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CONSTRUCTION OF SMOOTH CURVES OF $\mathbb{P}^3$ WITH ASSIGNED HILBERT FUNCTION AND GENERATORS' DEGREES

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The possible degrees for the generators of an irreducible arithmetically Cohen Macaulay curve of $\mathbb{P}^3$ are computed in terms of its Hilbert function. For any set of admissible degrees a smooth curve of this type, generated in those degrees is constructed.

1. Introduction.

Let H be a differentiable 0-dimensional 0-sequence, i.e. a set of positive integers which are the Hilbert function of a 0-dimensional variety of $\mathbb{P}^n$. In his paper (cfr. [1]) Campanella finds what are the possible degrees for any minimal set of genertors of a 0-dimensional variety which has H as Hilbert function.

Here we start with computing the admissible degrees for the

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generators of an irreducible arithmetically Cohen Macaulay curve of $\mathbb{P}^3$, obtaining a bound on the number of generators for its defining ideal (a Dubreil-type theorem).

Then we construct a special type of curves which are the join of suitable lines, called «partial intersection». We can build these curves for any assigned Hilbert function and any set of degrees for the generators according to Campanella's bounds; moreover they contain only singularities which allow the embedding into a smooth surface of «low» degree (cfr. [8]).

Using partial intersections, «liaison» techniques and Bertini-type theorems we can get a smooth irreducible curve for any assigned Hilbert function and any set of degrees for the generators according to the bounds previously found.

Finally, when a curve has the Hilbert function of a complete interesection, we prove that it is a complete intersection in the irreducible case and give a complete characterization in the arithmetically Cohen-Macaulay case.

Notation and basic facts.

Throughout the paper $\mathbb{P} = \mathbb{P}^n_k$ will denote the projective space over an algebraically closed field k of characteristic zero.

A «variety» $V \subset \mathbb{P}^n$ will mean a closed subscheme of pure dimension d with an assigned embedding; when $d = 1$ V will be called curve. Thus, $V = \mathrm{Proj}(S/I(V))$ where $S = k[x_0, \dots, x_n]$ and $I(V)$ is a homogeneous ideal of S.

When $S/I(V)$ is a Cohen-Macaulay ring V is called arithmetically Cohen-Macaulay (briefly ACM); for conditions equivalent to ACM see [5]. When V is the complete intersection of two hypersurfaces of degree a and b we will say that V is $CI(a, b)$ (of codimension 2).

We denote by $HF(V, i)$ the Hilbert function of V, i.e. $HF(V, i) = \dim_k (S/I(V))_i$, where $(S/I(V))_i$ is the i-th graded component of $S/I(V)$. When V has the Hilbert function of a complete intersection of type (a, b) we will write $HFCI(a, b)$ for $HF(V)$.

We will denote by $B_l(V)$ the k-vector space of the hypersurfaces of degree l vanishing on V and $Z_l(V)$ the base-locus of $B_l(V)$; when $\dim Z_l(V) > 0$, Γ_l will be the positive dimension part of $Z_l(V)$ and when this dimension is 1 we will refer to it as the «fixed curve».

For a zero-dimensional reduced subscheme $X \subset \mathbb{P}^2$ we use the following terminology:

$$h_i = HF(X, i) \qquad c_i = \Delta HF(X, i) = h_i - h_{i-1}$$

$$d_i = \Delta^2 HF(X, i) = c_i - c_{i-1} \quad e_i = \Delta^3 HF(X, i) = d_i - d_{i-1}$$

for $i \geq 1$, and $h_0 = c_0 = d_0 = e_0 = 1$;

$$a_1 = \min\{i \in \mathbb{N} | d_i \leq 0\}$$

$$a_2 = \min\{i \in \mathbb{N}\} | d_i < 0\}$$

$$t = \min\{i \in \mathbb{N} | c_{i+1} = 0\}.$$

Note that $\sum_{i=0}^{t+1} d_i = 0$ and $d_i = 1$ for $0 \leq i < a_1$.

