«LE MATEMATICHE»
Vol. XLII (1987) - Fasc. I-II, pp. 171-178

SOME FUNCTION SPACES AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

PIETRO ZAMBONI (Messina) (*)

In this paper we compare some function spaces which are relevant to the inequality

$$\int |f|u^2 dx \leq K \int |\nabla u|^2 dx \quad \forall u \in C_0^\infty(\mathbb{R}^n)$$

An unique continuation result for nonnegative solution of elliptic P.D.E.'s is also proved.

1. Introduction and preliminary results.

The purpose of this paper is to compare some function spaces which have been introduced by various Authors in connection with the inequality:

$$\int |f|u^2 dx \leq K \int |\nabla u|^2 dx \quad \forall u \in C_0^\infty(\mathbb{R}^n) \tag{1.1}$$

(*) Entrato in Redazione il 16 gennaio 1989

As an application we will extend a unique continuation result for nonnegative solutions of an elliptic second order operator originally proven in [4].

The equation we consider is of the form

$$Lu = -(a_{ij}u_{x_i})_{x_j} + b_i u_{x_i} + cu \tag{1.2}$$

where: $a_{ij}(x) = a_{ji}(x)$ $(i,j = 1,2,\dots,n)$ are bounded measurable functions in Ω, an open connected subset of $\mathbb{R}^n$. We assume the ellipticity condition

$$\nu^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \nu|\xi|^2 \tag{1.3}$$

to hold for some $\nu > 0$, $\forall\xi \in \mathbb{R}^n$, a.e. in Ω.

To introduce the assumptions on the low order terms and the notion of solution of (1.2) we need some definitions.

Let, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\varepsilon > 0$,

$$\eta(f;\varepsilon) = \sup_{x\in\mathbb{R}^n} \int_{|x-y|\le\varepsilon} \frac{|f(y)|}{|x-y|^{n-2}}\,dy.$$

We set

$$S = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \eta(f;\varepsilon) < +\infty \text{ for } \varepsilon > 0 \text{ and } \eta(f;\varepsilon) \searrow 0 \text{ as } \varepsilon \searrow 0\}$$

$$\tilde{S} = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \eta(f;\varepsilon) \text{ is finite } \forall\varepsilon_0 > 0\}$$

Also given $r \in \left]1, \frac{n}{2}\right]$, we set

$$F_r = \{f \in L^r_{\text{loc}}(\mathbb{R}^n) : \left(\fint_B |f|^r dx\right)^{\frac{1}{r}} \le C_{\varepsilon_0}|B|^{-\frac{2}{n}}$$

for any ball B with radius less then $\varepsilon_0 > 0$ and some positive constant $C_{\varepsilon_0}\}$(1).

(1) Here $|B|$ is the measure of B and

$$\fint_B f\,dx = \frac{1}{|B|}\int_B f\,dx$$

The class $\tilde{S}$ is a slight variant of the usual Stummel - Kato class S (see [1], [11], [12]) while F_r is a class slightly larger then the Fefferman - Phong class (see [2] and [7] where C_{ε_0} is substituted with a fixed $C > 0$).

To assume $f \in \tilde{S}$ or $f \in F_r$ is sufficient for the inequality (1.1) to hold with a constant dependent only on the support of u (see [2], [3], [7], for the case $f \in F_r$ and the following Lemma 3 for $f \in \tilde{S}$).

Coming back to equation (1.2) we assume either

$$b_i^2,\ c \in F_r \tag{1.4}$$

$$b_i^2,\ c \in \tilde{S} \tag{1.4)'$$

As customary we say that u belongs to $H^1(\Omega)$ $[H^1_{\text{loc}}(\Omega)]$ if u and $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$ $[L^2_{\text{loc}}(\Omega)]$, $i = 1, \ldots, n$.

We say that $u \in H^1_{\text{loc}}(\Omega)$ is a *local weak solution* of the equation

$$Lu = 0$$

if

$$\int_\Omega (a_{ij} u_{x_i} \varphi_{x_j} + b_i u_{x_i} \varphi + cu\varphi) dx = 0 \quad \forall \varphi \in H^1(\Omega),\ \text{supp}\varphi \subset\subset \Omega \tag{1.5}$$

Using Hölder inequality is immediately seen that (1.2) is meaningful because our assumptions (1.4) or (1.4)'.

To state our main result we need one more definition.

Let $w \in L^1_{\text{loc}}(\Omega)$, $w \geq 0$ a.e. in Ω. We say that w *has a zero of infinite order at* $x_0 \in \Omega$ if

$$\lim_{\sigma \to 0} \frac{\displaystyle\int_{B_\sigma(x_0)} w(x) dx}{|B_\sigma(x_0)|^k} = 0 \quad \forall k > 0 \qquad (^2)$$

(2) As usual $B_\sigma(x_0) = \{x : |x - x_0| < \sigma\}$

The following lemma is well known (see [8]).

