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ON CONTINUOUS PARAMETRIZATION OF A FAMILY OF SEPARATORS OF A LOCALLY CONNECTED CURVE

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The question (raised in [7]) whether every homogeneous family of separators of a locally connected metrizable space Y, which is a partition of Y and has the continuum power, admits a continuous parametrization, is studied in the realm of locally connected curves.

In [7], p. 224, B. Ricceri has asked the following question (Q) which is quoted below in a simpler, and slightly modified, form.

(Q) Let a topological space Y be locally connected and metrizable and let $\{C_i : i \in I\}$ be a homogeneous family of separators of Y which, moreover, is a partition of Y and has the continuum power. Then, is there some continuous function $f : Y \to \mathbb{R}$ for which the equality

(1) $$\{C_i : i \in I\} = \{f^{-1}(t) : t \in f(Y)\}$$

holds?

The question is related to some studies lying in the common border of topology and functional analysis, and arose in a natural

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way in B. Ricceri's research in this field (see [7] for details). In this paper a further study is made that concerns this topic. Namely for some acyclic as well as for cyclic locally connected curves we exhibit partitions of these spaces that do not admit any continuous function f satisfying (1). This answers (Q) in the negative. Furthermore, an upper semicontinuous monotone decomposition of an arbitrary dendrite is found for which a function f in matter does exist and which is minimal in the sense that it cannot be further refined. Also some questions are asked.

We start with necessary definitions which are recalled here after [7], p. 223. By a *partition*, or a *decomposition*, of a space Y we understand a family of pairwise disjoint closed subsets of Y whose union is Y. If all the members of the considered family are connected, then the decomposition is said to be *monotone*.

A subset C of a space Y is called a *separator* of Y if there exist two nonempty disjoint open sets A and B in Y such that $A \cup B = Y \setminus C$. The sets A and B are said to be *associated* to C. In other words a set $C \subset Y$ is a separator of Y if it is closed and $Y \setminus C$ is not connected. If, moreover, there is a connected set $S \subset Y$ such that $A \cap S \neq \emptyset \neq B \cap S$, then C is said to be a *strong separator*. We say that a family $\{C_i : i \in I\}$ of separators of Y is *homogeneous* provided there exist two families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ of subsets of Y such that for each $i \in I$ the sets A_i and B_i are associated to C_i, and for every open connected set $U \subset Y$ intersecting both A_i and B_i there is an open set V containing C_i and such that if $C_j \subset V$ for some $j \in I$, then $A_j \cap U \neq \emptyset \neq B_j \cap U$. To illustrate this last concept let us observe that all concentric circles with the center at a fixed point form a homogeneous family of strong separators of the plane $\mathbb{R}^2$. On the other hand, to see an example of a nonhomogeneous family of separators of the plane, put $L = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and, for an arbitrary positive integer n, let C_n denote the circle with center $(0, 2^{-n})$ and radius 2^{-n-2}. Then $\{L\} \cup \{C_n : n \in \mathbb{N}\}$ is the needed family.

A *continuum* means a compact connected metric space. A *dendrite* is defined as a locally connected continuum which contains no simple closed curve. Note that for locally connected continua this condition is equivalent to *hereditary unicoherence*, the property demanding that the intersection of any two subcontinua is connected (compare [8], (1.1), (v), p. 88).

Let m be a positive integer. A point p of a metric space X is said to be a *point of order m in X* (writing $\operatorname{ord}_p X = m$) provided there exist arbitrarily small neighbourhoods of p in X the boundary of each of which consists of m points and there are no such neighbourhoods whose boundaries consist of a less number of points. A point of order one is called an *end point* of the space. The set of all end points of a space X is denoted by $E(X)$. A point of order $m \geq 3$ is called a *ramification point*.

The reader is requested to follow Whyburn's book [8] for the definitions of other concepts used in the paper.

A. Acyclic spaces. In this section we deal with spaces which are homeomorphic to a connected subspace of a dendrite. Hence these spaces are *acyclic* in the sense that they contain no simple closed curve. For this class of spaces a negative answer to question (Q) can be seen by the following proposition.

