# THE NONLINEAR CONTINUUM TRAFFIC EQUILIBRIUM PROBLEM AND A RELATED DIRICHLET PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

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In the framework of continuum models, a system-optimization problem for transportation networks is studied in nonlinear case. Some results of convex analysis are used to prove existence theorems and to derive variational inequalities for optimal flow. A nonhomogeneous Dirichlet problem is proved to solve the minimization problem, provided that some non standard conditions on first derivatives of the solution are fulfilled.

## Introduction.

The evaluation of optimal flows in a transportation network is usually carried out in the setting of discrete models (see, e.g., [1] and the references quoted there). These models are appropriate for interurban transportation systems and, generally, when there exists a relatively low number of (Origin, Destination)-couples and interconnecting roads.

When the network, as in dense metropolitan case, contains a very high number of (Origin, Destination)-couples and paths, it is not only hard but also useless to evaluate the flows on the single arcs. In this case, we use continuum models of transportation whose main objective is to estimate optimal traffic densities for given directions, (see, e.g., [1]),

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at the points of a plane region containing the network.

In this work we are concerned with this class of models, whose mathematical formulation leads to minimization problems for a "cost" function on convex subsets, mainly characterized by a flow conservation law. We prove general existence theorems in the case of a nonlinear cost function (for the linear case, see [6]) and equivalence theorems of the minimization problem with suitable variational inequalities.

Then, considering the particular case of a rectangular domain and introducing suitable "potential" functions, we replace the initial problem with a quasilinear nonhomogeneous Dirichlet problem, with further conditions on the sign of the first derivatives of the solution, for which we provide some existence theorems.

## 1. Mathematical formulation: notations and hypotheses.

To describe the traffic equilibrium problem in the continuum case, let us consider a bounded open subset $\Omega$ of the plane $\mathbf{R}^2$ and, following [1], let us suppose that the traffic only flows, at any point $x = (x_1, x_2) \in \Omega$, in the direction of increasing axes $x_1$ and $x_2$.

We can describe the flux by a vectorial field $u(x)$, whose components $u_1(x), u_2(x)$ represent the traffic density along the directions $x_1$ and $x_2$, respectively; so, we have

$$u_1(x) \geq 0, \quad u_2(x) \geq 0 \tag{1.1}$$

and $u_1(x), u_2(x)$ have non negative fixed traces $\varphi_1(x)$ and $\varphi_2(x)$ on $\partial\Omega$ (or on a part of $\partial\Omega$). To this end, let us suppose that $\Omega$ has a suitably smooth boundary $\partial\Omega$, and that $u(x) \in H^1(\Omega, \mathbf{R}^2)$; the function $\varphi = (\varphi_1, \varphi_2)$ belongs to $H^{\frac{1}{2}}(\partial\Omega, \mathbf{R}^2)$. Then, if we associate to each point $x \in \Omega$ a scalar field $t(x) \in L^2(\Omega)$ representing the density of the flux with origin or destination at $x$, we can write the following form of the conservation law:

$$\int\int_D t\, dx + \oint_{\partial D} (u \mid n)\, ds = 0 \tag{1.2}$$

where $n = (n_1, n_2)$ is the normal at $x \in \partial\Omega$, $D$ is any subdomain of $\Omega$ with smooth boundary $\partial D$, and the symbol $(|)$ denotes the inner product in $\mathbf{R}^2$.

If we write (1.2) for $D = \Omega$ we obtain the following natural relation between the density $t$ and the trace $\varphi = (\varphi_1, \varphi_2)$:

$$\int\!\!\int_\Omega t\,dx + \oint_{\partial\Omega} (\varphi_1 n_1 + \varphi_2 n_2)\,ds = 0\,. \tag{1.3}$$

Moreover, from (1.2), since $u \in H^1(\Omega, \mathbf{R}^2)$, we obtain the differential form (in distributional sense) of the conservation law:

$$\operatorname{div} u + t(x) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + t(x) = 0 \quad , \quad \forall\ x \in \Omega\,. \tag{1.4}$$

The "cost" for crossing over the point $x \in \Omega$ in the direction $x_i$ and with a flow $u(x)$ will be represented by a function depending on $x$ and $u(x)$: the explicit dependence of $c_i$ on $x$ reflects possible non homogeneity of the network, while the dependence of $c_i$ on $u(x)$ points out congestion effects.

It is well known that the equilibrium problem of transportation network can be formulated by two different points of view:

I) the first produces a "user-optimizing" flow pattern with the equilibrium property that no user has any incentive to change unilaterally his decision;

II) the second is the "system-optimizing" point of view, in which the criterion for selecting paths is that of minimizing the global cost spent on the network.