LEMMA 1. *Let* $w \in L^1_{\text{loc}}(\Omega)$, $w \geq 0$ *a.e. in* Ω, $w \not\equiv 0$. *Assume that*

$$\exists C > 0 : \int_{B_{2\sigma}(x_0)} w(x)dx \leq C \int_{B_\sigma(x_0)} w(x)dx \quad \forall \sigma > 0$$

Then $w(x)$ *has no zero of infinite order in* Ω.

Our main result is the following

THEOREM 1. *Let* $u \in H^1(\Omega)$, $u \geq 0$, $u \not\equiv 0$, *a solution of* $Lu = 0$ *with assumptions (1.3) and (1.4) or (1.3) and (1.4)'.*

Then u *has no zero of infinite order in* Ω.

The proof of this theorem will be given in Section 2.

Some comments are now in order. Theorem 1 improves the result in [4] in two directions. First: [4] assumes $b_i = 0$; second, and more important, there are functions in both the classes F_r and $\tilde{S}$ which do not belong to $L^{\frac{n}{2},\infty}$(3) (which is the assumption of [4] on the potential term c). These examples will be discussed in a moment. First let us recall the definition of the classical Morrey space $L^{p,\lambda}(R^n)$ $p \geq 1$, $0 < \lambda \leq n$.

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \sup_{0<\varepsilon\leq\varepsilon_0} \left(\sup_{x\in\mathbb{R}^n} \frac{1}{\varepsilon^\lambda} \int_{B_\varepsilon(x)} |f(y)|^p dy \right)^{\frac{1}{p}} = C_{\varepsilon_0} < +\infty \right\}.$$

Now it is easy to see that $F_r \equiv L^{r,n-2r} \subsetneq L^{1,n-2}$ $\forall r > 1$ (see [10]).

Also, by Holder's inequality, $F_r \subsetneq F_{r_1}$ if $r > r_1$. Concerning the S and $\tilde{S}$ class we have:

LEMMA 2. $L^{1,\lambda} \subset S \subset \tilde{S} \subset L^{1,n-2}$ $\lambda > n-2$

Proof. For the inclusion $L^{1,\lambda} \subseteq S$ see [6].

(3) $f \in L^{\frac{n}{2},\infty} \Leftrightarrow \lambda^{\frac{n}{2}}|\{|f| > \lambda\}| \leq C$ for some constant $C > 0$, and $\forall\lambda \in]0, +\infty[$

Clearly S is contained in $\tilde{S}$.

To prove the final inclusion we consider $f \in \tilde{S}$ and a ball $B_\varepsilon(x_0)$ centered at x_0 with radius $\varepsilon \leq \varepsilon_0$. Then

$$\frac{1}{\varepsilon^{n-2}} \int_{B_\varepsilon(x_0)} |f(y)| dy \leq \int_{|x_0-y|\leq\varepsilon} \frac{|f(y)|}{|x_0-y|^{n-2}} dy \leq$$

$$\leq \sup_{x_0\in R^n} \int_{|x_0-y|\leq\varepsilon} \frac{|f(y)|}{|x_0-y|^{n-2}} dy =$$

$$= \eta(f;\varepsilon) < \eta(f;\varepsilon_0) = C_{\varepsilon_0}$$

Given this just recall the well known fact (see [10]) that for each Morrey space $L^{p,\lambda}$ $(p \geq 1, 0 < \lambda < n)$ it is possible to find a function $f \in L^{p,\lambda}$ such that $f \notin L^q$ $\forall q > p$.

This remark and the above recalled inclusions of Morrey spaces in $\tilde{S}$ show that there are functions in F_r (taking r small enough) and in $\tilde{S}$ which do not belong to $L^{\frac{n}{2},\infty}$.

For the sake of completeness we give a proof of inequality (1.1) assuming $f \in \tilde{S}$ (a case which, as for as we know, has not been considered in the literature).

LEMMA 3. *Let $f \in \tilde{S}$. Then for any $r_0 > 0$ it exists a positive constant C_{r_0} depending only on $\eta(f;r_0)$ and n such that*

$$\int_{\mathbb{R}^n} |f| u^2 dx \leq C_{r_0} \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

for any $u \in C_0^\infty(R^n)$ supported in B_{r_0}.

Proof. Let $u \in C_0^\infty(R^n)$. Consider

$$\int_{\mathbb{R}^n} |f| u^2 dx = \int_{B_{r_0}} |f| u^2 dx \leq C \int_{B_{r_0}} |f(x)u(x)| \int_{B_{r_0}} |\frac{\nabla u(y)|}{|x-y|^{n-1}} dy dx =$$

$$= C \int_{B_{r_0}} \left(\int_{B_{r_0}} \frac{|f(x)u(x)|}{|x-y|^{n-1}} dx \right) |\nabla u(y)| dx \leq$$

$$\leq C \left(\int_{B_{r_0}} |\nabla u(y)|^2 dy \right)^{\frac{1}{2}} \left(\int_{B_{r_0}} \left(\int_{B_{r_0}} \frac{|f(x)u(x)|}{|x-y|^{n-1}} dx \right)^2 dy \right)^{\frac{1}{2}}.$$

