'(y)^{-1})^t A(g(y))g'(y)^{-1}$ we have

$$\frac{1}{\alpha}\nu|\xi|^2 \leq \tilde{A}(y)\xi \cdot \xi \leq \alpha\nu^{-1}|\xi|^2. \tag{1.2}$$

(Obviously above M^{-1} and M^t denote the inverse and the transpose matrix of a matrix M).

(1.2) is a straightforward consequence of the quasiconformality of g. Indeed because Q.C. if $\lambda_i(y)$ are the eigenvalues of $g'(y)^t g'(y)$ there exists $\alpha > 0$ such that

$$\frac{1}{\alpha} \leq \frac{\lambda_i(y)}{\lambda_j(y)} \leq \alpha \quad \forall i, j, = 1, \dots, u.$$

We conclude that the original degenerate equation in Ω has been transformed through the performed Q.C. change of variables in the non degenerate equation in Ω

$$\operatorname{div}(\tilde{A}(y)Dv(y)) = 0, \tag{1.3}$$

where $v(y) = u(g(y))$.

Concerning solutions of (1.3) we have

THEOREM 1.4. (Harnack inequality on Quasi Circles). *Let* $S_r(y_0)$ *a quasicircle in* Ω', *such that* $S_{r(\lambda(\sqrt{n}+1))^k}(y_0) \equiv S_{\bar{c}r} \subseteq \Omega'$ *and suppose* v *is a local solution of (1.3)* $v \geq 0$ *in* $S_{\bar{c}r}$. *Then*

$$\max_{S_r} v \leq \gamma' \min_{S_r} v$$

for some positive constant γ' *independent of* r, v *and* x_0.

Proof. With the notation of Theorem 1.2 being $v \geq 0$ in $B_{s'}$ we have

$$\max_{B_{r'}} v \leq \gamma' \min_{B_{r'}} v \tag{1.4}$$

by the usual Moser's Harnack Theorem for non degenerate equations (γ' independent of r', y_0 and v). Then

$$\max_{S_r} v \leq \gamma' \min_{S_r} v.$$

We are now ready for the *Proof of Theorem* 1.3.

Let u be a local solution of $Lu = 0$ in Ω. Then $v(y) = u(g(y))$ is a local solution of (1.3) in Ω'. Let $c' = \bar{c}$, $\bar{c}$ as in Theorem 1.4. Because $u \geq 0$ in $B(x_0, c'r)$ we have $v \geq 0$ in $S_{c'r}$ and then, because Theorem 1.4

$$\max_{S_r} v \leq \gamma' \min_{S_r} v$$

which means, going back to u

$$\max_{B(x_0,r)} u \leq \gamma' \min_{B(x_0,r)} u.$$

2. The parabolic equation.

Let Ω a bounded open set in R^n, $T > 0$ and $Q = \Omega \times]0, T[$.

In the *cylinder* Q we will study the degenerate parabolic operator

$$Lu = \sum_{i,J=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \frac{\partial}{\partial t}(w(x)u)$$

with the same assumptions as in Section 1 on a_{ij} and w.

We will consider weak solutions u of $Lu = 0$ in Q where $u \in W$,

$$W = \{u \in L^2(0,T; H_0^1(\Omega; w)) : \frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega; w))\}.$$

The meaning of weak solution is the usual (see eg. [2], [5]).

We need the following imbedding theorem proved in [7] under our assumptions on the weight.

THEOREM 2.1. *It exists $k > 1$ such that for any ball $B(x_0, r) \subseteq \Omega$ and any $u \in H_0^1(B, w)$*

$$\left(\frac{1}{w(B)}\int_B |u|^{2k} w dx\right)^{1/2k} \leq cr \left(\frac{1}{w(b)}\int_B |Du|^2 w dx\right)^{1/2},$$

where $w(B) = \Omega \int_B w dx$ and $c > 0$ depends on n and w only.

From Theorem 2.1 it is immediately deduced

THEOREM 2.2. *It exists $\tilde{k} > 1$ such that for any $a, b \in R$, $a < b$, $B = B(x_0, r)$ and for any $u \in L^\infty\ (a, b; L^2(B; w)) \cap L^2(a, b; H_0^1(B; w))$ we have*

$$\left(\frac{1}{b-a}\int_a^b \frac{1}{w(B)}\int_B |u|^{2\tilde{k}} w dx dt\right)^{1/2\tilde{k}} \leq$$

$$\leq (cr)^{1/\tilde{k}} \left(\sup_{[a,b]} \left(\frac{1}{w(B)}\int_B u^2 w dx\right)^{1/2}\right)^{1-1/\tilde{k}}$$

$$\cdot \left(\frac{1}{b-a}\int_a^b \frac{1}{w(B)}\int_B |Du|^2 w dx dt\right)^{1/\tilde{k}},$$

where $c > 0$ depends on n and w only.

For any $(x_0, t_0) \in Q$, $\rho > 0$ we set

$$D_{x_0,t_0}(\rho) = \{(x,t) \in Q : |t - t_0| < \rho^2,\ |x - x_0| < 2\rho\}$$

$$D^+_{x_0,t_0}(\rho) = \{(x,t) \in Q : t_0 + \frac{3}{4}\rho^2 < t < t_0 + \rho^2, |x - x_0| < \rho/2\}$$

$$D^-_{x_0,t_0}(\rho) = \{(x,t) \in Q : t_0 - \frac{3}{4}\rho^2 < t < t_0 - \frac{1}{4}\rho^2, |x - x_0| < \rho/2\}.$$

We then have the following parabolic Harnack inequality.

THEOREM 2.3. *Let* $u(x,t)$ *be a positive solution of* $Lu = 0$ *in* $D_{x_0,t_0}(\rho)$. *Then*

$$\sup_{D^+_{x_0,t_0}(\rho)} u(x,t) \leq \gamma \inf_{D^+_{x_0,t_0}(\rho)} u(x,t) \tag{2.1}$$

$\forall \rho > 0$ *such that* $D_{x_0,t_0}(\rho) \subseteq Q$. γ *in (2.1) is a positive constant independent of* x_0, t_0, ρ *and* u.

Because we have the parabolic Sobolev embedding given in Theorem 2.2 all we need to prove (2.1) following the Moser's technique (see e.g. [10]) is an appropriate weighted local energy estimate with constant independent of ρ.

This is guaranteed, exactly as in [5], by the special form of the equation in which the same weight appears in both the sides. After these remarks the proof follows by the same steps as in [5].

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