In this paper we refer to this second point of view, and we look for a distribution flow $u^0(x)$ minimizing the "global cost":

$$F(u) = \int_\Omega c(x, u(x))\,dx \tag{1.5}$$

with

$$c(x, u(x)) = c_1(x, u(x)) + c_2(x, u(x)) \tag{1.6}$$

Here $c_i(x, u(x))$ represents the total travel cost spent in the neighbourhood of $x$, along the direction $x_i$ and with traffic density $u(x)$. For the case where this total cost can be perceived to be the sum of the "personal" costs $\widetilde{c}_i(x, u(x))$ , $i = 1, 2$, (1.6) can be rewritten us:

$$c(x, u(x)) = \widetilde{c}_1(x, u(x))u_1(x) + \widetilde{c}_2(x, u(x))u_2(x). \tag{1.7}$$

Also, if one supposes a linear dependence of $\widetilde{c}_i$ on $u(x)$, one has:

$$c(x, u(x)) = (Au \mid u) + (B \mid u) \tag{1.8}$$

where $A$ and $B$ are a matrix of order two and a column vector, respectively, whose elements are functions only of $x$. This last case is studied in [6].

We observe that the "natural" set for the flow distributions is the convex

$$K = \{u(x) \in H^1(\Omega, \mathbf{R}^2) \mid u(x) \geq 0 \ , \\ u(x)|_{\partial\Omega} = \varphi(x) \ , \ \mathrm{div} u + t(x) = 0 \ \text{ in } \ \Omega\}.$$

The functional $F$, given by (1.5), does not depend upon the derivatives of $u$, unlike the $H_1$-norm of $u$; this is remarkable for our purposes, because in this situation the "coerciveness" of $F(u)$ over $K$ is not guaranteed, and it is well known the rôle which some form of coerciveness plays in existence theorems.

Hence, following [6], we weaken the problem and we look for the minimum of $F$ not over $K$, but over the closure of $K$ in $L^2(\Omega, \mathbf{R}^2)$, $\overline{K}^{L^2(\Omega)}$, where the coerciveness of $F$ can be guaranteed.

## 2. Hypotheses and results.

To achieve the results of this work, we assume first the following hypotheses on the cost function:

(i) the function $c(x, y)$, $(x, y) \in \Omega \times \mathbf{R}^2$, is

– measurable with respect to $x$;

– there exist $\alpha(x) \in L^\infty(\Omega)$, $\beta(x) \in L^2(\Omega)$, $\alpha(x) \geq 0$, $\beta(x) > 0$, $\nu > 0$, such that

$$\nu|y|^2 \leq c(x,y) \leq \alpha(x)\,|y|^2 + \beta(x) \quad \text{a.e. on } \Omega \times \mathbf{R}^2$$

(ii) $c(x, \lambda y + (1-\lambda)\overline{y}) \leq \lambda c(x,y) + (1-\lambda)\, c(x,\overline{y})$ , $\forall\ y, \overline{y} \in \mathbf{R}^2$, $\lambda \in ]0,1[$, a.e. in $\Omega$. (1)

Then we have the following:

THEOREM 2.1. *Under the hypotheses* (i), (ii), *there exists a flow distribution* $u^0 \in \overline{K}^{L^2(\Omega)}$ *such that*

$$F(u^0) = \min_{u \in \overline{K}^{L^2(\Omega)}} \int_\Omega c(x, u(x))\, dx\,. \tag{2.1}$$

*The solution is unique if* (ii) *holds with strict inequality for* $y \neq \overline{y}$.

Let us start by observing that $\overline{K}^{L^2(\Omega)}$ is a non-empty convex closed subset of the relfexive Banach space $L^2(\Omega)$. By (ii), $F$ is convex (strictly convex, eventually) on $\overline{K}^{L^2(\Omega)}$. Furthermore, $F$ is continuous on $\overline{K}^{L^2(\Omega)}$; in fact if we consider a neighbourhood of $\overline{u} \in \overline{K}^{L^2(\Omega)}$:

$$I_\delta(\overline{u}) := \{u \in \overline{K}^{L^2(\Omega)} \mid \|u - \overline{u}\|_2 < \delta\}\,,$$

we have, for $u \in I_\delta(\overline{u})$:

$$\big|\|u\|_2 - \|\overline{u}\|_2\big| \leq \|u - \overline{u}\|_2 < \delta$$

and then, by (i):

$$\begin{aligned} 0 \leq F(u) &\leq \sup_\Omega \alpha(x) \int_\Omega |u(x)|^2\, dx + \int_\Omega \beta(x)\, dx \leq \\ &\leq \sup_\Omega \alpha(x) \|u\|_2^2 + \int_\Omega \beta(x)\, dx \leq \\ &\leq \sup_\Omega \alpha(x) (\|\overline{u}\|_2 + \delta)^2 + \int_\Omega \beta(x)\, dx\,. \end{aligned}$$

(1) Let us observe that the conditions i) and ii) imply the continuity of $c(x,y)$ in $y$ for a.e. $x \in \Omega$.