We will say that a set of points $X \subset \mathbb{P}^2$ has the Uniform Position Property (UPP for short, see [6]) if for every subset $X' \subset X$:

$$HF(X', i) = \min\{|X'|,\ HF(X, i)\}$$

where $|X'|$ is the cardinality of the set X'. For equivalent forms of this property see [7] Proposition 1.1.

Recall that, by Harris' Uniform Position Lemma, the generic plane section of a reduced irreducible curve $C \subset \mathbb{P}^3$ has the UPP.

1. Let $X \subset \mathbb{P}^2$ be a finite set of distinct points; every minimal set of generators of $I(X)$ has a number of elements, in each degree, bounded in terms of the Hilbert function $H = \{h_i\}$ of X. More precisely, let

$$(1) \qquad 0 \to \sum_{i=1}^{m-1} O_{\mathbb{P}}(-b_i) \to \sum_{i=1}^{m} O_{\mathbb{P}}(-a_i) \to \mathcal{J}(X) \to 0$$

be a minimal free resolution of the ideal sheaf $\mathcal{J}(X)$ of X, where $a_1 \leq a_2 \leq \ldots \leq a_m$ are the degrees of a minimal set of generators of $I(X)$ and $b_1 \leq b_2 \leq \ldots \leq b_{m-1}$ the degrees of a minimal set of generators for the syzygy module. Call α_j (resp. β_j) the number of the integers a_i (resp. b_i) equal to j; then the Hilbert function H of X forces any resolution of type (1) to have at least $-e_i$ generators of degree i, if $e_i < 0$, and at least e_i syzygies of degree i, if $e_i > 0$.

LEMMA 1.1. *With the above notation, if $X \subset \mathbb{P}^2$ is any finite set of points with minimal resolution (1), then $\beta_j = \alpha_j + e_j$ for every $j \in \mathbb{N}^+$.*

Proof. Let i_0 be the first integer for which the conclusion is false. Then $\beta_j = \alpha_j + e_j$ for $0 < j < i_0$, and computing $H(X, i_0)$ from (1) we get:

$$H(X, i_0) = \binom{i_0+2}{2} - \sum_{j=1}^{i_0} \alpha_j \binom{i_0+2-j}{2} + \sum_{j=1}^{i_0} \beta_j \binom{i_0+2-j}{2} =$$
$$= \sum_{j=0}^{i_0-1} e_j \binom{i_0+2-j}{2} + \beta_{i_0} - \alpha_{i_0} \neq \sum_{j=0}^{i_0} e_j \binom{i_0+2-j}{2}$$

the last inequality for the assumption on i_0; we have a contradiction since a simple computation shows that for every $i \in \mathbb{N}^+$

$$h_i = H(X, i) = \sum_{j=0}^{i} e_j \binom{i+2-j}{2}.$$

THEOREM 1.2. *Let $X \subset \mathbb{P}^2$ be a finite set of points with Hilbert function $H = \{h_i\}$, and minimal free resolution for its ideal sheaf $\mathcal{J}(X)$ of type (1).*

Then the following hold:

$$\alpha_{a_1} = -e_{a_1}, \alpha_{a_2} = -e_{a_2}, \ \max\{0, -e_i\} \leq \alpha_i \leq -d_i \quad \textit{for } i > a_2.$$

Proof. It can be deduced from [1], Theorem 2.1.

Remark 1.3. The theorem gives also the range for the numbers β_i, according with Lemma 1.1, precisely we have, for $i > a_2$:

$$\max\{0, e_i\} \leq \beta_i \leq e_i - d_i$$

For a given set $X \subset \mathbb{P}^2$ it can happen that $X \subseteq CI(a_1, a_2)$, a_1 and a_2 as defined in notation; this is the case for set of points with the UPP, or, in particular, for the generic plane sections of a reduced irreducible curve of $\mathbb{P}^3$ ([8], Prop. 1); then we can give better bounds for the numbers α_i, as stated in the following proposition.

