ON THE CONVERGENCE OF NONLINEAR BELTRAMI TYPE OPERATORS

RICCARDO DE ARCANGELIS (Salerno) *

One of the results proved is the following: if (f_h) is a sequence of K-quasiregular mappings, converging to f in L^1_{loc}, whose jacobians verify a weak integrability condition, then the solutions of Dirichlet problems for the nonlinear Laplace-Beltrami operator associated to each f_h converge to the solution of the Dirichlet problem for the nonlinear Laplace-Beltrami operator associated to f.

Such result is deduced as a particular case of a more general theorem concerning nonlinear operators.

The case of K-quasiconformal functions f_h is also treated.

A class of weighted Sobolev spaces associated to quasiconformal mappings is studied.

0. Introduction.

Let Ω_0 be an open set in $\mathbb{R}^n$. For every function $f = (f^1, \dots, f^n)$ in $(H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$ denote with $\dfrac{\partial f}{\partial x}(x)$ the jacobian matrix of f in x: i.e. the matrix having the vectors $D_x f^i(x)$ as rows (here $D_x f^i = (D_{x_1} f^i, \dots, D_{x_n} f^i)$) and with $\det \dfrac{\partial f}{\partial x}(x)$ its determinant.

In the following, when it is clear from the context, we will write simply D instead of D_x, besides we will adopt the notations of the usual matrix product.

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With Ω we generally denote an open set, $\Omega \subset\subset \Omega_0$ will mean that the closure of Ω is compact and contained in Ω_0.

Let $K \geq 1$, we say that a function f in $(H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$ is K-quasiregular on Ω_0 if

$$\left|\frac{\partial f}{\partial x}(x)\right|^n \leq K n^{n/2} \det \frac{\partial f}{\partial x}(x) \quad \text{a.e. in } \Omega_0$$

where $\left|\frac{\partial f}{\partial x}(x)\right| = \left(\sum_{i,j=1}^{n} (D_{x_j} f^i(x))^2\right)^{1/2}$.

It is known (see Proposition 1.1 in 1) that for every K-quasiregular mapping f the matrix $\left(\frac{\partial f}{\partial x}(x)\right)^{-1}$ exists almost everywhere.

Let f be a K-quasiregular mapping on Ω_0, in some papers [S2], [Sb], [DA D], the first of which is due to S. Spagnolo, has been started the study of the continuity (with respect to f) of the Euler operator Δ_f (linear operator of Laplace-Beltrami) of the functional

$$I_f(\Omega, \cdot) : u \in \mathrm{Lip}_{\mathrm{loc}} \to \int_\Omega \left| Du(x) \left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right|^2 \det \frac{\partial f}{\partial x}(x) dx$$

where $\Omega \subset\subset \Omega_0$ is open and $\mathrm{Lip}_{\mathrm{loc}}$ denotes the set of the locally Lipschitz functions on $\mathbb{R}^n$.

A possible result in this study is the following (see [DA D]).

Let (f_h) be a sequence of K-quasiregular mappings on Ω_0 converging to a function f in $(L^1_{\mathrm{loc}}(\Omega_0))^n$ and verifying

$$(0.1) \qquad 0 \leq w(x) \leq \left(\det \frac{\partial f_h}{\partial x}(x)\right)^{1-2/n} \leq \Lambda w(x) \quad \textit{a.e. in } \Omega_0, \ \Lambda \geq 1$$

with w, w^{-1} in $L^1_{\mathrm{loc}}(\Omega_0)$. Then f is itself K-quasiregular on Ω_0 and, for every open set $\Omega \subset\subset \Omega_0$, the solutions in $H^1_0(\Omega, w)$ (Sobolev space with weight w of functions with null trace on $\partial\Omega$) of the equations $\Delta_{f_h} u = \psi \in L^\infty(\Omega)$ converge in $L^1(\Omega)$ to the solution of $\Delta_f u = \psi$.

In [Sb] has been also proposed the problem of the study of the continuity of the Euler operator A_f (nonlinear Laplace-Beltrami operator) of the functional

$$J_f(\Omega, \cdot) : u \in \mathrm{Lip}_{\mathrm{loc}} \to \int_\Omega \left| Du(x) \left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right|^n \det \frac{\partial f}{\partial x}(x) dx.$$

This matter is treated in the present paper.

In particular one of the results obtained here is very similar to the one exposed above. In fact in Theorem 3.3 it is proved the following statement.

If (f_h) *is a sequence of* K*-quasiregular mappings on* Ω_0 *converging to a function* f *in* $(L^1_{\text{loc}}(\Omega_0))^n$ *and if*

$$\left\| \det \frac{\partial f_h}{\partial x} \right\|_{L^1(\Omega)} \leq Q(\Omega) < +\infty \quad \textit{for every open set } \Omega \subset\subset \Omega \tag{0.2}$$

then the function f *is* K*-quasiregular on* Ω_0 *and the solutions in* $H_0^{1,n}(\Omega)$ *of the equations* $A_{f_h}u = \psi \in L^\infty(\Omega)$ *converge in* $L^1(\Omega)$ *to the solution of the equation* $A_f u = \psi$.

To get a result of this kind we take into consideration a sequence (ϕ_h) of nonnegative convex functions on $\mathbb{R}^n$ with

$$0 \leq \Phi_h(\xi) \leq c(1 + |\xi|^p) \quad \text{for every } \xi \text{ in } \mathbb{R}^n,\ c \geq 1, p \geq 1$$

converging to a function Φ and a sequence (f_h) in $(H^{1,q}_{\text{loc}}(\Omega_0))^n$, $q \geq \max\{p, n\}$, converging in $(L^q_{\text{loc}}(\Omega_0))^n$ to a $(H^{1,q}_{\text{loc}}(\Omega_0))^n$ function f with $\det \frac{\partial f_h}{\partial x}(x) > 0$, $\det \frac{\partial f}{\partial x}(x) > 0$ a.e..

Then we construct the functionals

$$F_{f_h}(\Omega, \cdot) : u \in \text{Lip}_{\text{loc}} \to \int_\Omega \Phi_h\left(Du(x) \left(\frac{\partial f_h}{\partial x}(x) \right)^{-1} \right) \det \frac{\partial f_h}{\partial x}(x) dx$$

and prove that (Theorem 2.7), under suitable boundedness but not of quasiregularity hypotheses on (f_h), they $\Gamma^-(L^1(\Omega))$-converge to the functional F_f, associated analogously to f by means of the function Φ, for every open set $\Omega \subset\subset \Omega_0$.

This method, with the choice $\Phi_h(\xi) = \Phi(\xi) = |\xi|^p$, allows the study of the continuity properties of the Euler operator associated to the functional

$$L_f(\Omega, \cdot) : u \in \text{Lip}_{\text{loc}} \to \int_\Omega \left| Du(x) \left(\frac{\partial f}{\partial x}(x) \right)^{-1} \right|^p \det \frac{\partial f}{\partial x}(x) dx.$$

In order to get a theorem similar to the two ones stated above, is needed the following hypothesis that, at least in the case $1 < p \leq n$, looks to be intermediate between (0.1) and (0.2) ((0.2) being always implied by (0.1))

$$0 \leq w(x) \leq \left(\det \frac{\partial f_h}{\partial x}(x)\right)^{1-p/n} \leq \Lambda w(x) \quad \text{a.e. in } \Omega_0, \ \Lambda \geq 1$$

with w, $w^{-1/(p-1)}$ in $L^1_{\mathrm{loc}}(\Omega_0)$.

If the functions f_h are K-quasiconformal (K-quasiregular and one to one) the stated result can be lightly refined by using appropriately suitable weighted Sobolev spaces as in [DA D].

We wish to thank Professor B. Bojarski for some interesting discussions about quasiregular mappings theory.

1. Definitions and preliminaries.

For every $n \times n$ matrix $A = [a_{ij}]$ define:

$$|A| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}.$$

$$||A|| = \sup_{|z|=1} |Az|,$$

then it is well known that

$$\text{(1.1)} \qquad \frac{1}{\sqrt{n}}|A| \leq ||A|| \leq \sqrt{n}|A|.$$

Obviously the eigenvalues of $A^t A$ will be real and nonnegative: let $0 \leq \mu_1 \leq \ldots \leq \mu_n$ be these eigenvalues and define $\lambda_i = \lambda_i(A) = \sqrt{\mu_i}$.

