A REMARK ON CARATHÉODORY TYPE SELECTIONS

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We prove existence of Carathéodory type selections for multifunctions of two variables which are weakly lower semicontinuous with respect to one variable and measurable with respect to the other.

1. Introduction.

Let (T, Σ_T) be a measurable space. Let X be a locally compact separable metric space with metric d and let Y be a separable Banach space. Denote by $\mathcal{B}(X)$ the Borel σ-algebra of X. Let 2^Y be the collection of all nonempty subsets of Y. Set $C(Y) = \{A \in 2^Y | A \text{ is closed}\}$ and $C_C(Y) = \{A \in 2^Y | A \text{ is closed and convex}\}$. By $S(u, r)$ (resp. $K(u, r)$) denote the open (resp. closed) ball in Y of radius $r > 0$ and center u. Set $S = S(0; 1)$. By $\mathbb{N}$ (resp. $\mathbb{R}$) we denote the set of all natural (resp. real) numbers.

A multifunction $P : T \to 2^Y$ is called Σ_T-measurable (resp. lower semicontinuous or briefly l.s.c) if the set $P^- U = \{t \in T | P(t) \cap U \neq \emptyset\}$ belongs to Σ_T (resp. is open) for every open subset U of Y.

A multifunction $P : X \to 2^Y$ is called weakly Hausdorff lower semicontinuous or briefly H_W-l.s.c (see [4]) at x_0 if for every $\varepsilon > 0$ and every neighborhood V of x_0, there are a neighborhood U of x_0 ($U \subset V$) and a point $x' \in U$ such that $P(x') \subset P(x) + \varepsilon S$ for each point $x \in U$. P is called H_W-l.s.c in X if it is H_W-l.s.c at each point $x \in X$.

In general, a H_W-l.s.c. multifunction is not l.s.c.

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A function $p : T \times X \to Y$ is called a Carathéodory type selection of $P : T \times X \to 2^Y$ if (i) $p(t,x) \in P(t,x)$, for each $(t,x) \in T \times X$, (ii) $p(t,\cdot)$ is continuous for each $t \in T$, and (iii) $p(\cdot, x)$ is Σ_T-measurable for each $x \in X$.

The problem of a Carathéodory type selections, has been studied by several authors [2, 3, 5-7, 10, 12]. In this note we prove the existence of Carathéodory type selection for a multifunction $P : T \times X \to C_C(Y)$ which is $\Sigma_{T\times X}$-measurable and such that for each $t \in T$, $P(t,\cdot)$ is H_W-l.s.c. As an application of this result we obtain a random fixed point theorem.

2. Preliminaries.

LEMMA 1. *Let $F : X \to C_C(Y)$ be H_W-l.s.c and let $f : X \to Y$ be continuous. If $F(x) \cap S(f(x), R) \neq \emptyset$ for each $x \in X$, then for each $r > R$ the multifunction $F(\cdot) \cap K(f(\cdot), r)$ is H_W-l.s.c.*

Proof.

Claim. For every $\varepsilon > 0$ there exists a $\sigma = \sigma(\varepsilon, r, R) > 0$ such that

(0) $[F(x) + \sigma \cdot S] \cap [K(f(x), r) + \sigma \cdot S] \subset F(x) \cap K(f(x), r) + \varepsilon \cdot S, x \in X$

This claim can be proved as the Lemma in [4]. One should only take $r_\lambda = R \cdot \lambda$ in place of $r \cdot \dfrac{\lambda}{2}, \tilde{\lambda} = \dfrac{||y - f(x)|| + \sigma - r}{||y - f(x)|| - R}$ in place of $\dfrac{||y - f(x)|| + \sigma - r}{||y - f(x)|| - \frac{r}{2}}$ and σ such that $0 < \sigma < \min\{\varepsilon, r - R\}$, $\dfrac{3\sigma r + R\sigma + \sigma^2}{r - R - \sigma} < \varepsilon$.

Let $x_0 \in X$, $\varepsilon > 0$ and let V be a neighborhood of x_0. By Claim there is σ such that (0) holds. By the continuity of f at x_0, there is a neighborhood $W (W \subset V)$ of x_0 such that $K(f(x_1), r) \subset K(f(x_2), r) + \sigma \cdot S$, $x_1, x_2 \in W$. Since F is H_W-l.s.c at x_0, there are a neighborhood U of x_0 $(U \subset W)$ and a point $x' \in U$ such that $F(x') \subset F(x) + \sigma \cdot S$, $x \in U$.