PROPOSITION 1.4. *Let $X \subset \mathbb{P}^2$ be a finite set of distinct points with Hilbert function $H = \{h_i\}$ and minimal free resolution for $\mathcal{J}(X)$ of type (1). If $X \subseteq CI(a_1, a_2)$ then*

$$\alpha_{a_1} = -e_{a_1}, \alpha_{a_2} = -e_{a_2}, \quad \textit{and } \max\{0, -e_i\} \leq \alpha_i \leq -d_i - 1 \quad \textit{for } i > a_2.$$

Proof. Let $\bar{H} = \{\bar{h}_i\}$ be the Hilbert function of a $CI(a_1, a_2)$; we have, for the differences $\bar{c}_i$:

$$\Delta\bar{H}(i) = \bar{c}_i = \begin{cases} i+1 & \text{for } 0 \leq i < a_1 \\ a_1 & \text{for } a_1 \leq i < a_2 \\ L - i - 1 & \text{for } a_2 \leq i \leq L-2 \\ 0 & \text{for } i > L-2 \end{cases}$$

where $L = a_1 + a_2$.

Let $X' \subset \mathbb{P}^2$ be the set of points linked to X in a $CI(a_1, a_2)$; for the successive differences of the Hilbert function $H' = \{h'_i\}$ of X' it is ([3] «Hilbert function under liaison» Theorem):

$$\Delta H'(i) = c'_i = \begin{cases} i+1 & \text{for } 0 \leq i < L-2-t \\ i+1 - c_{L-i-2} & \text{for } L-2-t \leq i < a_1 - 1 \\ 0 & \text{for } i \geq a_1 - 1 \end{cases}$$

$$\Delta^2 H'(i) = d'_i = \begin{cases} 1 & \text{for } 0 \leq i < L-2-t \\ 1 + d_{L-i-1} & \text{for } L-2-t \leq i \leq a_1 - 1 \\ 0 & \text{for } i \geq a_1 \end{cases}$$

$$\Delta^3 H'(i) = e'_i = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } 0 < i < L-2-t \\ -e_{L-i} & \text{for } L-2-t \leq i \leq a_1 - 1 \\ -1 - d_{L-i-1} & \text{for } i = a_1 \\ 0 & \text{for } i > a_1 \end{cases}$$

By [9], §3, the degrees x_i, y_i of the generators and syzygies of the ideal $I(X')$ are:

$$x_i = L - b_{m-i} \quad i = 1, 2, \ldots, m-1$$

$$y_i = L - a_{m-i} \quad i = 0, 1, \ldots, m-3$$

Call α'_i and β'_i the number of generators and syzygies of degree i of $I(X')$. By definition:

$$\alpha_i = \beta'_{L-i}, i = 0, 1, \ldots, m-3.$$

On the other hand, by remark 1.3.:

$$\max\{0, e'_j\} \leq \beta'_j \leq e'_j - d'_j \text{ for } L-2-t < j < a_1$$

hence, for $j = L - i$:

$$\max\{0, -e_i\} \leq \alpha_i \leq -e_i - 1 - d_{i-1} = -1 - d_i \text{ for } i > a_2.$$

The first statement follows directly from Theorem 1.2.

When the set X is the generic plane section of an ACM curve $C \subset \mathbb{P}^3$ it is well known that $HF(X,i) = \Delta HF(C,i)$ for every i, hence the preceding results can apply easily to curves.

THEOREM 1.5. *Let $C \subset \mathbb{P}^3$ be a reduced irreducible ACM curve and X a generic plane section of C with Hilbert function $H = \{h_i\}$.*

Then the numbers α_i of elements of degree i in any minimal system of generators of $I(C)$ satisfy:

$$\alpha_{a_1} = -e_{a_1}, \alpha_{a_2} = -e_{a_2}; \max\{0, -e_j\} \leq \alpha_i \leq -d_i - 1 \textit{ for } i > a_2.$$

Proof. By Harris' Lemma the set X has the UPP; hence we can apply Prop. 1 of [8] to obtain $X \subseteq CI(a_1, a_2)$. Now the theorem follows from Prop. 1.4 and the just recalled property of ACM curves.