It will result

$$||A|| = \lambda_n$$

$$|\det A| = \lambda_1 \cdot \ldots \cdot \lambda_n.$$

Further if we suppose that $\lambda_1 \neq 0$ and define $\Lambda(A) = \lambda_n(A)/\lambda_1(A)$ we will have

$$\text{(1.2)} \quad |\det A| \leq \Lambda(A)|\det A| \leq ||A||^n \leq \Lambda(A)^{n-1}|\det A| \leq \Lambda(A)^n|\det A|.$$

We will often use the following simple estimates:

$$(1.3)\quad |zA^{-1}|^p|\det A| \leq \lambda_1^{-p}\lambda_1\ldots\lambda_n|z|^p \leq$$

$$\leq \lambda_1^{1-p}\lambda_n^{n-1}|z|^p \leq \lambda_n^{n-p}\Lambda^{p-1}|z|^p = ||A||^{n-p}\Lambda^{p-1}|z|^p.$$

and

$$(1.4)\quad |zA^{-1}|^p|\det A| \geq \lambda_n^{-p}\lambda_1\ldots\lambda_n|z|^p \geq$$

$$\geq \lambda_n^{1-p}\lambda_1^{n-1}|z|^p = \lambda_n^{n-p}\Lambda^{1-n}|z|^p = ||A||^{n-p}\Lambda^{1-n}|z|^p.$$

If we denote with

$$\Lambda(x) = \Lambda\left(\frac{\partial f}{\partial x}(x)\right) = \lambda_n\left(\frac{\partial f}{\partial x}(x)\right)/\lambda_1\left(\frac{\partial f}{\partial x}(x)\right)$$

for a K-quasiregular mapping, by virtue of (1.1) and (1.2), it must result

$$(1.5)\qquad \Lambda(x) \leq K \quad \text{a.e..}$$

A K-quasiregular mapping f on Ω_0 is said to be K-quasiconformal on Ω_0 if it is an isomorphism.

For a general exposition about quasiregular mappings see [B I] where the following estimates are proved:

PROPOSITION 1.1. *Let f be a nonconstant K-quasiregular mapping on Ω_0, then:*

1) $\det\dfrac{\partial f}{\partial x}(x) > 0$ *a.e. in Ω_0 (Theorem 7.2);*

2) f is differentiable a.e. in Ω_0 and the chain rule holds, i.e. for every ψ in $C^1(f(\Omega_0))$ $\psi \circ f$ is in $H^{1,n}_{\mathrm{loc}}(\Omega_0)$ and

$$D_x(\psi\circ f)(x) = \frac{\partial f}{\partial x}(x)(D_y\psi)(f(x)) \qquad \textit{(Theorem 5.3 and Lemma 9.6)};$$

3) f maps sets of measure zero into sets of measure zero (Theorem 8.1).

If further f is K-quasiconformal on Ω_0 then:

4) f^{-1} *is* K^{n-1}*-quasiconformal on* $f(\Omega_0)$ *(Theorem 9.1);*

5) $\left(\dfrac{\partial f}{\partial x}(x)\right)^{-1} = \dfrac{\partial f^{-1}}{\partial y}(f(x))$ *a.e. in* Ω_0*;*

6) for every u in $L^\infty(\mathbb{R}^n)$ and $\Omega \subset\subset \Omega_0$ it results

$$\int_\Omega u(f(x)) \det \frac{\partial f}{\partial x}(x)dx = \int_{f(\Omega)} u(y)dy \quad \textit{(Theorem 8.4).}$$

We now state the following result about L^p-integrability of the derivatives of quasiregular mappings (see [B I], [G2], [G M]).

THEOREM 1.2. *Let f be K-quasiregular on Ω_0. Then there exists $q > n$ depending only on n and K such that f is in $(H^{1,q}_{\text{loc}}(\Omega_0))^n$.*
Moreover for every compact subset S of $\Omega \subset\subset \Omega_0$ it results:

$$\left(\int_S \left|\frac{\partial f}{\partial x}\right|^q dx\right)^{1/q} \leq \frac{c_1(n,q,k)}{dist(S,\partial\Omega)^{1-n/q}} \left(\int_\Omega \left|\frac{\partial f}{\partial x}\right|^n dx\right)^{1/n} \tag{1.6}$$

The following result can be found in [B C O].

PROPOSITION 1.3. *Let f_h, f be functions in $(H^{1,n}_{\text{loc}}(\Omega_0))^n$ such that (f_h) converges weakly in $(H^{1,r}_{\text{loc}}(\Omega_0))^n$ to f for some $r > n^2/n+1$.*
Then for every ψ in $C_0^\infty(\Omega_0)$

$$\lim_{h\to\infty} \int_{\Omega_0} \det \frac{\partial f_h}{\partial x}(x)\psi(x)dx = \int_{\Omega_0} \det \frac{\partial f}{\partial x}(x)\psi(x)dx.$$

Finally define $Adj\,A$ as the matrix having as entries the algebraic complements of A^t. It will result

$$A \cdot Adj\,A = \det A\, I.$$

We will also need the following result (see for intance [B I]).

PROPOSITION 1.4. *Let f be a function in $(H^{1,n-1}_{\text{loc}}(\Omega_0))^n$, then the columns of the matrix $Adj\,\dfrac{\partial f}{\partial x}$ have null weak divergence.*

We now define Γ-convergence. We refer to [DG F] and [DG] for complete references.

Let (U, τ) be a topological space satisfying the first countability axiom and let F_h, F ($h \in \mathbb{N}$) be extended real functionals on U.

DEFINITION 1.5. *We say that*

$$F(u) = \Gamma^{-}(\tau) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \quad \textit{for every } u \in U$$

if and only if

i) for every $u \in U$ and for every $v_h \xrightarrow{\tau} u$

$$F(u) \leq \liminf_{h\to\infty} F_h(v_h)$$

ii) for every $u \in U$ there exists a sequence (u_h), $u_h \xrightarrow{\tau} u$, such that

$$F(u) = \lim_{h\to\infty} F_h(u_h).$$

In Γ-convergence theory the following result is fundamental (see [DG F]).

THEOREM 1.6. *Let (F_h) be a sequence of equicoercive functionals on U, i.e; for every real number c there exists a compact K_c in U such that $\{u \in U : F_h(u) \leq c\} \subseteq K_c$ for every $h \in \mathbb{N}$.*

Assume further that

$$F(u) = \Gamma^{-}(\tau) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \quad u \in U.$$

Then F has a minimum on U and

$$\min_{v\in U} F(v) = \lim_{h\to\infty} \inf_{v\in U} F_h(v).$$

Further if (u_h) is a sequence such that $u_h \xrightarrow{\tau} u$ and

$$\lim_{h\to 0}(F_h(u_h) - \inf_{v\in U} F_h(v)) = 0$$

then

$$F(u) = \min_{v\in U} F(v).$$

2. The continuity theorem.

Let $\Phi_h : \mathbb{R}^n \to [0, +\infty[$, $h \in \mathbb{N}$, be functions verifying

$$\text{(2.1)} \qquad \begin{cases} \Phi_h \text{ convex} \\ 0 \le \Phi_h(\xi) \le c_1(1 + |\xi|^p) \quad \text{for every } \xi \in \mathbb{R}^n \end{cases}$$

with $1 \le p < +\infty$, $c_1 \ge 1$ and such that

$$\text{(2.2)} \qquad \lim_{h\to\infty} \Phi_h(\xi) = \Phi(\xi) \quad \text{for every } \xi \in \mathbb{R}^n.$$

Obviously Φ will be a convex function and (2.1) will be verified by Φ.

Let (f_h) be a sequence of functions from Ω_0 to $\mathbb{R}^n$. We will suppose a certain degree of regularity on every f_h and on the convergence of the sequence (f_h) to a function f.

More precisely we will assume

$$\text{(2.3)} \qquad \begin{cases} f_h (H^{1,q}_{\mathrm{loc}}(\Omega_0))^n & q \ge \max\{p, n\} \\ f_h \to f \in (H^{1,q}_{\mathrm{loc}}(\Omega_0))^n & \text{in } (L^q_{\mathrm{loc}}(\Omega_0))^n. \end{cases}$$

Here the number q will represent for us an index of regularity of the functons f_h and of their convergence to f, while the number p is the order of growth of the functionals.