Hence

$$F(x') \cap K(f(x'), r) \subset [F(x) + \sigma \cdot S] \cap [K(f(x), r) + \\ + \sigma \cdot S] \subset F(x) \cap K(f(x), r) + \varepsilon \cdot S, \; x \in U.$$

This completes the proof.

One can show that the condition $r > R$ in Lemma 1 cannot be replaced by $r \geq R$.

Observation 1. Restriction of H_W-l.s.c function to open set is H_W-l.s.c function. The following example shows that this is not true for the closed sets.

Let $P : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$P(x) = \begin{cases} \{0\} & \text{for } x < 0 \\ [-1, 0] & \text{for } x = 0 \\ [0, 1] & \text{for } x > 1 \end{cases}$$

Obviously, P is H_W-l.s.c but $P_{|[0,\alpha)}$ is not H_W-l.s.c at 0.

Let (T, Σ_T), $(T \times X, \Sigma_{T\times X})$ be measurable spaces. Let π_T be the projection of $T \times X$ onto T. We say that the pair (T, X) is projective if $\pi_T(V) \in \Sigma_T$ for every $V \in \Sigma_{T\times X}$.

EXAMPLE 1. Let (T, Σ_T) be a complete measurable space and let $\Sigma_{T\times X} = \Sigma_T \times \mathcal{B}(X)$. Then (T, X) is projective (see [1, p. 75]).

EXAMPLE 2. Let T be a separable metrizable space and let μ be a positive measure on $\mathcal{B}(T)$. For $A \in 2^T$ we put $\mu^*(A) = \inf\{\mu(C)|C \in \mathcal{B}(T), A \subset C\}$. A set $A \subset T$ is μ^*-measurable if, for every $B \subset T$, $\mu^*(B) = \mu^*(A \cap B) + \mu^*(B \setminus A)$. By Σ_T-denote the σ-algebra of μ^*-measurable sets. Define the σ-algebra $\Sigma_{T\times X}$ by putting

$$\Sigma_{T\times X} = \{A \cup E | A \in \mathcal{B}(T \times X), E \in 2^{T\times X} \quad \text{and} \quad \mu^*(\pi_T(E)) = 0\}.$$

Then (T, X) is projective (see [8, p. 4]).

LEMMA 2. *Let (T, Σ_T) and $(T \times X, \Sigma_{T\times X})$ be measurable space such that the pair (T, X) is projective and $T \times U \in \Sigma_{T\times X}$ for every open set $U \subset X$. Let $\psi : T \times X \to Y$ and $P : T \times X \to C(Y)$ be $\Sigma_{T\times X}$-measurable maps. Then for every open set $U \subset X$, the set*

$$\{t \in T | P(t, x) \cap S(\psi(t, x), r) \neq \emptyset \quad \textit{for each} \quad x \in U\}$$

is Σ_T-measurable.

Proof. By [1 p. 67] there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ of $\Sigma_{T\times X}$ measurable functions, from $T\times X$ into Y, such taht $P(t,x)-\psi(t,x) = cl\{p_n(t,x)-\psi(t,x) : n\in\mathbb{N}\}$, $(t,x)\in T\times X$. Consequently, the set $W=\{(t,x)\in T\times X | P(t,x)\cap S(\psi(t,x),r)\neq\emptyset\} = \{(t,x)\in T\times X | (P(t,x)-\psi(t,x))\cap S(0,r)\neq\emptyset\}$ belongs to $\Sigma_{T\times X}$. Since $\{t\in T | P(t,x)\cap S(\psi(t,x),r)\neq\emptyset$ for each $x\in U\} = T\setminus\pi_T((T\times U)\setminus W)$ Lemma 2 follows.