PROPOSITION 1. *Let a nondegenerate dendrite X be different from an arc, and let $Y = X \setminus E(X)$. Then the family $\mathcal{F} = \{\{p\} : p \in Y\}$ of singletons of Y is a homogeneous family of strong separators of Y which is a partition of Y, has the continuum power, and for which there is no continuous function $f : Y \to \mathbb{R}$ satisfying condition (1).*

Proof. First note that Y is a connected subspace of the dendrite X. Hence Y is arcwise connected (see [8], (1.3), (ii), p. 89), and thus it has the continuum power. Second, observe that the considered family $\mathcal{F}$ of singletons of Y is a partition of Y. Third, note that each point of a dendrite X is either an end point of X or a cut point of X (see [8], (1.1), (ii), p. 88), whence it follows that if $p \in Y = X \setminus E(X)$, then $\{p\}$ is a strong separator of Y. Thus $\mathcal{F}$ is a family of strong separators of

Y. Since the decomposition of a dendrite into singletons is obviously continuous ([8], p. 129), the family $\mathcal{F}$ is homogeneous.

Let $f : Y \to f(Y)$ be any continuous function such that condition (1) is satisfied. Since the dendrite X is not an arc, there is a ramification point $r \in Y = X \setminus E(X)$. Thus r is a cut point of X and $X \setminus \{r\}$ has at least three components (see [8], (1.1), (ii) and (iv), p. 88) whose closures are subcontinua of X having r as the only common point. Hence there exist three arcs having r as their common end point, which are disjoint out of r and whose union is a triod $T \subset Y$. Since f is one-to-one, the partial mapping $f|T : T \to f(T) \subset f(Y)$ is a homeomorphism. Hence $f(Y)$ cannot be a subspace of $\mathbb{R}$. The proof is finished.

Recall that a continuum is called a *linear graph* provided it is the union of a finite set of points, called vertices, and of a finite number of open free arcs, called edges, such that two end points of an edge are distinct vertices. By a *generalized linear graph* we mean any connected subset of a linear graph. See [3], Theorem 1, p. 337, for some characterizations of these spaces, from which it follows that if a dendrite X has finitely many end points, then $X \setminus E(X)$ is an acyclic generalized linear graph, and inversely, any acyclic generalized linear graph admits a compactification which is a dendrite X with $E(X)$ finite (see (iii) of Theorem 1 of [3], p. 337). Thus as a consequence of Proposition 1 we have the following corollary.

COROLLARY 2. *Let an acyclic generalized linear graph Y be given which contains a ramification point and which has no end point. Then the decomposition of Y into singletons is a homogeneous family of strong separators, it has the continuum power and it admits no continuous function $f : Y \to \mathbb{R}$ satisfying condition (1).*

Now we give some positive results concerning a construction of special homogeneous families of separators of a dendrite each of which admits a continuous real-valued function satisfying condition (1).

Let a dendrite X be given, and let a and b be two distinct end points of X. Denote by ab the (unique) arc in X joining a with b. For each point $p \in ab$ let $C(p)$ mean the union of the singleton $\{p\}$ and of all components of $X \setminus \{p\}$ which contain neither a nor b. Observe that p belongs to the closure of each such component, and therefore $C(p)$ is a connected subset of X. Furthermore, since $X \setminus \{p\}$ is an open subset of a locally connected space X, its components are open ([8], (14.1), p. 20), whence it follows that $C(p)$ is a closed subset of X. Being both connected and closed, $C(p)$ is a subcontinuum of X. Recall that each subcontinuum of a dendrite is itself a dendrite (see [8], (1.3), (i), p. 89). So $C(p)$ is arcwise connected. Note that $C(a) = \{a\}$ and $C(b) = \{b\}$, and that $C(p) = \{p\}$ for $p \in ab \setminus \{a, b\}$ if and only if $\mathrm{ord}_p X = 2$. Since the set of all ramification points of a dendrite is at most countable ([8], (1.2), (iv), p. 89), the condition $\mathrm{ord}_p X = 2$ holds for all but at most countably many points p of the arc ab. Further, for each $p \in ab$ we have $C(p) \cap ab = \{p\}$.