COROLLARY 1.6. *With the same assumptions of Theorem 1.5, denoting by $\nu(I(C))$ the number of elements of a minimal set of generators of $I(C)$, we have $\nu(I(C)) \leq a_1 + a_2 - t$.*

Proof. Using Theorem 1.5. we have:

$$\nu(I(C)) \leq \sum_{i=a_2+1}^{t+1} (-d_i - 1) + 1 - d_{a_1} - d_{a_2} = \sum_{i=a_1}^{t+1} (-d_i) + 1 - (t + 1 - a_2) =$$
$$= a_1 + 1 - (t + 1 - a_2) = a_1 + a_2 - t.$$

Remark 1.7. This result generalizes Dubreil's Theorem (see, for instance, [2]), since $t \geq a_2 - 1$.

COROLLARY 1.8. *With the same assumptions of Theorem 1.5, if H is the Hilbert function of a $CI(a_1, a_2)$ then C is a $CI(a_1, a_2)$.*

Proof. By the assumption on H, $t = a_1 + a_2 - 2$, hence, by Corollary 1.6., $\nu(I(C)) = 2$.

2. Let $H = \{h_i\}$, $h_1 \leq 3$, be a differentiable 0-dimensional 0-sequence (see [10]), i.e. a sequence of positive integers such that there exists a set of point of $\mathbb{P}^2$ whose Hilbert function is just H (cf. [4] Cor. 3.4). Let us consider numbers $a_1 \leq a_2 \leq \ldots \leq a_m$ satisfying the conditions of Theorem 1.2 and define inductively

$$b_1 = \min \left\{ l \in \mathbb{N} : \binom{l+2}{2} - \sum_{i=1}^{d(l)} \binom{l - a_i + 2}{2} \neq h_l \right\}$$

and

$$b_k = \min\left\{ l \in \mathbb{N}; l \geq b_{k-1} : \binom{l+2}{2} - \right.$$
$$\left. - \sum_{i=1}^{d(l)} \binom{l-a_i+2}{2} + \sum_{j=1}^{k-1} \binom{l-b_j+2}{2} \neq h_l \right\}$$

where $d(l)$ is such that $a_{d(l)} \leq l < a_{d(l)+1}$.

Note that the numbers $\{b_i\}$ satisfy the conditions of Remark 1.3.

It is known that there exist perfect ideals of height two in $k[x_0, x_1]$ generated by polynomials of degree $a_1, \dots, a_m$ (see [1], Theorem 3.3).

We want to construct explicitly an ACM curve such that its generic plane section X has an ideal with minimal free resolution of type (1).

We will use a suitable set of connected lines of $\mathbb{P}^3$.

First we define, beginning from $\{a_i\}$ and $\{b_i\}$, the following two $(m-1)$-tuples of natural numbers:

$$\mathbf{p} = (p_1, \dots, p_{m-1}); \ \mathbf{q} = (q_1, \dots, q_{m-1}),$$

where

$$q_i = b_{m-i} - a_{m-i+1}, \ i = 1, 2, \dots, m-1$$

$$p_i = a_{m-i+1} - \sum_{j=1}^{i-1} q_j, \ i = 1, 2, \dots, m-1.$$

Note that $\sum_{i=1}^{m-1} q_i = \sum_{i=1}^{m-1} b_i - \sum_{i=2}^{m} a_i = a_1$ because $\sum_{i=1}^{m-1} b_i = \sum_{i=1}^{m} a_i$.

Furthermore it is $q_i > 0$ for each i: if it were $b_j = a_{j+1}$ for some j, since, by the minimality of the resolution (1), a syzygy of degree a_j cannot involve generators of the same degree, one would have j independent syzygies, of degree $b_1, \dots, b_j$, on at most j generators, of degree $a_1, \dots, a_j$. This is impossible for a matter of rank (every ideal of $k[x_0, x_1, x_2]$ has a minimal free resolution of length 2).