3. Result.

THEOREM 1. *Let* (T,Σ_T) *and* $(T\times X,\Sigma_{T\times X})$ *be measurable spaces. Suppose that the pair* (T,X) *is projective and, for every open set* U *in* X *we have* $T\times U\in\Sigma_{T\times X}$. *Let* $P : T\times X\to C_C(Y)$ *be a* $\Sigma_{T\times X}$*-measurable. Suppose that for every* $t\in T$, $P(t,\cdot)$ *is* H_W*-l.s.c. Then* P *admits a Carathéodory selection. Moreover, there is a multifunction* $\Psi : T\times X\to C_C(Y)$ *such that:*

- *(i)* $\Psi(t,x)\subset P(t,x)$, $((t,x)\in T\times X)$, Ψ *is* $\Sigma_T\times\mathcal{B}(X)$*-measurable and* $\Psi(t,\cdot),(t\in T)$ *is l.s.c.*
- *(ii)* p *is a Carathéodory selection of* P *iff* p *is a Carathéodory selection of* Ψ.
- *(iii) if* $\Phi : T\times X\to C_C(Y)$ *is* $\Sigma_T\times\mathcal{B}(X)$*-measurable,* $\Phi(t,\cdot)$, $(t\in T)$ *is l.s.c and* $\Phi(t,x)\subset P(t,x)$, $((t,x)\in T\times X)$ *then* $\Phi(t,x)\subset\Psi(t,x)$, $((t,x)\in T\times X)$.

Proof. For a given $x\in X$, let $r_x>0$ be such that $K(x,2r_x)$ is a compact subset of X. From Lindelöf theorem the family $\{S(x,r_x)\}_{x\in X}$ contains some countable family $\{S(x_i,r_i)\}_{i\in\mathbb{N}}$ which covers of X.

Let p_i be a seminorm on $C(X,Y)$ defined by $p_i(f)=\sup\{||f(x)|| \,|x\in K(x_i,r_i)\}$. It is well known that the space $C(X,Y)$ with topology determined by the family of seminorms $\{p_i\}_{i\in\mathbb{N}}$ is a Polish space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset of $C(X,Y)$. For $i,k,n\in\mathbb{N}$ set

$$U_{ikn}=\{f\in C(X,Y)|p_i(f-f_n)<1/k\}.$$

Note that the family $\mathcal{U}=\{U_{ikn}\}_{i,k,n\in\mathbb{N}}$ is a subbase of the topology in $C(X,Y)$ given by the family of seminorms $\{p_i\}_{i\in\mathbb{N}}$.

By [4] we can define a multifunction $\mathcal{P} : T\to 2^{C(X,Y)}$ given by

$$\mathcal{P}(t)=\{\varphi\in C(X,Y)|\varphi(x)\in P(t,x) \text{ for each } x\in X\}.$$

Obviously for each $t \in T$, the set $\mathcal{P}(t)$ is convex. We claim that for each $t \in T$, the set $\mathcal{P}(t)$ is closed. In fact, suppose on the contrary that there are $t_0 \in T$ and a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(t_0)$ such that $\varphi_n \to \varphi$ and $\varphi \notin \mathcal{P}(t_0)$. Let $x_0 \in X$, $\varepsilon > 0$, $i \in \mathbb{N}$ and $N \in \mathbb{N}$ be such that $\varphi(x_0) \notin P(t_0, x_0)$, $d(\varphi(x_0), P(t_0, x_0)) = \varepsilon$, $x_0 \in K(x_i, r_i)$ and $p_i(\varphi_n - \varphi) < \varepsilon$ for $n \geq N$. Then $\varepsilon > p_i(\varphi_n - \varphi) \geq ||\varphi_n(x_0) - \varphi(x_0)|| \geq d(\varphi(x_0), P(t_0, x_0)) = \varepsilon$ which gives a contradiction.

Claim 1. $\mathcal{P}$ is Σ_T-measurable iff $\mathcal{P}^- U \in \Sigma_T$ for every set

$$U = \{\varphi \in C(X,Y) | \sup\{||\varphi(x) - f(x)|| \, |x \in K(x^\circ, r^\circ)\} < t\},$$

where $f \in C(X,Y)$, $r > 0$, $x^\circ \in X$ and $r^\circ > 0$ is such that $K(x^\circ, 2 \cdot r^\circ)$ is compact.