In order to construct our functionals we will always suppose in the following that

$$\text{(2.4)} \qquad \det \frac{\partial f}{\partial x}(x) > 0, \quad \det \frac{\partial f_h}{\partial x}(x) > 0 \quad \text{a.e. in } \Omega_0.$$

Under these hypotheses we define the functions

$$\phi_h(x, z) = \Phi_h \left(z \left(\frac{\partial f_h}{\partial x}(x) \right)^{-1} \right) \det \frac{\partial f_h}{\partial x}(x) \quad x \text{ a.e. in } \Omega_0, z \in \mathbb{R}^n,$$

ϕ_h will be a Carathéodory function: i.e. measurable in x and convex in z and, by (2.1) and (1.3), will verify the following growth condition:

$$\text{(2.5)} \quad 0 \le \phi_h(x, z) \le c_1 \det \frac{\partial f_h}{\partial x}(x) + c_1 n^{(n-p)/2} \Lambda_h(x)^{p-1} \left| \frac{\partial f_h}{\partial x}(x) \right|^{n-p} |z|^p$$

for almost all x in Ω_0 and all z in $\mathbb{R}^n$, where $\Lambda_h(x) = \Lambda\left(\frac{\partial f_h}{\partial x}(x)\right)$.

For every open set $\Omega \subset\subset \Omega_0$ and u in $\mathrm{Lip}_{\mathrm{loc}}$ set

$$F_h(\Omega, u) = F_{f_h}(\Omega, u) = \int_\Omega \phi_h(x, Du)dx. \tag{2.6}$$

By (2.5) we easily get that

$$0 \leq F_h(\Omega, u) \leq c_1 \int_\Omega \det \frac{\partial f_h}{\partial x}(x)dx + \tag{2.7}$$

$$+c_1 n^{(n-p)/2} \int_\Omega \Lambda_h(x)^{p-1} \left|\frac{\partial f_h}{\partial x}(x)\right|^{n-p} |Du|^p dx$$

In order to control the behaviour of the functionals F_h when h tends to infinity we will suppose that
(2.8)

$$\lim_{h\to\infty} \int_\Omega \det \frac{\partial f_h}{\partial x}(x)dx = \int_\Omega \det \frac{\partial f}{\partial x}(x)dx \quad \text{for every open set } \Omega \subset\subset \Omega_0$$

and that

(2.9) the integrals $\int_\Omega \phi_h(x)^{p-1} \left|\frac{\partial f_h}{\partial x}(x)\right|^{n-p} dx$ are equiabsolutely continuous and, for every $\Omega \subset\subset \Omega_0$, uniformly bounded.

Observe that, by (2.9) and the Vitali - Hahn - Saks theorem, it follows the existence of a subsequence, still denoted by h, and of a function m in $L^1_{\mathrm{loc}}(\Omega_0)$ such that

$$\lim_{h\to\infty} \int_\Omega \Lambda_h(x)^{p-1} \left|\frac{\partial f_h}{\partial x}\right|^{n-p} dx = \int_\Omega m\, dx \quad \text{for every } \Omega \subset\subset \Omega_0. \tag{2.10}$$

From now onwords we will denote with the letter c various constants depending only on known quantities as n, p, c_1.

Fix an open set $\Omega_1 \subset\subset \Omega_0$ and define

$$\hat{\phi}_h(x, z) = \begin{cases} \phi_h(x, z) & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \mathbb{R}^n - \Omega_1 \end{cases} \tag{2.11}$$

then from (2.5) it follows that

$$0 \le \hat{\phi}_h(x,z) \le c\chi_{\Omega_1}(x)\left\{\det \frac{\partial f_h}{\partial x}(x) + \Lambda_h(x)^{p-1}\left|\frac{\partial f_h}{\partial x}(x)\right|^{n-p}|z|^p\right\} \tag{2.12}$$

for almost all x in $\mathbb{R}^n$, z in $\mathbb{R}^n$.

Define further

$$a_h(x) = \begin{cases} c\det \frac{\partial f_h}{\partial x}(x) + c\Lambda_h(x)^{p-1}\left|\frac{\partial f_h}{\partial x}(x)\right|^{n-p} & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \mathbb{R}^n - \Omega_1 \end{cases} \tag{2.13}$$

then, respectively by (2.12) and (2.8), (2.9) and (2.10), we get

$$0 \le \hat{\phi}_h(x,z) \le a_h(x)(1+|z|^p) \tag{2.14}$$

$$\int_\Omega a_h(x)dx \to \int_\Omega a(x)dx \tag{2.15}$$

for every bounded open set Ω where $a(x) = c\left(\det \frac{\partial f}{\partial x}(x) + m(x)\right)$.

Then, by a result of Carbone and Sbordone, [C S], we deduce the following theorem:

THEOREM 2.1. *Let (2.14) and (2.15) hold for $\hat{\phi}_h(x,z)$ defined in (2.11). Then there exist a subsequence (h_k) and a Carathéodory function $g : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[$ such that*

$$\int_\Omega g(x,Du)dx = \Gamma^-(M_0(\Omega)) \lim_{\substack{k\to\infty \\ v\to u}} \int_\Omega \hat{\phi}_{h_k}(x,Dv)dx = \tag{2.16}$$

$$= \Gamma^-(C_0^0(\Omega)) \lim_{\substack{k\to\infty \\ v\to u}} \int_\Omega \hat{\phi}_{h_k}(x,Dv)dx$$

for every bounded open set Ω of $\mathbb{R}^n$ and u in Lip_{loc}.

In Theorem 2.1 we have denoted with $M_0(\Omega)$ the topology of the convergence in measure on Ω and with $C_0^0(\Omega)$ the one deduced by the extended metric

$$\delta(u,v) = \begin{cases} \|u-v\|_{C^0(\bar{\Omega})} & \text{if } \operatorname{spt}(u-v) \subset\subset \Omega \\ +\infty & \text{otherwise} \end{cases}$$

Since we are going to identify the function $g(x, z)$ we can assume the convergence in (2.17) to hold for the whole sequence.

In particular from Theorem 2.1 it follows that

$$\int_\Omega g(x, Du)dx = \Gamma^-(M_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega \phi_h(x, Dv)dx = \tag{2.17}$$

$$= \Gamma^-(C_0^0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega \phi_h(x, Dv)dx$$

for every open set $\Omega \subset\subset \Omega_1$, u in $\mathrm{Lip}_{\mathrm{loc}}$.

By (2.16), (2.10), (2.8) it soon follows that the function $g(x, z)$ satisfies the following growth condition

$$0 \leq g(x, z) \leq c\left\{\det \frac{\partial f}{\partial x}(x) + m(x)|z|^p\right\}\chi_{\Omega_1}(x) \tag{2.18}$$

for almost all x in $\mathbb{R}^n$, $z \in \mathbb{R}^n$.

In order to identify the function g given by Theorem 2.1, a first step consists in the extension of (2.17) to the whole space $H^{1,q}_{\mathrm{loc}}(\Omega_0)$, that is to the space to which the components of f_h and f belong.

To carry out such an extension we will have to assume a further estimate on the f_h (at least if $q < +\infty$) in which the numbers q and p are linked:

$$\left\| \Lambda_h^{p-1} \left|\frac{\partial f_h}{\partial x}\right|^{n-p} \right\|_{L^{(q/q-p)}(\Omega)} \leq Q(\Omega) < +\infty \tag{2.19}$$

for every open set $\Omega \subset\subset \Omega_0$.

In (2.19) we have assumed

$$\frac{q}{q-p} = \begin{cases} 1 & \text{if } q = +\infty \\ +\infty & \text{if } q = p. \end{cases}$$

Remark 2.2. In order to prove (2.17) it sufficies to assume, in place of (2.3), that $\det \frac{\partial f_h}{\partial x}$ is in $L^1_{\mathrm{loc}}(\Omega_0)$ for every h.

On the other side hypothesis (2.19), for every $q \geq p$, implies that the functions $\left|\frac{\partial f_h}{\partial x}\right|$ are at least in $L^n_{\mathrm{loc}}(\Omega_0)$. This is the reason for which we have required the stronger hypothesis $q \geq n$ still in (2.3). ■

For $\Omega \subset\subset \Omega_0$ we extend the functionals F_h setting

$$G_h(\Omega, \cdot) : u \in H^{1,q}_{\mathrm{loc}}(\Omega_0) \to \int_\Omega \phi_h(x, Du)dx. \tag{2.20}$$

By (2.5), (2.19) and Hölder inequality it soon follows that $G_h(\Omega, \cdot)$ is convex and finite on $H^{1,q}_{\mathrm{loc}}(\Omega_0)$.