$F$ is coercive on $\overline{K}^{L^2(\Omega)}$; in fact (i) shows that:

$$u \in \overline{K}^{L^2(\Omega)} \Longrightarrow F(u) \geq \nu \|u\|_2^2 .$$

The proof of Theorem (2.1) then follows from classical results of convex analysis.

The solution of (2.1) can be characterized using variational inequalities. For this, let us introduce the following hypothesis:

(iii) the function $c(x, y)$, $(x, y) \in \Omega \times \mathbf{R}^2$, is a differentiable function with respect to $y$.

We observe that the gradient $\nabla_y c(x, y)$ turns out to be measurable with respect to $x$ and continuous in $y$ (see [7], Cor. 25.51); moreover from (i), (ii), it follows that there exist $\gamma(x) \in L^\infty(\Omega)$, $\delta(x) \in L^2(\Omega)$, $\gamma(x) \geq 0$, $\delta(x) > 0$, such that

$$|\nabla_y c(x, y)| \leq \gamma(x)\,|y| + \delta(x) \quad \text{a.e. on } \Omega \times \mathbf{R}^2 .$$

Then we have the following:

THEOREM 2.2. *Under the hypotheses* (i), (ii), (iii), *if* $u^0$ *is a solution of the optimization problem* (2.1) *if and only if* $u^0$ *is a solution of the variational inequality*:

$$\int_\Omega \nabla_u c(x, u^0(x)) \cdot (u - u^0)\, dx \geq 0 \ , \quad \forall \ u \in \overline{K}^{L^2(\Omega)} . \tag{2.2}$$

If we denote $F'(u, v)$ the derivative of $F$ in $u \in \overline{K}^{L^2(\Omega)}$ in the direction $v$, we have, taking into account (iii),

$$F'(u, v) = \int_\Omega \nabla_u c(x, u(x)) \cdot v(x)\, dx$$

with $\nabla_u c(x, u(x)) \in L^2(\Omega, \mathbf{R})$. If $u^0 \in \overline{K}^{L^2(\Omega)}$ is a solution of (2.1), classical results on variational inequalities ensure us that $u^0$ is a solution of (2.2), (see, e.g., [8]).

Conversely, any solution of (2.2) is also a solution of the optimization problem (2.1), by the convexity of $F$.

## 3. The traffic equilibrium problem in a rectangular domain and a related Dirichlet problem for quasilinear elliptic equations.

In this section we limit ourselves to the case of a rectangular domain $\Omega \subset \mathbf{R}^2$

$$\Omega \equiv ]0, a[\times]0, b[.$$

In this case we can assume the traces of $u$ on $\partial\Omega$ in the following way:

$$\begin{cases} u_1(0, x_2) = \varphi_1(x_2), & x_2 \in ]0, b[\,, \\ u_1(a, x_2) = \psi_1(x_2), & x_2 \in ]0, b[, \end{cases} \tag{3.1}$$

and

$$\begin{cases} u_2(x_1, 0) = \varphi_2(x_1), & x_1 \in ]0, a[, \\ u_2(x_1, b) = \psi_2(x_1), & x_1 \in ]0, a[. \end{cases} \tag{3.2}$$

We first consider the case in which $t(x) = 0$ in $\Omega$; then, the following theorem holds:

THEOREM 3.1. *For any* $u(x) \in K$ *there exists* $U(x) \in H_1(\Omega)$ *such that*:

$$\frac{\partial U}{\partial x_2} = u_1, \quad \frac{\partial U}{\partial x_1} = -u_2 \tag{3.3}$$

Following [6], let us start by considering a succession $\{u^{(k)}\} \subset C^1(\overline{\Omega}, \mathbf{R}^2)$ such that $u_1^{(k)}$, $u_2^{(k)}$, $\dfrac{\partial u_1^{(k)}}{\partial x_1}$, $\dfrac{\partial u_2^{(k)}}{\partial x_2}$ converge to $u_1$, $u_2$, $\dfrac{\partial u_1}{\partial x_1}$, $\dfrac{\partial u_2}{\partial x_2}$ in $L^2(\Omega)$ and $u_1^{(k)}|_{x_1=0} = \varphi_1^{(k)}(x_2)$, $u_1^{(k)}|_{x_1=a} = \psi_1^{(k)}(x_2)$, $u_2^{(k)}|_{x_2=0} = \varphi_2^{(k)}(x_1)$, $u_2^{(k)}|_{x_2=b} = \psi_2^{(k)}(x_1)$ converge to $\varphi_1, \psi_1, \varphi_2, \psi_2$ in $L^2$, respectively; moreover we can construct this succession so that $\operatorname{div} u^{(k)} = 0$. Then there exists a function $U^{(k)}$ such that $\dfrac{\partial U^{(k)}}{\partial x_2} = u_1^{(k)}$, $\dfrac{\partial U^{(k)}}{\partial x_1} = -u_2^{(k)}$ and, taking into account the estimate

$$(3.4)\quad \int_\Omega |U^{(n)} - U^{(m)}|^2\, dx \le$$
$$\le 2a^2 b \int_0^a |\varphi_2^{(n)} - \varphi_2^{(m)}|^2\, dx_1 + 2b^2 \int_\Omega |u_1^{(n)} - u_1^{(m)}|^2\, dx,$$

we obtain the thesis. (2)