Indeed, let $U_1, \dots, U_n, \dots$ be open sets in $C(X,Y)$ then

$$\mathcal{P}^-\left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} \mathcal{P}^- U_i. \tag{1}$$

Thus it is enough to show that

$$\mathcal{P}^-(W) \in \Sigma_T$$

where W is a finite intersection of members of $\mathcal{U}$.

If $W = \emptyset$ then $\mathcal{P}^- W = \emptyset \in \Sigma_T$, so we can assume that $W = \bigcap_{j=1}^{m} U_{i_j k_j n_j} \neq \emptyset$. For each $g \in W$ there is $r > 0$ such that $U_{i_j r g} \subseteq U_{i_j k_j n_j}$, $j = 1, \dots, m$ where $U_{i_j r g} = \{\varphi \in C(X,Y) | p_{i_j}(\varphi - g) < r\}$.

By Lindelöf theorem follows that there is a countable family $\left\{\bigcap_{j=1}^{m} U_{i_j r_n g_n}\right\}_{n\in\mathbb{N}}$ such that $\bigcup_{n=1}^{\infty} \bigcap_{j=1}^{m} U_{i_j r_n g_n} = W$. Thus by (1) follows that it is enough to show that

$$\mathcal{P}^-\left(\bigcap_{j=1}^{m} U_{jrg}\right) \in \Sigma_T.$$

Now we show that

$$\mathcal{P}^-\left(\bigcap_{j=1}^{m} U_{jrg}\right) = \bigcap_{j=1}^{m} \mathcal{P}^- U_{jrg}.$$

The inclusion « $\subseteq$ » is trivial, so we omit it. To prove the reverse inclusion, suppose that $t \in \bigcap_{j=1}^{m} \mathcal{P}^{-}U_{jrg}$. Let $\varphi_j \in \mathcal{P}(t) \cap U_{jrg}$, $j = 1, \ldots, m$. For each $j = 1, \ldots, m$ there is $\varepsilon_j > 0$ such that $\sup\{\|\varphi_j(x) - g(x)\| \,|\, x \in K(x_j, r_j + \varepsilon_j)\} < r$. Set

$$V_1 := S(x_1, r_1 + \varepsilon_1) \cup \left(X \setminus \bigcup_{j=1}^{m} K(x_j, r_j) \right)$$

$$V_j := S(x_j, r_j + \varepsilon_j),\ j = 2, \ldots, m.$$

For each $j = 1, \ldots, m$ define a continuous function

$$p_j(x) = \frac{d(x, X \setminus V_j)}{\sum_{i=1}^{m} d(x, X \setminus V_i)}.$$

Observe that $\sum_{i=1}^{m} p_i(x) = 1$ for each $x \in X$. Let us define $\varphi \in C(X, Y)$ by $\varphi = \sum_{j=1}^{m} p_j \cdot \varphi_j$. It is easy to see that $\varphi \in \mathcal{P}(t)$, $\varphi \in \bigcap_{j=1}^{m} U_{jrg}$ and the proof of Claim 1 is complete.

Claim 2. $\mathcal{P}$ is Σ_T-measurable.

By Claim 1 it is sufficient to show that $\mathcal{P}^{-}U \in \Sigma_T$. For $n \in \mathbb{N}$ set

$$U_n = \{\varphi \in C(X, Y) |\, \sup\{\|\varphi(x) - f(x)\| \,|\, x \in K(x^\circ, r^\circ)\} < r - 1/n\}.$$

Clearly,

$$\mathcal{P}^{-}U = \bigcup_{n=1}^{\infty} \mathcal{P}^{-}U_n. \tag{2}$$

For $n, k \in \mathbb{N}$ set

$$B_{n,k} = \{t \in T | P(t, x) \cap S(f(x), r - 1/n) \neq \emptyset \text{ for each } x \in S(x^\circ, r^\circ + 1/k)\}.$$

Choose $n_0, k_0 \in \mathbb{N}$ such that $1/n_0 < r$ and $1/k_0 \le r^\circ$. We will show that

$$\bigcup_{n=n_0}^{\infty} \mathcal{P}^{-}U_n = \bigcup_{n=n_0}^{\infty} \bigcup_{k=k_0}^{\infty} B_{n,k}. \tag{3}$$