To prove the extension result it will be useful to recall the following proposition from [DG F]:

PROPOSITION 2.3. *Let (U, τ) be a topological space, let σ be a topology on U finer than τ and let L be a σ-dense subset of U. Assume that the functionals of the sequence (F_h) are σ-continuous on U and that*

$$F(u) = \Gamma^-(\tau) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \quad u \in U.$$

Define

$$G_h(u) = \begin{cases} F_h(u) & \text{if } u \in L \\ +\infty & \text{if } u \in U - L, \end{cases}$$

then

$$F(u) = \Gamma^-(\tau) \lim_{\substack{h\to\infty \\ v\to u}} G_h(v) \quad u \in U.$$

THEOREM 2.4. *Assume that (2.17) and (2.19) hold. Then*

$$\begin{aligned} \int_\Omega g(x, Du)dx &= \Gamma^-(M_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} G_h(\Omega, v) = \\ &= \Gamma^-(C^0_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} G_h(\Omega, v) \end{aligned} \tag{2.21}$$

for every open set Ω of Ω_1 and u in $H^{1,q}_{\mathrm{loc}}(\Omega_0)$.

Proof. Define the functionals on $H^{1,q}_{\mathrm{loc}}(\Omega_0)$

$$F'_h(\Omega, u) = \begin{cases} \displaystyle\int_\Omega \phi_h(x, Du)dx & \text{if } u \in \mathrm{Lip}_{\mathrm{loc}} \\ +\infty & \text{if } u \in H^{1,q}_{\mathrm{loc}}(\Omega_0) - \mathrm{Lip}_{\mathrm{loc}}. \end{cases}$$

Let us argue with the topology $M_0(\Omega)$, the proof for the other being the same.

By general compactness theorems in Γ-convergence theory (see [DG F]) we have, eventually for a subsequence:

$$\text{(2.22)} \quad \Gamma^-(M_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} G_h(\Omega, v) \leq \int_\Omega g(x, Du)dx \leq$$

$$\leq \Gamma^-(M_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} F'_h(\Omega, v) = F'(\Omega, u)$$

for every u in $\mathrm{Lip}_{\mathrm{loc}}$.

Hence, by Proposition 2.3 with $\tau = M_0(\Omega)$, $\sigma = H^{1,q}(\Omega)$ and $L = \mathrm{Lip}_{\mathrm{loc}}$, and by (2.21) it follows that

$$\text{(2.23)} \qquad \int_\Omega g(x, Du)dx = F'(\Omega, u) \quad \text{for every } u \text{ in } \mathrm{Lip}_{\mathrm{loc}}.$$

By (2.18) and (2.19) both the functionals in (2.23) are convex and finite on $H^{1,q}_{\mathrm{loc}}(\Omega_0)$, hence they are continuous in this space, therefore the thesis follows from (2.23). ■

In order to complete the identification result we now recall the concept of local minimum for a functional and study the minimality properties of the f_h with respect to G_h.

We say that $u \in H^{1,q}_{\mathrm{loc}}(\Omega_0)$ is a local minimum on $H^{1,q}_{\mathrm{loc}}(\Omega_0)$ for the functional G_h if for every open subset $A \subset\subset \Omega_0$ and v in $H^{1,q}_0(A)$ it results

$$G_h(A, u) \leq G_h(A, u + v).$$

PROPOSITION 2.5. *For every $h \in \mathbb{N}$ and $\eta \in \mathbb{R}^n$ the function $u_h(x) = \eta \cdot f_h(x)$ is a local minimum on $H^{1,q}_{\mathrm{loc}}(\Omega_0)$ for the functional G_h.*

Proof. Since Φ_h is convex and verifies (2.1) it will be subdifferentiable everywhere in $\mathbb{R}^n$.

For every $\xi \in \mathbb{R}^n$ let $\partial^-\Phi_h(\xi)$ be the set of subdifferentials of Φ_h in ξ.

Let $\alpha_h \in \partial^-\Phi_h(\xi)$, $A \subset\subset \Omega_0$ and ψ in $H^{1,q}_0(A)$.

We will have:

$$Du_h(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1} = \eta\cdot\frac{\partial f_h}{\partial x}(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1} = \eta\cdot I = \eta,$$

and

$$\begin{aligned}(2.24)\qquad &\int_A \Phi_h\left(D(u_h+\psi)(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f_h}{\partial x}(x)dx \geq\\ &\geq \int_A \left(D\psi(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\alpha_h^t\right)\det\frac{\partial f_h}{\partial x}(x)dx+\\ &+\int_A \Phi_h\left(Du_h(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f_h}{\partial x}(x)dx\end{aligned}$$

Hence, by (2.24) and Proposition 1.4:

$$(2.25)\quad G_h(A,u_h+\psi)\geq\int_A D\psi(x)Adj\,\frac{\partial f_h}{\partial x}(x)\alpha_h^t dx+G_h(A,u_h)=G_h(A,u_h).$$

that is the thesis. ■

We now prove a first characterization of the function g.

LEMMA 2.6. *For every* $\eta\in\mathbb{R}^n$ *and every open set* $\Omega\subset\subset\Omega_1$ *it results*

$$(2.26)\qquad \int_\Omega g\left(x,\eta\cdot\frac{\partial f}{\partial x}\right)dx=\Phi(\eta)\int_\Omega\det\frac{\partial f}{\partial x}dx.$$

Proof. Fix an open subset Ω of Ω_1.

Set $u_h(x)=\eta\cdot f_h(x)$, $u(x)=\eta\cdot f(x)$.

Because $Du_h(x)=\eta\cdot\frac{\partial f_h}{\partial x}(x)$, $Du(x)=\eta\cdot\frac{\partial f}{\partial x}(x)$ and $\phi_h(x,Du_h(x))=\Phi_h(\eta)\det\frac{\partial f_h}{\partial x}(x)$ it will sufficies to prove that

$$\lim_{h\to\infty}\int_\Omega\phi_h(x,Du_h)dx=\int_\Omega g(x,Du)dx$$

and then use (2.2) and (2.8).

By (2.21) we can select a sequence (v_h) in $H^{1,q}_{\mathrm{loc}}(\Omega_0)$ such that $v_h \to u$ in $L^q(\Omega)$ and

$$\int_\Omega g(x, Du)dx = \lim_{h\to\infty} \int_\Omega \phi_h(x, Du_h)dx. \tag{2.27}$$

Let B, B_1, B_2 be three open subsets of Ω such that $B \subset\subset B_1 \subset\subset B_2 \subset\subset \Omega$ and meas $(\partial B) = 0$.

Let ψ in $C_0^1(B_1)$ such that

$$\begin{cases} 0 \leq \psi(x) \leq 1 & \text{for every } x \text{ in } \Omega \\ \psi(x) = 1 & \text{for every } x \text{ in } B \end{cases} \tag{2.28}$$

and, for every $t \in]0, 1[$ let γ^t be a function in $C^1(\bar{\Omega})$ such that

$$\gamma^t(x) = \begin{cases} 1 & \text{for every } x \text{ in } B_1 \\ 1/(1-t) & \text{for every } x \text{ in } \Omega - B_2, \end{cases} \tag{2.29}$$

$$|D\gamma^t(x)| \leq \frac{2t}{1-t} \cdot \frac{1}{dist(\partial B_1, \partial B_2)}. \tag{2.30}$$

Set

$$w_h^t(x) = (1-t)\gamma^t(x)(\psi(x)v_h(x) + (1-\psi(x))u_h(x)), \tag{2.31}$$

then $w_h^t \in u_h + H_0^{1,q}(\Omega)$ and, by Proposition 2.5

$$\int_\Omega \phi_h(x, Du_h)dx \leq \int_\Omega \phi_h(x, Dw_h^t)dx = \int_{B_1} \phi_h(x, Dw_h^t)dx + \tag{2.32}$$
$$+ \int_{\Omega - B_1} \phi_h(x, Dw_h^t)dx = a_h + b_h.$$