Finally it is $p_1 > p_2 > \ldots > p_{m-1}$; more precisely, for every $i = 1, 2, \ldots, m-2$:

$$p_i - p_{i+1} = a_{m-i+1} - \sum_{j=1}^{i-1} q_j - a_{m-i} + \sum_{j=1}^{i} q_j = a_{m-i+1} - a_{m-i} + q_i \geq q_i$$

since $a_{m-i} \leq a_{m-i+1}$.

Take now two sets of planes in $\mathbb{P}^3$, $\{L_i\}$ and $\{R_j\}$, $i = 1, 2, \ldots, a_1$, $j = 1, 2, \ldots, a_m$, with the following properties: $\bigcap_{i=1}^{a_1} L_i$ is a line l, $R_i \cap R_j \cap R_k$ is a point for every i, j, k and $l \cap R_i \cap R_j = \emptyset$ for every i, j. These conditions on the planes are given in order that the curve that we are going to construct have only plane singularities: this will allow us, in the next section, to find a smooth surface of a «low» degree containing the curve.

We put $l_{ij} = L_i \cap R_j$ and consider the curve C consisting of the lines $l_{ij(i)}$ with $i = 1, 2, \ldots, a_1$, $j(i) = 1, 2, \ldots, p_{r(i)}$, where $r(i) = \min\{s \in \mathbb{N} : \sum_{j=1}^{s} q_j \geq i\}$ (we have divided the planes L_i into $m-1$ groups of $q_1, \ldots, q_{m-1}$ planes respectively; the number $r(i)$ is the index of the «q» of the group which L_i belongs).

We will call the curve C just constructed a «partial intersection» of type $(\mathbf{p}, \mathbf{q})$. To prove that C is an ACM curve with the sequence H for Hilbert function we verify that its ideal sheaf has a free minimal resolution of length two.

PROPOSITION 2.1. *Any partial intersection C of type $(\mathbf{p}, \mathbf{q})$, constructed as above, is an ACM curve with a minimal free resolution for its ideal sheaf $\mathcal{I}(C)$:*

$$(2) \qquad 0 \to \sum_{i=1}^{m-1} \mathcal{O}_{\mathbb{P}}(-b_i) \to \sum_{i=1}^{m} \mathcal{O}_{\mathbb{P}}(-a_i) \to \mathcal{I}(C) \to 0.$$

In particular $HF(C, i) = h_i$ for every i.

Proof. By induction on the length $m-1$ of p or, equivalently, on the number m of generators.

If $m = 2$ C is a complete intersection of type $(q_1, p_1) = (a_1, a_m)$ and the proposition is true.

Suppose that the result is true for partial intersections of type $(\mathbf{p}', \mathbf{q}')$ with length of p' less than $m - 1$.

Call C' the partial intersection linked to C in the complete intersection $Y = \left(\bigcup_{i=1}^{a_1} L_i\right) \cap \left(\bigcup_{j=1}^{a_m} R_j\right)$; C' will consist of the lines $l_{i,j(i)}$ with $i = q_1 + 1, \ldots, a_1$ and $j(i) = p_{r(i)} + 1, \ldots, p_1$.

Renumber the planes as follows:

$$L'_i = L_{a_1+1-i}, \; i = 1, \ldots, a_1 - q_1$$

$$R'_j = R_{a_m+1-j}, \; j = 1, \ldots, a_m - p_{m-1}$$

and denote: $p'_i = p_1 - p_{m-i}$; $q'_i = q_{m-i}$, $i = 1, \ldots, m-2$.