Now let an arc $L \subset ab$ be given. Put $C(L) = \cup\{C(p) : p \in L\}$ and observe that $C(L)$ is a closed set for each arc L, just by the same argument used for $C(p)$; indeed, it is easy to check that $C(L)$ is the union of L and of all components of $X \setminus L$ which contain neither a nor b. Hence $C(L)$ is a subcontinuum of the dendrite X (and thus it is arcwise connected by the same argument as above). Consider an arbitrary monotone partition Π of the arc ab. Thus members of Π are either arcs (there are at most countably many of them) or single points of ab. Recall that, by the definition of a partition, members of Π are disjoint. A partition Π is said to be *nondegenerate* if it consists of more than one member. We denote by $L(p)$ the (unique) member of Π which contains the point p of the arc ab.

The following result holds true.

THEOREM 3. *Let a dendrite X with end points a and b be given. Then for each nondegenerate monotone partition Π of the arc ab the collection $D(\Pi)$ defined by*

$$D(\Pi) = \{C(L) : L \in \Pi\} \tag{2}$$

is an upper semicontinuous monotone decomposition of X such that

(3) $$F(\Pi) = \{C(L) : L \in \Pi\backslash\{L(a), L(b)\}\} \subset D(\Pi)$$

is a homogeneous family of strong separators of X which is a partition of $X\backslash(C(L(a)) \cup C(L(b)))$ of the continuum power, and for which a continuous function $f : X \to \mathbb{R}$ exists such that

(4) $$D(\Pi) = \{f^{-1}(t) : t \in f(X)\}.$$

In fact, $X/D(\Pi)$ with the quotient topology is an arc, so that such a function is just, up to a homeomorphism, the natural quotient mapping from X into $X/D(\Pi)$.

Proof. Observe that each element L of $\Pi\backslash\{L(a), L(b)\}$ separates the arc ab into two components. Thus, for each such L, the continuum $C(L)$ is a separator of X, and the open sets $A(L)$ and $B(L)$ associated to $C(L)$ are just the two components of $X\backslash C(L)$ containing the end points a and b respectively. Therefore $F(\Pi)$ is a family of strong separators of X (and of $X\backslash(C(L(a)) \cup C(L(b)))$ as well).

Since the dendrite X is hereditarily locally connected (i.e. each its subcontinuum is locally connected; see [8], (1.3), (i), p. 89) and since it can be embedded into the plane (cf. [5], §51, VI, (v), p. 300; compare [5], §61, I, Corollary 11, p. 509), the theorem of Gehman (see [5], §61, II, Theorem 13, p. 519) is applicable to X, from which it follows that each countable family of pairwise disjoint subcontinua of the dendrite X is a null-sequence. Since there are only countably many nondegenerate (i.e. distinct from singletons) members of $D(\Pi)$, the whole $D(\Pi)$ can be divided into two subcollections, one of which is a null-sequence and the other consists of single points. Now upper semicontinuity of $D(\Pi)$ follows from statement (1.11) of [8], p. 122.

As a consequence of upper semicontinuity of the decomposition $D(\Pi)$ we conclude that the quotient space $X/D(\Pi)$ (i.e., $D(\Pi)$ endowed with the quotient topology) is a (metric) continuum (see [8], Theorem (2.2), p. 123 and Corollary (3.11), p. 125). Now consider the natural quotient mapping $f : X \to f(X) = X\backslash D(\Pi)$ which by its definition shrinks each continuum $C(L(p))$, that obviously is an element of $D(\Pi)$, to a point. Of course f is both continuous and monotone. To describe its range space $f(X)$ observe that $f(X) = f(ab)$, which implies that the range spaces of f and of the partial mapping $f|ab$ are the same. Since a continuum X is hereditarily unicoherent if and only if for each monotone mapping $g : X \to g(X)$ and each subcontinuum K of X the partial mapping $g|K : K \to g(K) \subseteq g(X)$ is also monotone (see [2], Lemma 1, p. 932; compare [6], Corollary 3.2, p. 126), we conclude that $f|ab$ is a continuous monotone function defined on the arc ab, whence it follows that $f(ab)$ is either an *arc* or a point ([8], (1.1), p. 165). The latter possibility does not hold because the considered partition Π is nondegenerate. Thus $f(ab)$, and therefore $f(X)$, is an arc, and consequently the family $F(\Pi)$ has the continuum power.