We have, by an iterated use of the convexity of ϕ_h:

$$a_h = \int_B \phi_h(x, (1-t)Dv_h)dx +$$

$$+\int_{B_1-B}\phi_h(x,(1-t)\{\psi Dv_h+(1-\psi)Du_h+(v_h-u_h)D\psi\})dx\leq$$

$$\leq t\int_B\phi_h(x,0)dx+(1-t)\int_B\phi_h(x,Dv_h)dx+$$

$$+t\int_{B_1-B}\phi_h\left(x,\frac{1-t}{t}(u_h-v_h)D\psi\right)dx+$$

$$+(1-t)\left\{\int_{B_1-B}\psi(x)\phi_h(x,Dv_h)dx+\right.$$

$$\left.+\int_{B_1-B}(1-\psi(x))\phi_h(x,Du_h)dx\right\}.$$

By using (2.5) we obtain:

$$\text{(2.33)}\quad a_h\leq ct\int_B\det\frac{\partial f_h}{\partial x}(x)dx+\int_{B_1}\phi_h(x,Dv_h)dx+$$

$$+c\frac{(1-t)^p}{t^{p-1}}||D\psi||_{L^\infty(\Omega)}\int_{B_1-B}\Lambda_h^{p-1}\left|\frac{\partial f_h}{\partial x}\right|^{n-p}|u_h-v_h|^p dx+$$

$$+\int_{B_1-B}\phi_h(x,Du_h)dx.$$

Note that from (2.19) it soon follows that:

$$\text{(2.34)}\qquad \int_{B_1-B}\Lambda_h^{p-1}\left|\frac{\partial f_h}{\partial x}\right|^{n-p}|u_h-v_h|^p dx\leq Q(\Omega)||u_h-v_h||^p_{L^q(\Omega)}.$$

We observe now that:

$$\text{(2.35)}\quad b_h\leq\int_{\Omega-\bar B}\phi_h(x,Du_h)dx+\int_{\Omega-\bar B}\phi_h(x,(1-t)\{\gamma^t Du_h+u_hD\gamma^t\})dx.$$

By (2.27), (2.32), (2.33), (2.34) and (2.35) we deduce, passing to the limit as $h\to+\infty$ and remembering (2.8), that

$$\text{(2.36)}\quad \limsup_{h\to\infty}\int_\Omega\phi_h(x,Du_h)dx\leq\int_\Omega g(x,Du)dx+ct\int_{B_1}\det\frac{\partial f_h}{\partial x}(x)dx+$$

$$+\limsup_{h\to\infty}\left\{\int_{\Omega-\bar B}\phi_h(x,Du_h)+\phi_h(x,(1-t)\{\gamma^t Du_h+u_hD\gamma^t\})\right\}dx.$$

Let now

$$c_h = \int_{\Omega - \bar{B}} (\phi(x, Du_h) + \phi_h(x, (1-t)\{\gamma^t Du_h + u_h D\gamma^t\}))dx,$$

then by (2.1) it follows that

$$\begin{aligned} c_h \leq & 2c \int_{\Omega-\bar{B}} \det \frac{\partial f_h}{\partial x}(x)dx + \\ & + c\int_{\Omega-\bar{B}} \left| \eta \frac{\partial f_h}{\partial x}(x) \left(\frac{\partial f_h}{\partial x}(x) \right)^{-1} \right|^p \det \frac{\partial f_h}{\partial x}(x)dx + \\ & + c\int_{\Omega-\bar{B}} \left| (1-t)\gamma^t \eta \frac{\partial f_h}{\partial x}(x) \left(\frac{\partial f_h}{\partial x}(x) \right)^{-1} + \right. \\ & \left. + (1-t)u_h D\gamma^t \left(\frac{\partial f_h}{\partial x}(x) \right)^{-1} \right|^p \det \frac{\partial f_h}{\partial x}(x)dx. \end{aligned}$$

Recalling (1.3), (2.8), (2.30) and (2.19) we deduce that

$$\begin{aligned} \limsup_{h\to\infty} c_h \leq & 2c \int_{\Omega-\bar{B}} \det \frac{\partial f_h}{\partial x}(x)dx + \\ & + c(1+2^{p-1})|\eta|^p \int_{\Omega-\bar{B}} \det \frac{\partial f_h}{\partial x}(x)dx + \\ & + c2^{p-1} \frac{t^p}{dist(\partial B_1, \partial B_2)^p} Q(\Omega)||u||^p_{L^q(\Omega)}. \end{aligned} \tag{2.37}$$

Letting t go to zero, by means of (2.37), (2.36) becomes:

$$\begin{aligned} & \limsup_{h\to\infty} \int_\Omega \phi_h(x, Du_h)dx \leq \\ & \leq \int_\Omega g(x, Du)dx + c(2 + 2^{p-1}|\eta|^p) \int_{\Omega-\bar{B}} \det \frac{\partial f_h}{\partial x}(x)dx. \end{aligned} \tag{2.38}$$

If meas $(\Omega - B) \to 0$ we get by (2.21) and (2.38)

$$\text{(2.39)} \quad \int_\Omega g(x, Du)dx \leq \liminf_{h\to\infty} \int_\Omega \phi_h(x, Du_h)dx \leq$$

$$\leq \limsup_{h\to\infty} \int_\Omega \phi_h(x, Du_h)dx \leq \int_\Omega g(x, Du)dx$$

and the lemma is proved. ■

We can prove now the main result of this paper.

THEOREM 2.7. *Let Φ_h, Φ, f_h, f verify (2.1), (2.2), (2.3), (2.4), (2.8) and (2.19). Then*

$$\text{(2.40)} \quad \int_\Omega \Phi\left(Du(x)\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x)dx =$$

$$= \Gamma^-(M_0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega \Phi_h\left(Dv(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f_h}{\partial x}(x)dx =$$

$$= \Gamma^-(C_0^0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega \Phi_h\left(Dv(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f_h}{\partial x}(x)dx$$

for every open set $\Omega \subset\subset \Omega_0$ and u in $H^{1,q}_{\mathrm{loc}}(\Omega_0)$.

Proof. Let H be the set of the Lebesgue points in Ω_0 of the functions $\det\frac{\partial f}{\partial x}(x)$ and $g\left(x, \eta\frac{\partial f}{\partial x}(x)\right)$ for every η in $\mathbb{Q}^n$. Obviously meas$(H) = 0$.

By (2.26), for every x in $\Omega_1 - H$ and η in $\mathbb{Q}^n$ we get

$$\text{(2.41)} \qquad g\left(x, \eta\frac{\partial f}{\partial x}(x)\right) = \Phi(\eta)\det\frac{\partial f}{\partial x}(x)$$

and, by the continuity in z of g and the arbitrary choice of Ω_1, (2.41) will be valid for every x in $\Omega_0 - H$ and η in $\mathbb{R}^n$.

Since $\det\frac{\partial f}{\partial x}(x) > 0$ a.e., for every z in $\mathbb{R}^n$ and for almost all x in Ω_0 we will have

$$\text{(2.42)} \qquad g(x, z) = \Phi\left(z\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x),$$

and from (2.21) the theorem follows. ∎

Remark 2.8. Assumptions (2.19), that links q and p, and (2.3), from which the equiboundedness of $\left(\det \frac{\partial f_h}{\partial x}\right)$ in $L^1(\Omega)$ follows, imply the equiboundedness of the sequence (f_h) in some $(H^{1,r}(\Omega))^n$.

In fact let $t, s \in \mathbb{R}$, $0 < s < n$, to be chosen later; from (1.1), (1.2) and Hölder inequality it follows that

$$\text{(2.43)} \quad \int_\Omega \left|\frac{\partial f_h}{\partial x}\right|^{(1+t)s} dx \leq$$

$$\leq n^{s/2} \left(\int_\Omega \det \frac{\partial f_h}{\partial x} dx\right)^{s/n} \left(\int_\Omega \Lambda_h^{s(1-1/n)n/(n-s)} \left|\frac{\partial f_h}{\partial x}\right|^{stn/(n-s)} dx\right)^{1-s/n}$$

Now if we choose t and s so that (see (2.19))

$$\begin{cases} s\left(1 - \dfrac{1}{n}\right) \dfrac{n}{n-s} = (p-1)\dfrac{q}{q-p} \\[2ex] st\dfrac{n}{n-s} = (n-p)\dfrac{q}{q-p} \end{cases}$$

and if we further suppose that $f_h \to f$ in $(L^1(\Omega))^n$, by the $L^1(\Omega)$-equiboundedness of $\left(\det \frac{\partial f_h}{\partial x}\right)$ we get the equiboundedness of (f_h) in $(H^{1,r}(\Omega))^n$ with

$$r = (1+t)s = \frac{n(n-2)+p}{q(p-1)+(n-1)(q-p)}. \qquad ∎$$

3. Weighted Sobolev spaces and convergence of minima.

In this section we want to study the behaviour of the solutions of minimum problems related to the functionals $\int_\Omega \phi_h(x, Du)dx$.