Let $t \in \bigcup_{n=n_0}^{\infty} \mathcal{P}^{-}U_n$. Let $n \geq n_0$ be such that $t \in \mathcal{P}^{-}U_n$. Let $\varphi \in \mathcal{P}(t) \cap U_n$. Let $n_1 > n$ and $\varepsilon = \frac{1}{2n} - \frac{1}{2n_1}$. Since the restrictions of φ and f to $K(x^\circ, 2r^\circ)$ are uniformly continuous, there is a $\delta > 0$ such that $\|\varphi(x) - \varphi(x')\| < \varepsilon$ and $\|f(x) - f(x')\| < \varepsilon$ for each $x, x' \in K(x^\circ, 2r^\circ)$ such that $d(x, x') < \delta$. Let $k \in \mathbb{N}$ be such that $1/k < \delta$. For $x \in S(x^\circ, r^\circ + 1/k)$ there exist $x' \in K(x^\circ, r^\circ)$ such that $d(x, x') < 1/k$ and $\|\varphi(x) - f(x)\| \le \|\varphi(x) - \varphi(x')\| + \|\varphi(x') - f(x')\| + \|f(x') - f(x)\| \le r - 1/n_1$. Hence $t \in B_{n_1,k}$. This implies

$$\bigcup_{n=n_0}^{\infty} \mathcal{P}^{-}U_n \subset \bigcup_{n=n_0}^{\infty} \bigcup_{k=k_0}^{\infty} B_{n,k}.$$

To prove the reverse inclusion suppose that $t \in \bigcup_{n=n_0}^{\infty} \bigcup_{k=k_0}^{\infty} B_{n,k}$. Clearly $t \in B_{n,k}$ for some $n \geq n_0$ and $k \geq k_0$. Let $n_1 > n$. Consider the multifunction $G : S(x^\circ, r^\circ + 1/k) \to C_C(Y)$ given by

$$G(x) = P(t, x) \cap K(f(x), r - 1/n_1).$$

By Observation 1 and Lemma 1, G is H_W-l.s.c. Hence by [4] G and $P(t, \cdot)$ admit continuous selections, say φ_1 and φ_2. Let p_1 and p_2 be continuous functions from X into $[0, 1]$ defined by

$$p_1(x) = \begin{cases} 1 & \text{if } x \in K(x^\circ, r^\circ) \\ k \cdot \left[r^\circ + \frac{1}{k} - d(x, x^\circ)\right] & \text{if } x \in S\left(x^\circ, r^\circ + \frac{1}{k}\right) \setminus (x^\circ, r^\circ) \\ 0 & \text{otherwise.} \end{cases}$$

$$p_2(x) = \begin{cases} 0 & \text{if } x \in K(x^\circ, r^\circ) \\ k \cdot [d(x, x^\circ) - r^\circ] & \text{if } x \in S\left(x^\circ, r^\circ + \frac{1}{k}\right) \setminus (x^\circ, r^\circ) \\ 1 & \text{otherwise.} \end{cases}$$

Note that $p_1(x)+p_2(x)=1$ for each $x \in X$. The facts that f is continuous function and $K(x^\circ, 2r^\circ)$ is a compact set implay that φ_1 is a bounded function on $S(x^\circ, r^\circ+1/k)$.

Let $\tilde{\varphi}_1$ be a function from X to Y defined by

$$\tilde{\varphi}_1(x) = \begin{cases} \varphi_1(x), & x \in S\left(x^\circ, r^\circ + \frac{1}{k}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the function $\varphi : X \to Y$ given by $\varphi = p_1 \cdot \tilde{\varphi}_1 + p_2 \cdot \varphi_2$. It is easy to see that φ is a continuous function such that $\varphi(x) \in P(t,x)$ for each $x \in X$ and $\sup\{\|\varphi(x) - f(x)\| \,|\, x \in K(x^\circ, r^\circ)\} \leq r - \frac{1}{n_1}$. Hence $t \in \bigcup_{n=n_0}^{\infty} \mathcal{P}^- U_n$ and the proof of (3) is complete.

Let ψ be a function from $T \times X$ into Y defined by $\psi(t,x) := f(x)$. Observe that ψ is $\Sigma_{T\times X}$-measurable. By Lemma 2, $B_{n,k} \in \Sigma_T$ for each $n \geq n_0$ and $k \geq k_0$. Thus, by (2) and (3) we have $\mathcal{P}^- U \in \Sigma_T$. This completes the proof of Claim 2.