As a consequence of (3.1) and (3.2), the traces of $U(x)$ on $\partial\Omega$, which of course belong to $H^{\frac{3}{2}}$, become:

$$(3.5)\qquad \begin{cases} U(0, x_2) = \displaystyle\int_0^{x_2} \varphi_1(t)\, dt + c_1 = \Phi_1(x_2)\ , \ x_2 \in ]0, b[\ , \\ U(a, x_2) = \displaystyle\int_0^{x_2} \psi_1(t)\, dt + d_1 = \Psi_1(x_2)\ , \ x_2 \in ]0, b[\ , \end{cases}$$

$$(3.6)\qquad \begin{cases} U(x_1, 0) = -\displaystyle\int_0^{x_1} \varphi_2(t)\, dt + c_2 = \Phi_2(x_1)\ , \ x_1 \in ]0, a[\ , \\ U(x_1, b) = -\displaystyle\int_0^{x_1} \psi_2(t)\, dt + d_2 = \Psi_2(x_1)\ , \ x_1 \in ]0, a[\ . \end{cases}$$

Imposing the compatibility conditions for $U(x_1, x_2)$ at the vertex of $\Omega$:

$$\Phi_1(0) = \Phi_2(0)\ , \quad \Psi_1(0) = \Phi_2(a)\ , \quad \Psi_2(0) = \Phi_1(b)\, ,$$

after some calculations, we obtain

$$(3.7)\qquad c_1 = c_2 = c\ , \quad d_1 = c - \int_0^a \varphi_2(x_1)\, dx_1\ , \quad d_2 = c + \int_0^b \varphi_1(x_2)\, dx_2\, ,$$

where $c$ is an arbitrary constant, while the condition

$$\Psi_1(a) = \Psi_2(b)$$

(2) We observe that theorem (3.1) also holds for $(u_1, u_2) \in \bar{K}^{L^2(\Omega)}$, because one can shows an estimate of the type (3.4), where the first term of the right hand side is zero.

is equivalent to the conservation law on $\partial\Omega$.

Moreover, for every $u \in \overline{K}^{L^2(\Omega)}$ there exists a function $U \in H^1(\Omega)$ such that (3.3) holds (see Note (2)). Then, if we consider another function $v \in \overline{K}^{L^2(\Omega)}$ and denote by $V$ the function of $H^1(\Omega)$ for which (3.3) holds, the function $U - V$ belongs to $H^1_0(\Omega)$.

Taking into account the above consideration, the variational inequality (2.2) becomes the following: to find $U^0 \in H^1(\Omega)$ such that

$$(3.8)\quad \int_\Omega \left\{ \frac{\partial\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_1} \frac{\partial (U - U^0)}{\partial x^2} - \right.$$

$$\left. - \frac{\partial c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_2} \frac{\partial (U - U^0)}{\partial x_1} \right\} dx \geq 0,$$

$$\forall\ U \in H^1(\Omega) \text{ s.t. } \left(\frac{\partial U}{\partial x_2}, \frac{-\partial U}{\partial x_1}\right) \in \overline{K}^{L^2(\Omega)}.$$

Now, we wish to associate to (3.8) a suitable Dirichlet problem: to this end, let us assume the following hypotheses on the cost function (we recall that these assumptions are natural, roughly speaking, for quasilinear elliptic equations):

(v) $c_{y_i y_j} \in L^\infty(\Omega \times \mathbf{R}^2)$ for $i, j = 1, 2$ and satisfies:

$$c_{y_2y_2}\xi_1^2 + c_{y_1y_1}\xi_2^2 - (c_{y_1y_2} + c_{y_2y_1})\xi_1\xi_2 \geq \nu|\xi|^2,$$

$$\forall\ \xi = (\xi_1, \xi_2) \in \mathbf{R}^2)\ ,\ (\nu > 0).$$

Then, if we denote by $U^0$ the solution of class $H^2(\Omega)$ of the Dirichlet problem:

$$
\begin{cases}
\dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_1^2} \dfrac{\partial^2 U^0}{\partial x_2^2} + \dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_2^2} \dfrac{\partial^2 U^0}{\partial x_1^2} - \\
\quad - \left[\dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_1 \partial u_2} + \dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_2 \partial u_1}\right] \dfrac{\partial^2 U^0}{\partial x_1 \partial x_2} = \\
\quad = \dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_2 \partial x_1} - \dfrac{\partial^2 c\left(x, \dfrac{\partial U^0}{\partial x_2}, \dfrac{-\partial U^0}{\partial x_1}\right)}{\partial u_1 \partial x_2}, \quad \text{a.e. on } \Omega, \\
U^0(0, x_2) = \Phi_1(x_2) \\
U^0(a, x_2) = \Psi_1(x_2) \\
U^0(x_1, 0) = \Phi_2(x_1) \\
U^0(x_1, b) = \Psi_2(x_1)
\end{cases}
$$

and we suppose that the following conditions

$$
(3.9) \qquad \frac{\partial U^0}{\partial x_2} \geq 0 \quad , \quad \frac{\partial U^0}{\partial x_1} \leq 0 \quad , \quad \text{a.e. on} \quad \Omega,
$$

are fulfilled, the vector

$$
(3.10) \qquad u^0 = \left(\frac{\partial U^0}{\partial x_2}, -\frac{\partial U^0}{\partial x_1}\right)
$$

is a solution of the variational inequality (2.2).

To achieve this result, one can use the Gauss-Green formula, taking into account that $U - U^0$ belongs to $H_0^1(\Omega)$ and the assumption (v).

The existence of the solutions of problem (3.8) for example is guaranteed under standard assumption on data (see [4], p. 347), if we have existence and uniqueness theorems and a priori estimates for the following linear Dirichlet problem:

$$
(3.11) \qquad \begin{cases} L(U^0) \equiv a(x_1, x_2)\dfrac{\partial^2 U^0}{\partial x_1^2} + b(x_1, x_2)\dfrac{\partial^2 U^0}{\partial x_2^2} + \\ \\ +2c(x_1, x_2)\dfrac{\partial^2 U_0}{\partial x_1 \partial x_2} = f(x_1, x_2) \quad \text{a.e. on} \quad \Omega \\ U^0(0, x_2) = \Phi_1(x_2) \\ U^0(a, x_2) = \Psi_1(x_2) \\ U^0(x_1, 0) = \Phi_2(x_1) \\ U^0(x_1, b) = \Psi_2(x_1) \end{cases}
$$

Problem (3.11) usually is decomposed in a problem with homogeneous boundary conditions and in a non-homogeneous boundary problem for the Laplacian operator. If we denote by $v \in H^2(\Omega)$ the solution of the problem:

$$
(3.12) \qquad \begin{cases} \Delta v = 0 & \text{a.e. on } \Omega \\ v(0, x_2) = \Phi_1(x_2) \\ v(a, x_2) = \Psi_1(x_2) \\ v(x_2, 0) = \Phi_2(x_1) \\ v(x_1, b) = \Psi_2(x_1) \end{cases}
$$

and by $w \in H^2(\Omega)$ the solution of the problem

$$
(3.13) \qquad \begin{cases} Lw = f - Lv \,, & \text{a.e. on } \Omega \\ w(0, x_2) = 0 \\ w(a, x_2) = 0 \\ w(x_1, 0) = 0 \\ w(x_1, b) = 0 \end{cases}
$$

the function $U^0 = v + w$ is a solution of (3.11).

It is well known that the solution of problem (3.13) exists and it is unique; for problem (3.12) the solution is ensured, since, in our case, the compatibility conditions indicated by [3], Th. 3.1.2.4, p. 260, become conditions (3.7).

Now, let us suppose $t(x) \neq 0$ and belonging to $H^1(\Omega)$. Following [6], we write condition

$$
(3.14) \qquad \operatorname{div} u + t(x) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0
$$

where

$$v_1 = u_1 + \frac{1}{2}\int_0^{x_1} t(\tau_1, x_2)\, d\tau_1 \,,$$

$$v_2 = u_2 + \frac{1}{2}\int_0^{x_2} t(x_1, \tau_2)\, d\tau_2 \,.$$

If we put

$$v = (v_1, v_2) \quad , \quad t_1(x_1, x_2) = \frac{1}{2}\int_0^{x_1} t(\tau_1, x_2)\, d\tau_1 \,,$$

$$t_2(x_1, x_2) = \frac{1}{2}\int_0^{x_2} t(x_1, \tau_2)\, d\tau_2 \quad , \quad \vartheta = (t_1, t_2)\,,$$

our functional $F$ becomes

$$F^*(v) = F(v - \vartheta) = \int_\Omega c(x, v(x) - \vartheta(x))\, dx \tag{3.15}$$

and the convex $K$ is transformed into

$$\begin{aligned} K^* = \{ & v \in H^1(\Omega, \mathbf{R}^2) \; ; \; v(x) \geq \vartheta(x) \,, \\ & v_1(0, x_2) = \varphi_1(x_2) \,, \; v_1(a, x_2) = \psi_1(x_2) + t_1(a, x_2) \,, \\ & v_2(x_1, 0) = \varphi_2(x_1) \,, \; v_2(x_1, b) = \psi_2(x_1) + t_2(x_1, b) \,, \\ & \mathrm{div v} = 0 \;, \; \text{a.e. on } \Omega \} \,. \end{aligned}$$