I

Suppose that the functions Φ_h verify (2.2) and

$$(3.1) \qquad \begin{cases} \Phi_h \text{ strictly convex} \\ |\xi|^p \leq \Phi_h(\xi) \leq c_1(1+|\xi|^p) \quad \text{for every } \xi \text{ in } \mathbb{R}^n,\ p>1,\ c_1 \geq 1. \end{cases}$$

Let f_h, $h \in \mathbb{N}$, be functions in $(H^{1,n}_{\text{loc}}(\Omega_0))^n$ with $\det \dfrac{\partial f_h}{\partial x}(x) > 0$ a.e., then the functions

$$\phi_h(x,z) = \Phi_h\left(z\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right) \det \frac{\partial f_h}{\partial x}(x)$$

verify, by (1.3) and (1.4):

$$(3.2) \qquad \Lambda_h(x)^{1-n}\left\|\frac{\partial f_h}{\partial x}(x)\right\|^{n-p}|z|^p \leq \phi_h(x,z) \leq$$
$$\leq c_1 \det \frac{\partial f_h}{\partial x}(x) + c_1\Lambda_h(x)^{p-1}\left\|\frac{\partial f_h}{\partial x}(x)\right\|^{n-p}|z|^p$$

for almost all x in Ω_0 and every z in $\mathbb{R}^n$.

In order to have, for the functionals $\displaystyle\int_\Omega \phi_n(x,Du)dx$, finiteness and equicoeciveness in the same space we are forced to require the existence of a positive function w on Ω_0 and of a positive real number M such that

$$(3.3) \qquad \begin{cases} w(x) \leq \Lambda_h(x)^{1-n}\left\|\frac{\partial f_h}{\partial x}(x)\right\|^{n-p} & \text{a.e. in } \Omega_0 \\ \Lambda_h(x)^{p-1}\left\|\frac{\partial f_h}{\partial x}(x)\right\|^{n-p} \leq Mw(x) & \text{a.e. in } \Omega_0. \end{cases}$$

By (3.3) it easily follows that

$$(3.4) \qquad \Lambda_h(x) \leq M^{1/(n+p-2)} \quad \text{a.e. in } \Omega_0,$$

this means, by (1.2), that the functions f_h are K-quasiregular on Ω_0 with $K = M^{(n-1)/(n+p-2)}$.

Define now $\nu = \max\{p, n\}$, then, because of (3.4) and (1.2), the estimates in (3.2) can be rewritten as

$$K^{1-\nu}\left(\det\frac{\partial f_h}{\partial x}(x)\right)^{1-p/n}|z|^p \le \phi_h(x,z) \le \tag{3.5}$$

$$\le c_1 \det\frac{\partial f_h}{\partial x}(x) + c_1 K^{\nu-1}\left(\det\frac{\partial f_h}{\partial x}(x)\right)^{1-p/n}|z|^p$$

for almost all x in Ω_0 and every z in $\mathbb{R}^n$.

Because of (3.3) we will assume (see (1.2)) that

(3.6)
$$\begin{cases} w(x) \le \left(\det\dfrac{\partial f_h}{\partial x}(x)\right)^{1-p/n} \le Mw(x) & x \text{ a.e. in } \Omega_0 \quad \text{if } p \ne n \\ \left\|\det\dfrac{\partial f_h}{\partial x}\right\|_{L^1(\Omega)} \le Q(\Omega) < +\infty & \text{for every } \Omega \subset\subset \Omega_0 \quad \text{if } p = n \end{cases}$$

We now introduce a class of weighted Sobolev spaces in which define the functionals $\int_\Omega \phi_h(x, Du)dx$ in order to have minimum points.

For every positive function σ in $L^1_{\text{loc}}(\Omega_0)$ and every $\Omega \subset\subset \Omega_0$ define $H_0^{1,p}(\Omega,\sigma)$ as the completion of $C_0^1(\Omega)$ in the topology induced by the norm

$$||u||_{H_0^{1,p}(\Omega,\sigma)} = \left(\int_\Omega |Du|^p \sigma(x)dx\right)^{1/p}.$$

Observe that if $\sigma^{-1/(p-1)}$ is in $L^1_{\text{loc}}(\Omega_0)$ then $H_0^{1,p}(\Omega,\sigma)$ embeds continuously in $H_0^{1,1}(\Omega)$ and compactly in $L^1(\Omega)$ for every $\Omega \subset\subset \Omega_0$.

In the following we will consider the case in which $\sigma(x) = w(x)$, w being given by (3.6), $w \equiv 1$ if $p = n$.

In order to obtain the result on the convergence in $L^1(\Omega)$ of the minimum points we will need first of all a good definition of the space $H_0^{1,p}(\Omega, w)$.

This is ensured if (f_h) is a sequence of K-quasiregular mappings on Ω_0 such that

$$f_h \to f \quad \text{in } (L^1_{\text{loc}}(\Omega_0))^n \tag{3.7}$$

and

$$\text{(3.8)} \qquad \text{if } p < n \quad w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega_0)$$

$$\text{(3.9)} \qquad \text{if } p > n \quad w \in L^1_{\mathrm{loc}}(\Omega_0).$$

In fact observe that from (3.6) it follows that $w \in L^1_{\mathrm{loc}}(\Omega_0)$ if $p < n$ and that $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega_0)$ if $p > n$. This holds since f_h is in $(H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$ and hence $w^{n/(n-p)}$ if $p < n$ and $w^{-n.(p-n)}$ if $p > n$ are in $L^1_{\mathrm{loc}}(\Omega_0)$.

If $p = n$ obviously $H^{1,p}_0(\Omega, w) = H^{1,n}_0(\Omega)$.

In conclusion in any case $H^{1,p}_0(\Omega, w)$ continuously embeds in $H^{1,1}_0(\Omega)$.

Thanks to the above hypotheses on f_h we soon get informations on f.

LEMMA 3.1. *Let (f_h) be a sequence of K-quasiregular mappings on Ω_0 verfying (3.6) and (3.7).*

Then (f_h) converges weakly to f in $(H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$ and f itself is K-quasiregular on Ω_0.

Proof. Let $\Omega \subset\subset \Omega_0$. By (3.6) and the K-quasiregularity we get that $f(_h)$ is bounded in $(H^{1,n}(\Omega))^n$ and, by (3.7), that (f_h) converges weakly to f in $(H^{1,n}(\Omega))^n$.

Therefore for every nonnegative function ψ in $C^\infty_0(\Omega)$ by Proposition 1.3 it results

$$\int_\Omega \left|\frac{\partial f}{\partial x}\right|^n \psi(x)dx \leq \liminf_{h\to\infty} \int_\Omega \left|\frac{\partial f_h}{\partial x}\right|^n \psi(x)dx \leq$$
$$\leq n^{n/2} K \liminf_{h\to\infty} \int_\Omega \det \frac{\partial f_h}{\partial x}(x)\psi(x)dx = n^{n/2} K \int_\Omega \det \frac{\partial f}{\partial x}(x)\psi(x)dx.$$

By the arbitrary choice of ψ we get

$$\text{(3.10)} \qquad \frac{\partial f}{\partial x}(x) \leq K n^{n/2} \det \frac{\partial f}{\partial x}(x) \quad \text{a.e. in } \Omega_0,$$

i.e. f is K-quasiregular on Ω_0. ■

In order to apply Theorm 2.7 we will need some further hypotheses, at least in the case $p > n$.

Such hypotheses are the following

$$(3.11) \qquad \begin{cases} \text{if } p > n \ f_h,\ f \in (H^{1,q}_{\text{loc}}(\Omega_0))^n & \text{with } q \geq p \\ w \in L^{q/(q-p)}_{\text{loc}}(\Omega_0). \end{cases}$$

Observe that (3.11) implies (3.9).

We are now able to apply Theoem 2.7 and to extend it to the whole $H^{1,p}_0(\Omega, w)$.

LEMMA 3.2. *Let Φ_h, Φ verify (3.1) and (2.2). Let (f_h) be a sequence of K-quasiregular mappings on Ω_0 verifying (3.6), (3.7), (3.8) and (3.11).*

Then f is K-quasiregular on Ω_0 and

$$\int_\Omega \Phi\left(Du(x)\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x)dx = \Gamma^-(M_0(\Omega))\lim_{\substack{h\to\infty\\ v\to u}}\int_\Omega \phi_h(x, Dv)dx =$$

$$= \Gamma^-(C^0_0(\Omega))\lim_{\substack{h\to\infty\\ v\to u}}\int_\Omega \phi_h(x, Dv)dx$$

for every open set $\Omega \subset\subset \Omega_0$ and u in $H^{1,p}_0(\Omega, w)$.