Let p be a Σ_T-measurable selection of $\mathcal{P}$ [1, p. 67]. Consider the function $p : T \times X \to Y$ given by $p(t,x) := (p(t))(x)$. We claim that p is a Carathéodory selection of P.

Indeed, for each $t \in T$, $p(t,\cdot)$ is continuous and $p(t,x) \in P(t,x), (t,x) \in T \times X$. For each $x \in X$ we define a continuous function $\gamma_x : C(X,Y) \to Y$ by $\gamma_x(f) := f(x)$. Observe that $p(\cdot,x) = \gamma_x \circ p$, so $p(\cdot,x)$ is Σ_T-measurable.

Let $\{p_n\}_{n\in \mathbb{N}}$ be a sequence of Σ_T-measurable selections of $\mathcal{P}$ such that

$$\mathcal{P}(t) = cl\{p_n(t) | n \in \mathbb{N}\}, \text{ [1, p. 67].}$$

For each $n \in \mathbb{N}$ define the function $p_n : T \times X \to Y$ by $p_n(t,x) := (p_n(t))(x)$. By [1, p. 70] p_n is $\Sigma_T \times \mathcal{B}(X)$-measurable. Now define the multifunction $\Psi : T \times X \to C_C(Y)$ by

$$\Psi(t,x) := cl\{p_n(t,x) | n \in \mathbb{N}\}.$$

We shall see that Ψ satisfy the conditions (i)-(iii). The condition (i) follows by [5, Theorem 1]. To see (ii), let p be a Carathéodory type selection of P. For each $t \in T$ there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k}(t,\cdot) \to p(t,\cdot)$, if $n_k \to \infty$. This implies $p(t,x) \in \Psi(t,x)$,

$(t,x) \in T \times X$ and (ii) is prove. Finally, to see (iii), let $\Phi : T \times X \to C_C(Y)$ be $\Sigma_T \times \mathcal{B}(X)$-measurable and such that $\Phi(t,\cdot)$ is l.s.c for each $t \in T$, and $\Phi(t,x) \subset P(t,x)$, $(t,x) \in T \times X$. By [5, Theorem 1] there is a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ of Carathéodory type selections of Φ such that $\Phi(t,x) = cl\{\phi_n(t,x) | n \in \mathbb{N}\}$, $(t,x) \in T \times X$. By (ii), ϕ_n is a Carathéodory selection of Ψ. This completes the proof.

Remark 1. The assumptions of Theorem 1 are fulfilled if (T, Σ_T) and $(T \times X, \Sigma_{T \times X})$ are as in Example 1 or Example 2.

Remark 2. Note that Theorem 1 fails without the assumption that (T, X) is projective.

Indeed, let $X = [0,1]$ and $T = [0,1]$. Let Σ be the σ-algebra of Lebesgue measurable subsets of [0,1]. Set $\Sigma_T = \Sigma$ and $\Sigma_{T \times X} = \Sigma \times \Sigma$. Observe that the pair (T, X) is not projective. Let A be a nonmeasurable subset of [0,1]. Let P be the multifunction from $[0,1] \times [0,1]$ into $C_C(\mathbb{R})$ given by

$$P(t,x) = \begin{cases} \{1\} & \text{if } t \in A,\ x = 0 \\ \{0\} & \text{if } t \in [0,1] \backslash A,\ x = 0 \\ [0,1] & \text{otherwise.} \end{cases}$$

Clearly P is $\Sigma_{T \times X}$-measurable and for each $t \in [0,1]$, $P(t,\cdot)$ is H_W-l.s.c. It is easy to see that P does not admit a Carathéodory type selection.

Remark 3. If $P : T \times X \to C_C(Y)$ is such that $P(\cdot, x)$ is Σ_T-measurable for each $x \in X$ and $P(t, \cdot)$ is H_W-l.s.c for each $t \in T$. Then P need not admit Carathéodory type selection.