The minimum $v^0$ of $F^*$ on $K^*$ satisfies the variational inequality:

$$\int_\Omega \nabla_u c(x, v^0(x) - \vartheta(x)) \cdot (v(x) - v^0(x))\, dx \geq 0 \,, \qquad \forall \; v \in \overline{K^*}^{L^2(\Omega)} \,. \tag{3.16}$$

If we repeat the procedure we used above and introduce the potential $V^0$ associated by means of (3.3) to $v^0$, we can consider for $V^0$ the

Dirichlet problem:

$$
(3.17)\quad \begin{cases}
\dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, -\dfrac{\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_1^2} \dfrac{\partial^2 V^0}{\partial x_2^2} + \\
+ \dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, -\dfrac{\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_2^2} \dfrac{\partial^2 V^0}{\partial x_1^2} - \\
- \left[ \dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, -\dfrac{\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_1 \partial u_2} + \right. \\
\left. + \dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, -\dfrac{\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_2 \partial u_1} \right] \dfrac{\partial^2 V^0}{\partial x_1 \partial x_2} = \\
= \dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, -\dfrac{\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_2 \partial x_1} - \\
- \dfrac{\partial^2 c\left(x, \dfrac{\partial V^0}{\partial x_2} - \vartheta_1, \dfrac{-\partial V^0}{\partial x_1} - \vartheta_2\right)}{\partial u_1 \partial x_2} \\
V^0(0, x_2) = \Phi_1(x_2) \\
V^0(a, x_2) = \Psi_1(x_2) + \displaystyle\int_o^{x_2} t_1(a, \tau_2)\, d\tau_2 \\
V^0(x_1, 0) = \Phi_2(x_1) \\
V^0(x_1, b) = \Psi_2(x_1) + \displaystyle\int_0^{x_1} t_2(\tau_1, b)\, d\tau_1 ,
\end{cases}
$$

and, also in the case $t(x) \neq 0$, we can affirm that the vector

$$\left( \frac{\partial V^0}{\partial x_2}, -\frac{\partial V^0}{\partial x_1} \right)$$

is the unique solution of our problem, provided that

$$\frac{\partial V^0}{\partial x_2} \geq t_1(x) \qquad \frac{\partial V^0}{\partial x_1} \leq t_2(x) .$$

**4. The case of linear "personal cost".**

Let us suppose that $c(x, u(x))$ depends on $u(x)$ as in (1.7) and that the network is the rectangular grid $\Omega = ]0, a[\times]0, b[$. In particular we suppose that $t(x) = 0$ and assume that $c(x, u(x))$ according to (1.7) is given by

$$c(x, u(x)) = (Au \mid u) + (B \mid u) \tag{4.1}$$

where $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with $a_i > 0$, $b_i \geq 0$, $i = 1, 2$. Moreover we assume that $u_1(x), u_2(x)$ satisfy the boundary conditions

$$\begin{cases} u_1(0, x_2) = \varphi_1(x_2)\,, & x_2 \in ]0, b[ \\ u_1(a, x_2) = \psi_1(x_2)\,, & x_2 \in ]0, b[ \\ u_2(x_1, 0) = \varphi_2(x_1)\,, & x_1 \in ]0, a[ \\ u_2(x_1, b) = \psi_2(x_1)\,, & x_1 \in ]0, a[ \end{cases} \tag{4.2}$$

where $\varphi_1, \psi_1, \varphi_2, \psi_2$ are non negative functions belonging to $C^\infty$ and verifying condition (1.3) that, in our case, becomes:

$$\int_0^a \varphi_2(x_1)\, dx_1 - \int_0^a \psi_2(x_1)\, dx_1 = \int_0^b \psi_1(x_2)\, dx_2 - \int_0^b \varphi_1(x_2)\, dx_2\,. \tag{4.3}$$