Proof. By Lemma 3.1 f is K-quasiregular on Ω_0.

The sequence (f_h) converges uniformly to f. In fact, by (3.6), the functions f_h are equibounded in $(H^{1,n}(\Omega))^n$, then by Theorem 1.2 they are equibounded in $H^{1,q}(\Omega)$ with $q > n$ and hence equi-Hölder continuous.

Since they converge to f in $L^1(\Omega)$ the uniform convergence follows.

Hypothesis (2.4) follows directly from Proposition 1.1 and Lemma 3.1.

In order to obtain (2.8) we can apply first Lemma 3.1 and after Proposition 1.3 getting

(3.12)

$$\int_{\Omega_0} \psi(x)\det\frac{\partial f_h}{\partial x}(x)dx \to \int_{\Omega_0} \psi(x)\det\frac{\partial f}{\partial x}dx \quad \text{for every } \psi \text{ in} C^\infty_0(\Omega_0).$$

From the equiboundedness of (f_h) in $H^{1,q}(\Omega)$ for some $q > n$ follows the one of $\left(\det\frac{\partial f_h}{\partial x}\right)$ in $L^{q/n}(\Omega)$. Hence (2.8) easily follows from (3.12).

Finally (2.19) follows directly from (3.11) if $p > n$ or from (3.6) and Theorem 1.2 if $p < n$.

By the above considerations we can apply Theorem 2.7.

To get the thesis we have now to argue as in the proof of Theorem 2.4 working with $H_0^{1,p}(\Omega, w)$ instead of $H_{\text{loc}}^{1,q}(\Omega_0)$ and with $\text{Lip}_{\text{loc}} \cap H_0^{1,p}(\Omega, w)$ instead of Lip_{loc}.

Here we only have to observe that, by (3.5) and (3.6) the functional $\int_\Omega \Phi\left(Du(x)\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right) \det \frac{\partial f}{\partial x}(x)dx$ is finite on $H_0^{1,p}(\Omega, w)$. ■

We can prove now the result on the convergence of minima.

THEOREM 3.3. *Let Φ_h, Φ verify (3.1) and (2.2). Let (f_h) be a sequence of K-quasiregular mappings on Ω_0 verifying (3.6), (3.7), (3.8) and (3.11).*

Then f is K-quasiregular on Ω_0.

Further for every open set $\Omega \subset\subset \Omega_0$ and every g in $L^\infty(\Omega)$ the minimum points of the problems

$$(3.13)\quad \min_{v \in H_0^{1,p}(\Omega,w)} \left\{ \int_\Omega \Phi_h\left(Dv(x)\left(\frac{\partial f_h}{\partial x}(x)\right)^{-1}\right) \det \frac{\partial f_h}{\partial x}(x)dx + \int_\Omega g(x)v(x)dx \right\}$$

converge in $L^1(\Omega)$ to the minimum point of the problem

$$(3.14)\quad \min_{v \in H_0^{1,p}(\Omega,w)} \left\{ \int_\Omega \Phi\left(Dv(x)\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right) \det \frac{\partial f}{\partial x}(x)dx + \int_\Omega g(x)v(x)dx \right\},$$

and the convergence of the minimum values holds.

Proof. First of all observe that, by Lemma 3.1, the function f verifies

$$(3.15)\qquad w(x) \le \left(\det \frac{\partial f}{\partial x}(x)\right)^{1-p/n} \le M w(x) \quad \text{a.e. in } \Omega_0$$

and hence, by (3.5) written with ϕ instead of ϕ_h, follows that problem (3.14) is well posed.

Fix an open set $\Omega \subset\subset \Omega_0$ and g in $L^\infty(\Omega)$.

By Lemma 3.2 and the $L^1(\Omega)$-continuity of $\int_\Omega gu dx$ it soon follows that (see also [DG F] Proposizione 1.11)

$$(3.16)\quad \int_\Omega \Phi\left(Du(x)\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x)dx + \int_\Omega g(x)u(x)dx =$$
$$= \Gamma^-(L^1(\Omega))\lim_{\substack{h\to\infty \\ v\to u}}\left\{\int_\Omega \phi_h(x, Dv)dx + \int_\Omega g(x)v(x)dx\right\}$$

for every u in $H_0^{1,p}(\Omega, w)$.

By (3.5) and (3.6) the functionals in (3.15) are equicoercive in the topology of $L^1(\Omega)$ in the sense of Theorem 1.6; hence the thesis follows from Theorem 1.6. ■

Remark 3.4. If g is in the dual space of $H_0^{1,p}(\Omega, w)$ the same result of Theorem 3.3 continues to hold: in fact it will be sufficient to use, in the proof of Theorem 3.3, the $L^1(\Omega)$-continuity on bounded subsets of $H_0^{1,p}(\Omega, w)$ of the functional $u \in H_0^{1,p}(\Omega, w) \to \langle g, u\rangle$.

Observe that in general by (3.6) $w^{n/(n-p)}$ is in $L^1_{\text{loc}}(\Omega_0)$, hence the following continuous embeddings hold

$$-\text{if } p < n \quad H_0^{1,n}(\Omega) \subseteq H_0^{1,p}(\Omega, w)$$

$$-\text{if } p = n \quad H_0^{1,p}(\Omega, w) = H_0^{1,n}(\Omega)$$

$$-\text{if } p > n \quad H_0^{1,p}(\Omega, w) \subseteq H_0^{1,n}(\Omega).$$

Further, in the case of K-quasiregular mappings and $p > n$, by Theorem 1.2, we get that $w^{-n/(p-n)}$ is in $L^s_{\text{loc}}(\Omega_0)$ for some $s > 1$; hence we deduce that $H_0^{1,p}(\Omega, w)$ continuously embeds in $H_0^{1,n+\varepsilon}(\Omega)$ for suitable $\varepsilon > 0$.

Therefore in this case, by the equiboundedness in $H_0^{1,p}(\Omega, w)$ of the solutions of (3.13) one infers their convergence to the solution of (3.14) at least in some topology $C^{0,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$.

II

In this part we consider the case of K-quasiconformal mappings f_h and prove that, at least if $1 < p < n$, hypothesis (3.6), with no assumptions on w^{-1}, is sufficient to prove Theorem 3.3.

This result is obtained by means of the study of a class of weighted Sobolev spaces associated to quasiconformal mappings in the same order of ideas of [DA D].

If $p \geq n$ from (3.6) we get informations on w^{-1}, hence the results of part I apply.

For every positive function w in $L^1_{\mathrm{loc}}(\Omega_0)$, every $\Omega \subset\subset \Omega_0$ and $p \geq 1$ define

$$L^p(\Omega, w) = \{u \text{ measurable on } \Omega : \|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u|^p w(x)dx\right)^{1/p} < +\infty\}.$$

Denote with $G_0(\Omega)$ the set of the n-tuples of functions $v = (v_1, \dots, v_n)$ verifying $v(x) = Du(x)$ for some u in $C_0^1(\Omega)$.

Define $H_0^{1,p}(\Omega, w)$ as the closure of $G_0(\Omega)$ in $(L^p(\Omega, w))^n$, denote with $H^{-1,p'}(\Omega, w)$ its dual ($p' = p/(p-1)$, $p' = \infty$ if $p = 1$) and with $\langle \cdot, \cdot \rangle$ the duality among them.

Observe that $H_0^{1,p}(\Omega, w)$ is well defined since, by the positivity of w, every Cauchy sequence in $(L^p(\Omega, w))^n$ has an unique limit in $(L^p(\Omega, w))^n$.

Therefore, in general, $H_0^{1,p}(\Omega, w)$ will be only a closed subspace of $(L^p(\Omega, w))^n$.

To prove that $H_0^{1,p}(\Omega, w)$ is in fact a Sobolev space we must show that each of its elements is a gradient in a strong sense.

To this aim let us consider the case in which

$$w(x) = \left(\det \frac{\partial f}{\partial x}(x)\right)^{1-p/n} \quad \text{a.e. in } \Omega_0 \tag{3.17}$$

with f K-quasiconformal on Ω_0 and $1 \leq p < n$.