We also have:

$$\int_{B_{r_0}} \left(\int_{B_{r_0}} \frac{|f(x)u(x)|}{|x-y|^{n-1}} dx \right)^2 dy \leq$$

$$\leq \int_{B_{r_0}} \left(\int_{B_{r_0}} \frac{|f(z)|}{|z-y|^{n-1}} dz \right) \left(\int_{B_{r_0}} \frac{|f(x)|u^2(x)}{|x-y|^{n-1}} dx \right) dy =$$

$$= \int_{B_{r_0}} \int_{B_{r_0}} [|f(z)|\,|f(x)|u^2(x)] \left(\int_{B_{r_0}} \frac{dy}{|x-y|^{n-1}|z-y|^{n-1}} \right) dxdz \leq$$

$$\leq \int_{B_{r_0}} \int_{B_{r_0}} [|f(z)|\,|f(x)|u^2(x)] \left(\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-1}|z-y|^{n-1}} \right) dxdz \leq$$

$$\leq C_n \int_{B_{r_0}} \int_{B_{r_0}} |f(z)|\,|f(x)|u^2(x) \frac{1}{|z-x|^{n-2}} dxdz =$$

$$= C_n \int_{B_{r_0}} \left(\int_{B_{r_0}} \frac{|f(z)|}{|z-x|^{n-2}} dz \right) |f(x)|u^2(x)dx \leq$$

$$\leq C_n \eta(f; r_0) \int_{B_{r_0}} |f(x)|u^2(x)dx.$$

Hence:

$$\int_{\mathbb{R}^n} |f|u^2 dx \leq C_n \eta(f; r_0) \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

2. Proof of Theorem. 1.

Following the «central» step in the Moser's proof of Harnack inequality we use equation (1.2) (obviously intended in the weak form) taking as a test function $\alpha^2 u^{-1}$(4) where $\alpha \in C_0^\infty(\Omega)$.

We obtain

$$\int_\Omega \left(a_{ij} u_{x_i} \frac{2\alpha\alpha_{x_j} u - \alpha^2 u_{x_j}}{u^2} + b_i u_{x_i} \frac{\alpha^2}{u} + cu \frac{\alpha^2}{u} \right) dx = 0$$

(4) In fact we should substitute to u in our argument, $u + \varepsilon$ ($\varepsilon > 0$), which is positive in Ω and after obtaining estimates independent of ε go to the limit as $\varepsilon \searrow 0$.

and after some standard calculations

$$\begin{aligned}(1.5)\qquad \nu\int_\Omega \alpha^2\frac{|\nabla u|^2}{u}dx \leq& \nu^{-1}\varepsilon\int_\Omega \alpha^2|\nabla \log u|^2dx+\\ &+\nu^{-1}K(\varepsilon)\int_\Omega |\nabla\alpha|^2dx+\int_\Omega |b_i u_{x_i}\frac{\alpha^2}{u}|dx+\int_\Omega |c|\alpha^2dx.\end{aligned}$$

($\varepsilon > 0$, $K(\varepsilon)$ a positive constant depending only on ε).

We now majorize the last two terms using (1.4) or (1.4)'.

$$\begin{aligned}\int_\Omega |b_i u_{x_i}\frac{\alpha^2}{u}|dx &\leq \left(\int_\Omega \alpha^2|\nabla\log u|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega b_i^2\alpha^2dx\right)^{\frac{1}{2}}\leq\\ &\leq K(b,\mathrm{supp}\alpha)\left(\int_\Omega \alpha^2|\nabla\log u|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\alpha^2|dx\right)^{\frac{1}{2}};\end{aligned}$$

$$\int_\Omega |c|\alpha^2dx \leq K(c,\mathrm{supp}\alpha)\int_\Omega |\nabla\alpha|^2dx.$$

Introducing these in (1.5) we obtain

$$\int_\Omega \alpha^2|\nabla\log u|^2dx \leq K(\nu,b,c,\mathrm{supp}\alpha,n)\int_\Omega |\nabla\alpha|^2dx.$$

Finally we fiw $x_0\in\Omega$, a cube Q_0 centered at x_0 and such that the concentric cube with double side length Q_1 is still contained in Ω.

Then we consider any subcube Q_r of Q_0, and an $\alpha(x)\in C_0^\infty(Q_{2r})$ (where Q_{2r} is the cube concentric with Q_r but having double side lenght) such that $\alpha=1$ in Q_r and $|\nabla\alpha|\leq\dfrac{K}{r}$.

With this choise of α we obtain

$$\int_{Q_r} |\nabla\log u|^2dx \leq K(\nu,b,c,Q_1,n)r^{n-2}$$

and by the Poincarè inequality and the John - Nirenberg Lemma we deduce (as in [9]) that $\log u \in$B.M.O.(Q_0). This in turn implies $u^\delta\in A_2$ for some $\delta>0$. Now it is well known (see [5]) that A_p implies the doubling property for u^δ (i.e. the assumption of Lemma 1) and the conclusion follows for u^δ and hence also for u.

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Università di Messina
Dipartimento di Matematica
Contrada Papardo
98010 S. Agata (Messina)