Then taking into account the results of Section 3, we can say that the solution $U^0$ of the following Dirichlet problem

$$\begin{cases} a_2 \dfrac{\partial^2 U^0}{\partial x_1^2} + a_1 \dfrac{\partial^2 U^0}{\partial x_2^2} = 0 \\ U^0(0, x_2) = \Phi_1(x_2) \\ U^0(a, x_2) = \Psi_1(x_2) \\ U^0(x_1, 0) = \Phi_2(x_1) \\ U^0(x_1, b) = \Psi_2(x_1)\,, \end{cases} \tag{4.4}$$

where $\Phi_1, \Psi_1, \Phi_2, \Psi_2$ are given by (3.5) and (3.6), is such that the vector

$$u^0 = (u_1^0, u_2^0) = \left( \frac{\partial U^0}{\partial x_2}, -\frac{\partial U^0}{\partial x_1} \right) \tag{4.5}$$

is the optimal flow if

$$\frac{\partial U^0}{\partial x_2} \geq 0 \quad , \quad \frac{\partial U^0}{\partial x_1} \leq 0 \text{ in } \Omega . \tag{4.6}$$

In order to achieve a comparison with other results, let us observe that a necessary and sufficient condition for the relation $\frac{\partial U^0}{\partial x_2} \geq 0$ to hold in $\Omega$ is that [3]

$$\partial U^0 \frac{(x_1, 0)}{\partial x_2} = \chi_2(x_1) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{\frac{n\pi}{b}}{shk \frac{a}{b} n\pi} \cdot \tag{4.8}$$

$$\left[ sh \frac{kn\pi x_1}{b} \int_0^b \Psi_1(t) \sin \frac{n\pi}{b} t \, dt + \right.$$

---

[3] We recall that the solution of problem (4.4) is given by the Fourier series

$$U^0(x_1, x_2) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x_2}{b}}{sh \frac{n\pi a \sqrt{a_1}}{b\sqrt{a_2}}} \cdot \tag{4.7}$$

$$\left[ sh \frac{n\pi\sqrt{a_1} x_1}{\sqrt{a_2} b} \int_0^b \Psi_1(t) \sin \frac{n\pi}{b} t \, dt + \right.$$

$$\left. + sh \frac{n\pi\sqrt{a_1}(a - x_1)}{\sqrt{a_2} b} \int_0^b \Phi_1(t) \sin \frac{n\pi t}{b} dt \right] +$$

$$+ \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{a} x_1}{sh \frac{nb\sqrt{a_2}}{a\sqrt{a_1}}} \cdot \left[ sh \frac{n\pi\sqrt{a_2} x_2}{\sqrt{a_1} a} \int_0^a \Psi_2(t) \sin \frac{n\pi}{a} t \, dt + \right.$$

$$\left. + sh \frac{n\pi\sqrt{a_2}(b - x_2)}{\sqrt{a_1} a} \int_0^a \Phi_2(t) \sin \frac{n\pi t}{b} dt \right] ,$$

and, hence, $U^0(x_1, x_2)$ belongs to $C^\infty(\Omega)$.

$$+sh\,\frac{kn\pi(a-x_1)}{b}\int_0^b \Phi_1(t)\,\sin\frac{n\pi t}{b}\,dt\Bigg]+$$

$$+\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin\dfrac{n\pi}{a}x_1}{sh\,\dfrac{nb\pi}{ka}}\cdot\Bigg[\frac{n\pi}{ka}\int_0^a \Psi_2(t)\,\sin\frac{n\pi}{a}t\,dt-$$

$$-\frac{n\pi}{ka}\,ch\,\frac{n\pi b}{ka}\int_0^a \Phi_2(t)\,\sin\frac{n\pi t}{b}\,dt\Bigg]\geq 0$$

$$\partial U^0\frac{(x_1,b)}{\partial x_2}=\zeta_2(x_1)=\frac{2}{b}\sum_{n=1}^{\infty}\frac{\dfrac{n\pi}{b}\cos n\pi}{shk\,\dfrac{a}{b}\,n\pi}\cdot \tag{4.9}$$

$$\Bigg[sh\,\frac{kn\pi x_1}{b}\int_0^b \Psi_1(t)\,\sin\frac{n\pi}{b}t\,dt+$$

$$+sh\,\frac{kn\pi(a-x_1)}{b}\int_0^b \Phi_1(t)\,\sin\frac{n\pi t}{b}\,dt\Bigg]+$$

$$+\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin\dfrac{n\pi}{a}x_1}{sh\,\dfrac{nb\pi}{ka}}\cdot\Bigg[\frac{n\pi}{ka}\,ch\,\frac{n\pi b}{ka}\int_0^a \Psi_2(t)\sin\frac{n\pi}{a}t\,dt-$$

$$-\frac{n\pi}{ka}\int_0^b \Phi_2(t)\sin\frac{n\pi t}{b}\,dt\Bigg]\geq 0$$

where $k=\sqrt{\dfrac{a_1}{a_2}}$.