Finally, to show that $F(\Pi)$ is homogeneous, fix an element L of $\Pi\backslash\{L(a), L(b)\}$ and let $A(L)$ and $B(L)$ be the two open sets associated to the separator $C(L) \in F(\Pi)$. Let U be an open and connected subset of X such that $U \cap A(L) \neq \emptyset \neq U \cap B(L)$. Note that the sets U, $A(L)$ and $B(L)$ are arcwise connected ([8], (5.3), p. 38). Pick two points $c \in U \cap A(L)$ and $d \in U \cap B(L)$, consider the unique (possibly degenerate) arcs $ac \subset A(L)$ and $bd \subset B(L)$, and denote by c' and d' the points in the arc ab such that $ac \cap ab = ac'$ and $bd \cap ab = bd'$. It follows that $cd = cc' \cup c'd' \cup d'd$. Since $cd \subset U$, we see that c' and d' are in U, so that

$$U \cap A(L) \cap ab \neq \emptyset \neq U \cap B(L) \cap ab. \tag{5}$$

Also, again by arcwise connectedness of U, we infer that the intersection $S = U \cap ab$ is a connected open subset of ab. Then it follows by (5) that $L \subset S$. Put $V = \cup\{C(p) : p \in S\}$. It is easy to show that V is an open subset of X. Obviously $L \subset V$. Moreover, if $C(L') \subset V$ for some $L' \in \Pi\backslash\{L(a), L(b)\}$, then we have $L' \subset S$, and it is easy to verify, as a consequence of this, that S, and hence V, meets both $A(L')$ and $B(L')$. The proof is then complete.

Remark 4. Note that the conclusion of Theorem 3 is no longer true if we omit the assumption of monotoneity of the partition Π.

This can be seen by the following example. Let X be the closed interval $[a,b]$ of reals. Take two distinct points u and v in $]a,b[$, and consider the partition Π of X that consists of all singletons $\{t\}$, where $t \in [a,b]\setminus\{u,v\}$ and of the set $\{u,v\}$. Then $D(\Pi) = \Pi$ and $F(\Pi) = \Pi\setminus\{\{a\},\{b\}\}$, and it is easy to recognize that no continuous function $f : Y =]a,b[\to \mathbb{R}$ exists such that $\{f^{-1}(t) : t \in f(Y)\} = F(\Pi)$.

Note that in this example $F(\Pi)$ is a homogeneous family of strong separators of $Y =]a,b[$. So we have another easy example which solves question (Q) in the negative, and which complements the negative answer to this question furnished by Proposition 1. In fact, in that proposition the negative answer is essentially due to the nature of the space, while in the example above it is just the nature of the family which works.

Taking in Theorem 3 the collection of singletons of the arc ab as the partition Π we get the following corollary.

COROLLARY 5. *Let a dendrite X with end points a and b be given. Then*

$$D_0 = \{C(p) : p \in ab\} \tag{6}$$

is an upper semicontinuous monotone decomposition of X such that

$$F = \{C(p) : p \in ab\setminus\{a,b\}\} \subset D_0 \tag{7}$$

is a homogeneous family of separators of X which is a partition of $X\setminus\{a,b\}$ of the continuum power. The function $f : X \to ab$ defined by

$$f^{-1}(p) = C(p) \quad \textit{for each} \quad p \in ab \tag{8}$$

is a continuous monotone retraction satisfying condition (1). In particular we have

$$f^{-1}(a) = \{a\} \quad \textit{and} \quad f^{-1}(b) = \{b\}. \tag{9}$$

Proof. We already know that the quotient space X/D_0 is an arc. Neglecting a homeomorphism between X/D_0 and the arc ab, we can treat $f : X \to ab$ as the quotient mapping. Now, only the new property of f, namely being a retraction, needs an argumentation. But it follows from the definition of f that the partial mapping $f|ab : ab \to ab$ is the identity. Finally (9) follows from (8) and the definition of $C(p)$.

Remark 6. A mapping from one space to another is said to be 1^0: *open*, if it maps open subsets of the domain onto open subsets of the range; 2^0: *inductively open*, if there is a subset of the domain such that the partial mapping restricted to this subset is open, and it maps the subset onto the whole range space ([1], Chapter 2, p. 234). Note that each retraction is an inductively open mapping. So, under the assumptions of the above corollary, f is inductively open. This fact is related in a way to Theorem 3.5 of [7], p. 232, where the conclusion of inductive opennes of f needs – unlike in our situation – an additional assumption, namely the nowhere density of the separators.