Indeed, let X, T and Σ_T be as in Remark 2. Let A be a Vitali (Lebesgue) nonmeasurable subset of [0,1] (see [11, p. 33]. For $x, y \in [0,1]$ we writte $x \sim y$ to denote that $x - y$ is a rational number. It is evident that « $\sim$ » is an equivalence relation. The equivalence class which contains the element $x \in [0,1]$ is denoted by $[x]$. Note that the class $[x]$ is countable. For each $a \in A$ there is an increasing sequence $\{w_n^a\}_{n \in \mathbb{N}} \subset [0,1]$ of rational numbers such that $w_n^a \to a$. For each $a \in A$, we fix the sequence $\{w_n^a\}$. Let P be the multifunction from $[0,1] \times [0,1]$

into $C_C(\mathbb{R})$ given by

$$P(t,x) = \begin{cases} \{1\} & \text{for } x = 1 + w_n^a - a \text{ and } t = a \in A \\ \{0\} & \text{for } x = 1 + w_n^a - a \text{ and } t \in [a] \setminus \{a\}, a \in A \\ [0,1] & \text{otherwise} \end{cases}$$

It is routine to see that $P(t,\cdot)$ is H_W-l.s.c for each $t \in [0,1]$ and $P(\cdot,x)$ is Σ_T-measurable for each $t \in [0,1]$. Suppose that P admits a Carathéodory type selection p. Thus we have $p(a, 1 + w_n^a - a) = 1$ for each $a \in A$, $n \in \mathbb{N}$ and $p(t, 1 + w_n^a - a) = 0$ for each $t \in [a] \setminus \{a\}$, $n \in \mathbb{N}$. Hence by the continuity of $p(t,\cdot)$, we have $p(t,1) = 1$, for $t \in A$ and $p(t,1) = 0$, for $t \in [0,1] \setminus A$. Clearly $p(\cdot,1)$ is not measurable, which gives a contradiction.

Let T be any measure space and X be a nonempty subset of any linear topological space. Let P be a multifunction from $T \times X$ into 2^X. The multifunction P is said to have a random fixed point if there exists a measurable function $x : T \to X$ such that $x(t) \in P(t, x(t))$ for almost all t in T.

THEOREM 2. *Let (T, Σ_T, μ) be a complete finite measure space and X be a nonempty compact convex subset of a separate Banach space Y. Suppose that the pair (T,X) is projective and for every open set U in X we have $T \times U \in \Sigma_{T\times X}$. Let $P : T \times X \to C_C(X)$ be such that*

(i) for each $t \in T$, $P(t,\cdot)$ is H_W-l.s.c

(ii) P is $\Sigma_{T\times X}$-measurable

Then P has a random fixed point.

Proof. By Theorem 1 there exists the multifunction $\Psi : T \times X \to C_C(Y)$ such that $\Psi(t,x) \subset P(t,x)$ for each $(t,x) \in T \times X$, Ψ is $\Sigma_T \times \mathcal{B}(X)$-measurable and $\Psi(t,\cdot)$ is l.s.c for each $t \in T$. Hence by [6, Theorem 3.3] Ψ has a random fixed point. This complete the proof.

Remark 4. Unlike in [6], the Theorems 1 and 2 are not true without closed valueness of $P : T \times X \to 2^Y$ even if both

(i) Y is finite dimensional and

(ii) $P(t,x)$ has a nonempty interior for all $(t,x) \in T \times X$.

Indeed, let $X = [-1,1]$ and $T = [-1,1]$. Let Σ_T be the σ-algebra of

Lebesgue measurable subset of $[-1,1]$. Let P be the multifunction from $[-1,1]\times[-1,1]$ into $2^{[-1,1]}$ defined by

$$P(t,x)=\begin{cases}[-tx,0] & \text{for } x>0, t>0\\ (0,1] & \text{otherwise}\end{cases}$$

It is easy to see that for each $t \in T$, $P(t,\cdot)$ is H_W-l.s.c and P is $\Sigma_T \times \mathcal{B}(X)$-measurable. Suppose that P admits a Carathéodory type selection p. Thus we have $p(1,x)<0$ for each $x>0$ and $p(1,x)>0$ for each $x \leq 0$, which gives a contrafiction because $p(1,\cdot)$ is continuous.

For each function $x: T \rightarrow X$ we have that for $t>0$, $x(t) \notin P(t,x(t))$. From this follows that P does not have a random fixed point.

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