In fact, since $\dfrac{\partial U^0}{\partial x_2}$ is a solution of the problem

$$\begin{cases} a_1\,\dfrac{\partial^2\,\dfrac{\partial U^0}{\partial x_2}}{\partial x_1^2}+a_2\,\dfrac{\partial^2\,\dfrac{\partial U^0}{\partial x_2}}{\partial x_2^2}=0 \\[2ex] \dfrac{\partial U^0(0,x_2)}{\partial x_1}=\varphi_1(x_2) \qquad \dfrac{\partial U^0(a,x_2)}{\partial x_2}=\psi_1(x_2) \\[2ex] \dfrac{\partial U^0(x_1,0)}{\partial x_2}=\chi_2(x_1) \qquad \dfrac{\partial U^0(x_1,b)}{\partial x_2}=\zeta_2(x_1)\,, \end{cases} \tag{4.10}$$

if (4.8), (4.9) are fulfilled, also the relation $\dfrac{\partial U^0}{\partial x_2} \geq 0$ is fulfilled because the maximum principle holds. Analogously we find that $\dfrac{\partial U^0}{\partial x_1} \leq 0$ in $\Omega$ if and only if the following conditions are fulfilled:

(4.11)
$$\partial U^0 \frac{(0, x_2)}{\partial x_1} = \chi_1(x_2) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{\sin \dfrac{n\pi x_2}{b}}{sh \, \dfrac{kna\pi}{b}} \cdot$$
$$\cdot \left[ k \frac{n\pi}{b} \int_0^b \Psi_1(t) \sin \frac{n\pi}{b} t \, dt - \right.$$
$$\left. -k \frac{n\pi}{b} \, ch \, \frac{kn\pi a}{b} \int_0^b \Phi_1(t) \sin \frac{n\pi t}{b} \, dt \right] +$$
$$+\frac{2}{a} \sum_{n=1}^{\infty} \frac{\dfrac{n\pi}{a}}{sh \, \dfrac{n\pi b}{ka}} \cdot \left[ sh \, \frac{n\pi x_2}{ka} \int_0^b \Psi_2(t) \sin \frac{n\pi}{a} t \, dt + \right.$$
$$\left. +sh \, \frac{n\pi(b - x_2)}{ka} \int_0^a \Phi_2(t) \sin \frac{n\pi}{b} t \, dt \right] \leq 0$$

(4.12)
$$\partial U^0 \frac{(a, x_2)}{\partial x_1} = \zeta_1(x_2) =$$
$$\frac{2}{b} \sum_{n=1}^{\infty} \frac{\sin \dfrac{n\pi x_2}{b}}{sh \dfrac{kna\pi}{b}} \cdot \left[ k \frac{n\pi}{b} \, ch \, \frac{kn\pi a}{b} \int_0^b \Psi_1(t) \sin \frac{n\pi}{b} t \, dt - \right.$$
$$\left. -k \frac{n\pi}{b} \int_0^b \Phi_1(t) \sin \frac{n\pi t}{b} \, dt \right] +$$
$$+\frac{2}{a} \sum_{n=1}^{\infty} \frac{\dfrac{n\pi}{a} \cos n\pi}{sh \, \dfrac{n\pi b}{ka}} \cdot \left[ sh \, \frac{n\pi x_2}{ka} \int_0^a \Psi_2(t) \sin \frac{n\pi}{a} t \, dt + \right.$$
$$\left. +sh \, \frac{n\pi(b - x_2)}{ka} \int_0^a \Phi_2(t) \sin \frac{n\pi}{b} t \, dt \right] \leq 0.$$

In particular, if we choose

$$\Omega = ]0, a[\times]0, \pi[$$

$$\begin{aligned}
\varphi_1(x_2) &= 1 \\
\psi_1(x_2) &= 1 - \cos x_2 \\
\varphi_2(x_1) &= 0 \\
\psi_2(x_1) &= 0
\end{aligned}$$

we obtain

$$U^0(x_1, x_2) = x_2 - \frac{shkx_1}{shka} \sin x_2$$

and, since

$$\begin{aligned}
\chi_2(x_1) &= 1 - \frac{shkx_1}{shka} \geq 0 \ , \quad x_1 \in [0, a] \\
\zeta_2(x_1) &= 1 + \frac{shkx_1}{shka} \geq 0 \ , \quad x_1 \in [0, a] \\
\chi_1(x_2) &= -k \frac{1}{shka} \sin x_2 \leq 0 \qquad x_2 \in [0, \pi] \\
\zeta_2(x_2) &= -k \frac{chka}{shka} \sin x_2 \leq 0 \ , \quad x_2 \in [0, \pi] \, ,
\end{aligned}$$

the vector $u^0 = \left( \frac{\partial U^0}{\partial x_2}, - \frac{\partial U^0}{\partial x_1} \right)$ is the equilibrium flow. In this way we obtain, as a particular case, the result of [1], p. 300.

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