Consider now a family $\mathcal{F}$ of some decompositions of a given space X. Let two decompositions D_1 and D_2, both members of $\mathcal{F}$, be given. Then we write $D_1 \leq D_2$ if every element of D_1 is contained in some element of D_2, i.e., if D_1 refines D_2. Clearly $\leq$ defines a partial ordering on the family $\mathcal{F}$. A decomposition $D \in \mathcal{F}$ is called 1^0: the *smallest* member of $\mathcal{F}$, if D refines each other member of $\mathcal{F}$, and 2^0: a *minimal* member of $\mathcal{F}$, if no member of $\mathcal{F}$ properly refines D (compare [4], p. 32). Obviously, the smallest member of a given family is unique (when it exists), and if a member is the smallest one, then it is minimal. The inverse implication does not hold in general, as the reader can see by easy examples.

Let us recall that there is a one-to-one correspondence between upper semicontinuous monotone decompositions of a given continuum X and monotone mappings defined on X (see [8], Theorem (4.1), p. 127). Namely to an upper semicontinuous monotone decomposition D of a continuum X corresponds the quotient mapping $q : X \to X/D$

which is monotone; and to each monotone mapping $f : X \to f(X)$ corresponds the upper semicontinuous monotone decomposition $D = \{f^{-1}(y) : y \in f(X)\}$ of X for which f serves as the quotient mapping.

THEOREM 7. *Given a dendrite X, let $\mathcal{M}$ be the family of all upper semicontinuous monotone decompositions of X such that for each D in $\mathcal{M}$ the decomposition space X/D is an arc. Then for each two end points a and b of X the decomposition D_0 of X which corresponds to the quotient mapping f defined by (8) is a minimal member of $\mathcal{M}$.*

Furthermore, if $\mathcal{M}^* \subset \mathcal{M}$ denotes the family of all decompositions D in $\mathcal{M}$ for which

$$(10) \qquad X/D \quad \text{is an arc from } q(a) \text{ to } q(b),$$

where $q : X \to X/D$ is the natural quotient mapping, then the decomposition D_0 described above is the smallest member of $\mathcal{M}^*$.

Proof. Assume that a decomposition $D \in \mathcal{M}$ of X refines D_0. Thus if $q : X \to X/D$ is the natural quotient mapping that corresponds to D, then for each $t \in X/D$ there exists a point $p \in ab$ such that

$$(11) \qquad q^{-1}(t) \subset f^{-1}(p).$$

Note that the point p is uniquely determined by inclusion (11). In particular, for $t \in \{q(a), q(b)\} \subset X/D$ we have

$$a \in q^{-1}(q(a)) \subseteq f^{-1}(p) = C(p) \quad \text{for some} \quad p \in ab,$$

whence $p = a$. Therefore $q^{-1}(q(a)) = \{a\}$, and similarly $q^{-1}(q(b)) = \{b\}$. So $q(a)$ and $q(b)$ are distinct points of the arc X/D. It is easy to verify that they are end points of X/D. In fact, if not, then there is an end point y of X/D such that $q(a)\neq y\neq q(b)$. If $q(a) \in yq(b) \subset X/D$, take $x \in q^{-1}(y)$ and note that the arc xb does not contain the point a. Since the partial mapping $q|xb$ is monotone ([2], Lemma 1, p. 932), and since monotone mappings preserve end points of arcs ([8], (1.1), p. 165; cf. [5], §48, I, Theorem 3, p. 192), we have $q(xb) = yq(b)$.

Therefore the condition $q(a) \in yq(b)$ contradicts to $q^{-1}(q(a)) = \{a\}$. Similarly if $q(b) \in yq(a)$, we apply the same arguments to get a contradiction with $q^{-1}(q(b)) = \{b\}$. Therefore $X/D = q(a)q(b) = q(ab)$. So, for each $t \in X/D$ we have $ab \cap q^{-1}(t) \neq \emptyset$, whence $p \in q^{-1}(t)$ by (11).

Suppose by contradiction that for some $t \in X/D$ the inclusion in (11) is proper. Thus there exists a point $x \in f^{-1}(p) \setminus q^{-1}(t)$. Putting $t_0 = q(x) \in X/D \setminus \{t\}$ we see that $q^{-1}(t_0) \subset f^{-1}(p)$. Thus $p \in q^{-1}(t_0)$, and so $q^{-1}(t_0) \cap q^{-1}(t) \neq \emptyset$, a contradiction.