By 5) of Proposition 1.1 it follows that

$$\det \frac{\partial f^{-1}}{\partial y}(f(x)) = \left(\det \frac{\partial f}{\partial x}(x)\right)^{-1} \quad \text{a.e. in } \Omega_0. \tag{3.18}$$

Further we have for every z in $\mathbb{R}^n$

$$K^{-(1-1/n)}\left(\det \frac{\partial f}{\partial x}(x)\right)^{1/n}|z| \le \left|z\frac{\partial f}{\partial x}(x)\right| \le \le K^{1-1/n}\left(\det \frac{\partial f}{\partial x}(x)\right)^{1/n}|z|. \tag{3.19}$$

Then, arguing as in [DA D] Proposition 2.1, it can be proved that

PROPOSITION 3.5. *Let f be a K-quasiconformal mapping on Ω_0 and let Ω be an open set with $\Omega \subset\subset \Omega_0$.*

Let $1 \le p < n$ and assume (3.17), then there exists a constant $c = c(n, \Omega, K, p)$ such that for every u in $C_0^1(\Omega)$ it results

$$\int_\Omega |u|^p w(x)^{n/(n-p)}\,dx \le c\int_\Omega |Du|^p w(x)dx.$$

Proposition 3.5 proves that if the sequences $(D\psi_h')$ and $(D\psi_h'')$ converge to v in $(L^p(\Omega, w))^n$ then the sequences (ψ_h') and (ψ_h'') are converging sequences and converge to the same measurable function in $L^p(\Omega, w^{n/(n-p)})$.

Therefore to each v in $H_0^{1,p}(\Omega, w)$ can be uniquely associated a function u in $L^p(\Omega, w^{n/(n-p)})$ of which v can be considered the «gradient».

In the following we will say that a measurable function u is in $H_0^{1,p}(\Omega, w)$ if there exists a sequence (ψ_h) in $C_0^1(\Omega)$ such that (ψ_h) converges to u in measure and $(D\psi_h)$ is a Cauchy sequence in $(L^p(\Omega, w))^n$.

As a consequence we get the continuous embedding of $H_0^{1,p}(\Omega, w)$ in $L^p(\Omega, w^{n/(n-p)})$.

From these considerations and Proposition 3.5 we deduce easily

COROLLARY 3.6. *Let f be a K-quasiconformal mapping on Ω_0 and let Ω be an open set with $\Omega \subset\subset \Omega_0$.*

Let $1 \leq p < n$ *and assume (3.16), then there exists a constant* $c = c(n, \Omega, K, p)$ *such that for every* u *in* $H_0^{1,p}(\Omega, w)$ *it results*

$$\int_\Omega |u|^p w(x)^{n/(n-p)}\, dx \leq c \int_\Omega |Du|^p w(x) dx.$$

Moreover, arguing as in [DA D] Proposition 2.3, it is possible to characterize $H_0^{1,p}(\Omega, w)$ in the following way.

PROPOSITION 3.7. *Let* f *be a* K*-quasiconformal mapping on* Ω_0 *and let* Ω *be an open set with* $\Omega \subset\subset \Omega_0$.

Let $1 \leq p < n$ *and assume (3.17), then a function* u *is in* $H_0^{1,p}(\Omega, w)$ *if and only if the function* $U = u \circ f^{-1}$ *is in* $H_0^{1,p}(f(\Omega))$.

Besides there exists a constant c *independent on* u *such that*

$$\text{(3.20)} \qquad \begin{cases} c^{-1}||u||_{H_0^{1,p}(\Omega,w)} \leq ||U||_{H_0^{1,p}(f(\Omega))} \leq c||u||_{H_0^{1,p}(\Omega,w)} \\ c^{-1}||u||_{L^p(\Omega,w^{n/(n-p)})} \leq ||U||_{L^p(f(\Omega))} \leq c||u||_{L^p(\Omega,w^{n/(n-p)})} \end{cases}$$

As a consequence the following compactness result holds (see [DA D] Corollary 2.4).

PROPOSITION 3.8. *In the same assumptions of Proposition 3.7 the embedding of* $H_0^{1,p}(\Omega, w)$ *in* $M_0(\Omega)$ *is compact.*

We can prove now the analogous of Theorem 3.3.

THEOREM 3.9. *Let* Φ_h, Φ *verify (3.1) and (2.2) with* $1 \leq p < n$.

Let (f_h) *be a sequence of* K*-quasiconformal mappings on* Ω_0 *verifying (3.7) and (3.6) with* w *positive.*

Then f *is a nonconstant* K*-quasiconformal mapping on* Ω_0 *and verifies (3.15).*

Further for every open set $\Omega \subset\subset \Omega_0$ *and every* g *in* $H^{-1,p'}(\Omega, w)$ *the minimum points of the problems*

$$\text{(3.21)} \qquad \min_{v \in H_0^{1,p}(\Omega,w)} \left\{ \int_\Omega \Phi_h \left(Dv(x) \left(\frac{\partial f_h}{\partial x}(x)(x) \right)^{-1} \right) \det \frac{\partial f_h}{\partial x}(x) dx + \langle g, v \rangle \right\}$$

converge in $M_0(\Omega)$ to the minimum point of the problem

$$(3.22)\qquad \min_{v\in H_0^{1,p}(\Omega,w)}\left\{\int_\Omega \Phi\left(Dv(x)\left(\frac{\partial f}{\partial x}(x)(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x)dx+\langle g,v\rangle\right\}$$

and the convergence of the minimum values holds.

Proof. By Lemma 3.1 f is K-quasiregular on Ω_0 and verifies (3.15). By the positivity of w we deduce further that f is nonconstant.

The K-quasiconformality of f now follows from a result of Gehring (see [G1] and [C], see also [DA D] Proposition 1.4).

By (3.6) and (3.15) it is not restrictive to assume that

$$w(x)=\left(\det\frac{\partial f}{\partial x}(x)\right)^{1-p/n}.$$

Set, for sake of brevity, $\phi(x,z)=\Phi\left(z\left(\frac{\partial f}{\partial x}(x)\right)^{-1}\right)\det\frac{\partial f}{\partial x}(x).$

Fix an open set $\Omega\subset\subset\Omega_0$ and g in $H^{-1,p'}(\Omega,w)$.

Let us prove that from Lemma 3.2 it follows that

$$(3.23)\qquad \int_\Omega\phi(x,Du)dx+\langle g,u\rangle=\Gamma^-(M_0(\Omega))\lim_{\substack{h\to\infty\\ v\to u}}\left(\int_\Omega\phi_h(x,Dv)dx+\langle g,v\rangle\right)$$

for every u in $H_0^{1,p}(\Omega,w)$.

In fact, by Lemma 3.2, let $u_h\to u$ in $M_0(\Omega)$ such that

$$\int_\Omega\phi(x,Du)dx=\lim_{h\to\infty}\int_\Omega\phi_h(x,Du_h)dx,$$

then, by (3.6), (u_h) is bounded in $H_0^{1,p}(\Omega,w)$ and $u_h\rightharpoonup u$ in $H_0^{1,p}(\Omega,w)$, therefore we get

$$(3.24)\qquad \int_\Omega\phi(x,Du)dx+\langle g,u\rangle=\lim_{h\to\infty}\left(\int_\Omega\phi_h(x,Du_h)dx+\langle g,u_h\rangle\right).$$

Now let $v_h \to u$ in $M_0(\Omega)$ such that

$$\liminf_{h\to\infty} \int_\Omega \phi_h(x, Dv_h)dx < +\infty$$

then, at least for a subsequence, we deduce from (3.6) that (v_h) is bounded in $H_0^{1,p}(\Omega, w)$ and hence $v_h \rightharpoonup u$ in $H_0^{1,p}(\Omega, w)$.

By Lemma 3.2 we deduce

$$\int_\Omega \phi(x, Du)dx + \langle g, u\rangle \leq \liminf_{h\to\infty} \left(\int_\Omega \phi_h(x, Dv_h)dx + \langle g, v_h\rangle \right). \tag{3.25}$$

Now (3.23) follows from (3.24) and (3.25).

By (3.6) and Proposition 3.8 one sees that the functionals in (3.23) are equicoercive in the topology of $M_0(\Omega)$ in the sense of Theorem 1.6; hence the thesis follows from Theorem 1.6. ■

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Istituto di Fisica Matematica
ed Informatica
Facoltà di Ingegneria
84084 Fisciano (SALERNO)