With these new indices one sees that C' is a partial intersection of type $(\mathbf{p}', \mathbf{q}')$ with $\mathbf{p}' = (p'_1, \ldots, p'_{m-2})$ and $\mathbf{q}' = (q'_1, \ldots, q'_{m-2})$. By the inductive hypothesis C' has a free resolution with degrees:

$$a'_1 = \sum_{j=1}^{m-2} q'_j,$$

$$a'_i = p'_{m-i} + \sum_{j=1}^{m-1-i} q'_j, \; i = 2, \ldots, m-2,$$

$$b'_i = p'_{m-1-i} + \sum_{j=1}^{m-1-i} q'_j, \; i = 1, 2, \ldots, m-2.$$

We use now the relations among the degrees of generators and syzygies of two linked ACM curves ([9] §3) to conclude the proof:

$$\begin{aligned} a_1 + a_m - b'_{m-i} &= a_1 + a_m - p'_{i-1} - \sum_{j=1}^{i-1} q'_j = \\ &= a_1 + a_m - p_1 + p_{m-i+1} - \sum_{j=1}^{i-1} q_{m-j} = \\ &= p_{m-i+1} + \sum_{j=1}^{m-i} q_j = a_i \quad \text{for } i = 2, \ldots, m-1. \end{aligned}$$

$$a_1 + a_m - a'_{m-i} = a_1 + a_m - p'_i - \sum_{j=1}^{i-1} q'_j = a_1 + a_m - p_1 + p_{m-i} - \sum_{j=1}^{i-1} q_{m-j} =$$
$$= p_{m-i} + \sum_{j=1}^{m-i} q_j = b_i \quad \text{for } i = 1, \dots, m-1.$$

3. Let be given a 0-dimensional differentiable 0-sequence $H = \{h_i\}$, $h_1 \leq 3$; we call H of «decreasing type» if $a_1 > c_{a_2} > \dots > c_t > 0$ (cf. [8]).

We recall that a smooth irreducible ACM curve of $\mathbb{P}^3$ has the Hilbert function of its generic plane section of decreasing type (cf. [8] Cor. 2).

In the same paper it was proved that for any differentiable 0-dimensional 0-sequence H of decreasing type there exists an ACM smooth irreducible curve of $\mathbb{P}^3$ whose generic plane section has H as Hilbert function. Here, we want to show that for such a sequence and for any set of positive integers $a_1 \leq \dots \leq a_m$ satisfying the bounds of Prop. 1.4, there exists an irreducible smooth ACM curve C with a minimal free resolution of type (2), where the b_i's are obtained from H and $\{a_i\}$ as we said in n. 2.

THEOREM 3.1. *Let H be a 0-dimensional differentiable 0-sequence of decreasing type; $\{a_1, \dots, a_m\}$ integers satisfying the conditions of Prop. 1.4. Then there exists a smooth irreducible ACM curve C with minimal free resolution of type (2).*

Proof. Let H' be the 0-sequence linked to H in a $CI(a_1, a_2)$; the integers $a'_i = a_1 + a_2 - b_i$, $i = 1, \dots, m-1$, satisfy the conditions of Theorem 1.2 by the assumption on the integers $\{a_i\}$. We now construct, using H' and the integers $\{a'_i\}$ a partial intersection C' of type $(\mathbf{p}', \mathbf{q}')$ as we did in n. 2.

In this way any curve linked to C' in a $CI(a_1, a_2)$ will be generated by elements with the assigned degrees $\{a_i\}$. We get now the conclusion just using the same arguments as in the proof of

Theorem 4 in [8].

Now we study the particular case of curves which have the Hilbert function of a complete intersection.

PROPOSITION 3.2. *Let $C \subset \mathbb{P}^3$ be a reduced irreducible curve such that $HF(C) = HFCI(a,b)$. Then C is a $CI(a,b)$.*

Proof. Since $HF(C) = HFCI(a,b)$, deg $C = ab$; if X is a generic plane section of C, $|X|$ =deg C.

We know that $\Delta HF(C,i) \geq HF(X,i)$ for every i (see [5]); if it were $\Delta HF(C,i) > HF(X,i)$ for some i, since $HF(X)$ is of decreasing type (X has the UPP) it would be $|X|$ =deg$C < ab$. Hence C is an ACM curve; now the conclusion follows from Cor. 1.8.