To prove the second part of the conclusion, take an arbitrary decomposition $D \in \mathcal{M}^*$, i.e. such that (10) holds. We have to show that $D_0 \leq D$, i.e., that for each point $p \in ab$ there is a point $t \in X/D = q(a)q(b)$ such that

$$f^{-1}(p) \subset q^{-1}(t). \tag{12}$$

With this end in view we define an auxiliary mapping $h : ab \to q(a)q(b)$ by $h = q|ab$. Note that h is monotone as a restriction of a monotone mapping q to a subcontinuum of the hereditarily unicoherent continuum X (see again Lemma 1 of [2], p. 932). We claim that $q = hf$. In fact, for a point $x \in X$ let $p = f(x) \in ab$. Then $h(f(x)) = h(p) = (q|ab)(p) = q(p)$. Further, $p \in ax \cap xb$. Since a monotone image of an arc is an arc and end points of the domain are mapped to end points of the range ([8], (1.1), p. 165), we conclude that $q(p) \in q(ax \cap xb) \subset q(ax) \cap q(xb) = q(a)q(x) \cap q(x)q(b) = \{q(x)\}$, whence the equality $h(f(x)) = q(x)$ follows for each x in X. So the claim is proved.

Now, given a point $p \in ab$, let us put $t = q(p) \in X/D$. To show (12) take a point $x \in f^{-1}(p)$. Thus $f(x) = p$ and we have $q(x) = h(f(x)) = h(p) = (q|ab)(p) = q(p) = t$. So $x \in q^{-1}(t)$. The proof is then complete.

Remark 8. We shall show that neither the former part of Theorem 7 can be sharpened replacing «minimal» by «the smallest» nor assumption (10) can be deleted from the latter part of the theorem. This can be seen by the following example. Consider a dendrite

$X = A \cup B \cup C$ in the plane $\mathbb{R}^2$, where

$$A = \{(x, 0) : x \in [0, 2]\},$$

$$B = \{(x, 1) : x \in [0, 2]\},$$

$$C = \{(1, y) : y \in [0, 1]\}.$$

Put $a = (0, 0)$ and $b = (2, 0)$, and let $f : X \to ab$ be given by formula (8), i.e.,

$$f(p) = p \quad \text{for} \quad p \in A \quad \text{and} \quad f(B \cup C) = \{(1, 1)\}.$$

Now observe that another monotone mapping $g : X \to ab$ satisfying condition (1) is given by

$$g((x, 1)) = (x, 0) \quad \text{if} \quad (x, 1) \in B \quad \text{and} \quad g(A \cup C) = \{(1, 0)\}.$$

Since $g(a) = g(b) = (1, 0)$, we see that the decomposition $D = \{g^{-1}(p) : p \in ab\}$ does not satisfy (10). Further, the decomposition $D_0 = \{f^{-1}(p) : p \in ab\}$ does not refine D because $f^{-1}((1, 0)) \in D_0$ is contained in no element of D, and thereby D_0 is not the smallest member of the family $\mathcal{M}$.

It is natural to ask if a converse to Theorem 3 is true in the following sense.

Question 9. Assume D^* is an upper semicontinuous monotone decomposition of a dendrite X which has the following two properties.

(13) D^* contains a homogeneous family $F^* = \{C_i : i \in I\}$ of separators of X;

(14) there exists a continuous function $f : X \to \mathbb{R}$ for which condition (1) holds true.

Do there exist an arc ab in X with $a, b \in E(X)$ and a partition Π of this arc such that $D^* = D(\Pi)$ and $F^* = F(\Pi)$ according to (2) and (3)?

B. Cyclic spaces. The question (Q) recalled here in the very beginning of the paper has a negative answer not only in the case when Y is an acyclic space (see Proposition 1 and Corollary 2), but also if Y is cyclic, e.g. a simple closed curve. However, since no singleton disconnects a simple closed curve X, the separators of X belonging to the family $\mathcal{F}$ cannot be chosen as simple as it was done in Proposition 1 and Corollary 2. But it is shown below that for each integer $k \geq 2$ there is a suitable family of separators, each one consisting of exactly k points, for which no continuous function $f : X \to \mathbb{R}$ exists satisfying equality (1).