Recall now the following: if $C \subset \mathbb{P}^3$ is an ACM reduced curve with $HF(C) = HFCI(a,b)$, then the degrees of a minimal set of generators of $I(C)$ allowed by Theorem 1.2 are $\{a, b, \varepsilon_1, \ldots, \varepsilon_r\}$ where $\varepsilon_1 < \ldots < \varepsilon_r$ is any subset of $\{b+1, \ldots, a+b-1\}$.

THEOREM 3.3. *Let $C \subset \mathbb{P}^3$ be a reduced ACM curve such that $HF(C) = HFCI(a,b)$. If a minimal set of generators of $I(C)$ has elements of degrees $a, b, \varepsilon_1, \ldots, \varepsilon_r$, with $a \leq b = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_r < a+b$ then all the surfaces in $B_{\varepsilon_i}(C)$, $i = 0, 1, \ldots, r-1$, split in a fixed component S_i of degree $a+b-\varepsilon_{i+1}$; furthermore dim$(S_i \cap C) = 1$ and deg$(S_i \cap C) = \varepsilon_{i+1}(a+b-\varepsilon_{i+1})$, $i = 0, 1, \ldots, r-1$.*

Proof. Let X be a generic plane section of C. Observe that, since C is an ACM curve, the base locus of $B_{\varepsilon_i}(C)$ contains a surface S_i of degree $a+b-\varepsilon_{i+1}$ if and only if $B_{\varepsilon_i}(X)$ has base locus containing a fixed curve Γ_i of degree $a+b-\varepsilon_{i+1}$; similarly dim$(S_i \cap C) = 1$ and deg$(S_i \cap C) = \varepsilon_{i+1}(a+b-\varepsilon_{i+1})$ if and only if the set $X \cap \Gamma_i$ has $\varepsilon_{i+1}(a+b-\varepsilon_{i+1})$ points.

So it will be sufficient to prove the analogous theorem for the section X.

PROPOSITION 3.4. *Let $X \subset \mathbb{P}^2$ be a set of ab distinct points with*

$HF(X) = HFCI(a,b)$. If $I(X)$ has a minimal set of generators with elements of degrees $a, b, \varepsilon_1, \ldots, \varepsilon_r$, with $a \leq b = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_r < a+b$, then $B_{\varepsilon_i}(X)$ has base locus of positive dimension, $i = 0, 1, \ldots, r-1$, its fixed curve Γ_i has degree $deg\Gamma_i = a+b-\varepsilon_{i+1}$ and $|X \cap \Gamma_i| = \varepsilon_{i+1}(a+b-\varepsilon_{i+1})$.

Proof. First note that if $\dim Z_s(X) = 0$, $s \geq b$, then $Z_s(X) = X$; in fact, by [7] Prop. 2.1:

$$|X| \leq |Z_s(X)| \leq \binom{s+1}{2} - \left[\binom{s+1}{2} - ab + \binom{a+b-s}{2}\right] + \\ + \binom{a+b-s}{2} = ab.$$

On the other hand, $\dim Z_s(X) > 0$ for any s such that $b \leq s < \varepsilon_r$: otherwise X would be s-free (i.e. $Z_s(X) = X$) and, since $I(X)$ is reduced, by Hilbert's Nullstenllensatz every polynomial of $B_l(X)$, $l \geq s$, could be expressed by means of generator of degree $\leq s$: this contradicts the existence of a generator of degree $\varepsilon_r > s$. Thus, we see that ε_r is the lowest degree s such that $\dim Z_s(X) = 0$. One more remark: for $\varepsilon_i < s < \varepsilon_{i+1}$, $i = 0, 1, \ldots, r-1$, it is $\Gamma_s = \Gamma_i$; in fact, by assumption $I(X)$ has no generators of degree s, hence the elements of $B_s(X)$ can be expressed in terms of generators of degree $< \varepsilon_{i+1}$ all of which contain as common factor the polynomial defining Γ_i. So we have $\Gamma_s = \Gamma_i$.

After these introductory remarks we apply Remark 2.3 ii) of [7] to get $|X_s| \leq x_s(a+b-x_s)$, where $X_s = X \cap \Gamma_s$ and $x_s = \deg\Gamma_s$.

From the same paper, Prop. 2.4,2), we have $x_i \leq a+b-\varepsilon_{i+1}$.

Let us consider now Γ_{r-1}, the fixed curve of $B_{\varepsilon_{r-1}}(X)$; call $Y_{r-1} = X - X \cap \Gamma_{r-1}$. We have:

$$\dim B_l(X) = \dim B_{l-x_{r-1}}(Y_{r-1}) \quad \text{for } x_{r-1} \leq l < \varepsilon_r.$$

Hence, in this range, the set Y_{r-1} has the Hilbert function of $CI(a-x_{r-1}, b-x_{r-1})$. Note that for $l \geq \varepsilon_r$ $\Delta HF(Y_{r-1})$ decreases: otherwise $B_{l-x_{r-1}}(Y_{r-1})$ would have base locus of positive dimension for some $l \geq \varepsilon_r$ (see [7] Theorem 2.9); call $\bar{\Gamma}$ its fixed curve. In this case the base locus of $B_{\varepsilon_r - x_{r-1}-1}(Y_{r-1})$ would contain $\bar{\Gamma}$ and this is in contradiction with the fact that $Z_{\varepsilon_r-1}(X)$ has base locus Γ_{r-1} and consequently $Z_{\varepsilon_r-1-x_{r-1}}(Y_{r-1})$ has dimension zero. So we have

$$|Y_{r-1}| \leq (a - x_{r-1})(b - x_{r-1})$$

and since $|X_{r-1}| \leq x_{r-1}(a + b - x_{r-1})$ equality holds in both cases.

Since $\dim Z_{\varepsilon_r}(X) = 0$, we have $|X_{r-1}| \leq x_{r-1}\varepsilon_r$; by substitution we get: $x_{r-1}(a + b - x_{r-1}) \leq x_{r-1}\varepsilon_r$ from which $a + b - x_{r-1} \leq \varepsilon_r$.

This last inequality, together with its inverse which we stated above, gives

$$\deg\Gamma_{r-1} = x_{r-1} = a + b - \varepsilon_r \text{ and } |X_{r-1}| = \varepsilon_r(a + b - \varepsilon_r).$$

We now apply the same reasoning as above to Y_{r-1}: let us call Γ'_{r-2} the fixed curve of $B_{\varepsilon_{r-2}-x_{r-1}}$ (Y_{r-1}), x'_{r-2} =deg Γ'_{r-2}, $X'_{r-2} = Y_{r-1} \cap \Gamma'_{r-2}$; $Y'_{r-2} = Y_{r-1} - X'_{r-2}$. We get

$$x'_{r-2} = a - x_{r-1} + b - x_{r-1} - (\varepsilon_{r-1} - x_{r-1}) = \varepsilon_r - \varepsilon_{r-1}.$$

So we have:

$$|X'_{r-2}| = (\varepsilon_{r-1} - x_{r-1})(a - x_{r-1} + b - x_{r-1} - \varepsilon_{r-1} - x_{r-1}) =$$

$$= (\varepsilon_{r-1} - x_{r-1})(a + b - x_{r-1} - \varepsilon_{r-1}).$$

Finally:

$$x_{r-2} = x_{r-1} + x'_{r-2} = a + b - \varepsilon_r + \varepsilon_r - \varepsilon_{r-1} = a + b - \varepsilon_{r-1}.$$

$$|X_{r-2}| = |X_{r-1}| + |X'_{r-2}| = \varepsilon_r(a + b - \varepsilon_r) + (\varepsilon_r - \varepsilon_{r-1})(\varepsilon_{r-1} - x_{r-1}) =$$

$$= \varepsilon_{r-1}(\varepsilon_r - \varepsilon_{r-1} + a + b - \varepsilon_r) = \varepsilon_{r-1}(a + b - \varepsilon_{r-1}).$$

The proof can be concluded with the same arguments for the remaining $r - 2$ steps.

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