Let an integer $k \geq 2$ be fixed. To each point z of the unit circle $X = \{z : |z| = 1\}$ in the complex plane we assign a subset $C(z) = \{z_0, z_1, \dots, z_{k-1}\}$ of X consisting of k points such that $z_0 = z$ and that these points divide X into k equal parts. In other words we have $\arg z_j = \arg z + 2\pi j/k$ for each $j \in \{0, 1, \dots, k-1\}$. Obviously each $C(z)$ is a strong separator of X. It is readily seen that the family

$$\mathcal{F} = \{C(z) : z \in X\} \tag{15}$$

is homogeneous, and is a continuous decomposition of X whose decomposition space is again X. In fact, the quotient mapping $f : X \to f(X)$, which is uniquely determined by the condition $f^{-1}(y) = C(z)$ for each $y \in f(X)$ and some $z \in X$, wraps the domain space X just k times onto itself, so that we can assume $f(X) = X$ without loss of generality. But obviously this range space $f(X)$, being topologically a simple closed curve, cannot be embedded into the reals. So there is no possibility of defining a continuous function f from X into $\mathbb{R}$ satisfying condition (1). Thus we have proved the following proposition.

PROPOSITION 10. *For each integer $k \geq 2$ the family $\mathcal{F}$ defined by (15) is a homogeneous family of strong separators of the unit circle X for which there is no continuous function f from X into $\mathbb{R}$ that satisfies (1).*

The reader can certainly find other examples of cyclic graphs X

and homogeneous families of (finite) strong separators of X for which the conclusion of Proposition 10 holds. On the other hand, it is readily seen that for the projection f of the unit circle $X = \{z : |z| = 1\}$ onto its diameter $\{z = x + iy : x \in [-1, 1] \text{ and } y = 0\}$ defined by $f(z) = x$, the family

$$\mathcal{F} = \{f^{-1}(x) : x \in]-1, 1[\}$$

is a homogeneous family of two-point separators of X, and that f satisfies (1). Thus it is natural to ask about a characterization of these homogeneous families $\mathcal{F}$ of (strong) separators of a graph, a cyclic graph, or – in general – a locally connected metric space X which admit a continuous function $f : X \to \mathbb{R}$ such that $\mathcal{F}$ coincides with the family

$$\{f^{-1}(t) : t \in f(X)\}$$

or with the family

$$\{f^{-1}(t) : t \in f(X) \setminus \{\inf f(X), \sup f(X)\}\}.$$

C. Final remarks. As the reader had certainly observed, the families $\mathcal{F}$ of separators of spaces X considered in sections A and B of the paper were not only homogeneous, but also either continuous or (at least) upper semicontinuous. It was so because the definitions of homogeneity and of upper semicontinuity of a given family are formulated in a rather similar way. However, they are not identical. Thus the following questions seem to be of some interest.

Questions 11. What are relations between homogeneity of a given family of separators of a space and (either lower or upper) semicontinuity of this family? Under what conditions (regarding the space as well as the family) one of these properties implies the other? In particular, does lower semicontinuity of a decomposition D of a connected and locally connected space into separators imply homogeneity of D?

REFERENCES

[1] Arhangel'skii A.V., *Open and near open mappings. Connections between spaces,* Trans. Moscow Math. Soc. **15** (1966), 204-294.

[2] Charatonik J.J., *Fixed point property for monotone mappings of hereditarily stratified λ-dendroids,* Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. **16** (1968), 931-936.

[3] Charatonik J.J., Miklos S., *Generalized graphs and their open mappings,* Rend. Mat. (7) **2** (1982), 335-354.

[4] Kelley J.L., *General topology,* van Nostrand 1953.

[5] Kuratowki K., *Topology,* vol. 2, Academic Press and PWN 1968.

[6] Maćkowiak T., *The hereditary classes of mappings,* Fund. Math. **97** (1977) 123-150.

[7] Ricceri B., *On multiselections,* Le Matematiche **38** (1983), 221-235.

[8] Whyburn G.T., *Analytic topology,* Amer. Math. Soc. Colloq. Publ. vol. **28** (1942).

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