SYMMETRIZATION IN A NEUMANN PROBLEM

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Let w be a weak solution of the Neumann problem for a second order elliptic equation in divergence form, in a bounded open subset G of R^n. In the case that the right hand side of the equation is a continuous linear functional on $H^1(G)$, we give some symmetrization results and an estimate of the norm of w.

Let G be an open bounded subset of R^n satisfying the cone property. For the sake of simplicity we suppose $n \geq 3$; the case $n = 2$ can be studied modifying in the usual way the assumptions. Let $w \in H^1(G)$ be a weak solution, which is unique modulo constants (see [11]), of the problem

$$(1) \qquad \int_G a_{ij} w_{x_i} \varphi_{x_j} dx = \int_G f_i \varphi_{x_i} dx + \int_G h\varphi dx \quad \forall \varphi \in H^1(G)$$

where $a_{ij}(x) \in L^\infty(G)$, $f_i \in L^p(G)$, $p \geq 2$, $h \in L^s(G)$, $s \geq \dfrac{2n}{n+2}$ and:

$$(2') \qquad a_{ij}\xi_i\xi_j \geq |\xi|^2 \quad \text{a.e. in } G, \quad \forall \xi \in R^n$$

$$(2'') \qquad \int_G h dx = 0.$$

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The aim of this note is to study, like in [12], [2], [7], [8], [9], [10], the problem (1) from the point of view of the symmetrization (see Th. 1). Then we will use the results to obtain estimates for w (see Th. 2), in which we give the explicit value of the costants.

Among other things, the problem (1) is interesting because, in the regular case, it is the weak formulation of the Neumann problem with «natural» boundary conditions:

$$(3)\qquad \begin{cases} -(a_{ij}u_{x_i})_{x_j} = -(f_i)_{x_i} + h & \text{in } G \\ (a_{ij}u_{x_i} - f_i)n_j = 0 & \text{on } \partial G \end{cases}$$

where n_j are the components of the external normal to ∂G. When $f_i = 0$ the problem (3) is reduced to the homogeneous Neumann problem, for which Maderna and Salsa obtained symmetrization results in [9], for the linear case, and in [10], for the nonlinear case; the same problem for degenerate elliptic operators has been studied in [13]. A similar analysis for the Dirichlet problem can be found, for example, in [12], [2], [7] and [8].

Following [9], we will assume that the open set G has a fixed measure $|G|$ and that it satisfies a relative isoperimetric inequality, that is, there exists a constant Q such that, for each measurable subset E of G,

$$(4)\qquad (\min\{|E|,\, |G\backslash E|\})^{1-1/n} \leq QP_G(E)$$

where $P_G(E)$ is the perimeter of E relative to G (see [5] and [6]). Now, let k be a constant, which surely exists (see [9]), such that, if $u = w - k$, we have

$$(5)\qquad |\{x \in G : u_i(x) > 0\}| = |\text{sprt}(u_i)| \leq \frac{|G|}{2} \quad i = 1, 2$$

where u_1 and u_2 are the positive and the negative part of u respectively ($u_1 = \max(u, 0), u_2 = \max(-u, 0)$). If we choose $E_i = \{x \in G : |u_i| > t\}$, $t \in]0, +\infty[$, $i = 1, 2$, from (5) we have:

$$\min\{|E_i|,\, |G\backslash E_i|\} = \mu_i(t)$$

where $\mu_i(t) = |\{x \in G : u_i > t\}|$ is the distribution function of u_i. Then, from (4) and from the equality

$$-\frac{d}{dt}\int\limits_{u_i>t} |Du_i|dx = P_G\{x \in G : u_i(x) > t\},$$

which can be obtained in a standard way from the Fleming-Rishel formula (see [6]), we get:

$$-\frac{d}{dt}\int_{u_i>t} |Du_i|dx \geq Q^{-1}\mu_i(t)^{1-1/n}. \tag{6}$$

On the other hand, by Schwartz's inequality, we have

$$-\frac{d}{dt}\int_{u_i>t} |Du_i|dx \leq (-\mu_i'(t))^{1/2}\left(-\frac{d}{dt}\int_{u_i>t} |Du_i|^2dx\right)^{\frac{1}{2}}$$

which, with (6), gives:

$$1 \leq Q(-\mu_i'(t))^{1/2}\mu_i(t)^{-1+1/n}\left(-\frac{d}{dt}\int_{u_i>t} |Du_i|^2dx\right)^{\frac{1}{2}}. \tag{7}$$

To estimate the last factor in the right hand side of (7), we have to go back to (1). Choosing the test functions

$$\varphi_i = \max(u_i(x) - t, 0) \quad i = 1, 2$$

$\varphi_i \in H^1(G)$ (for example, see [11]), we get:

$$\int_{u_i>t} a_{jk}u_{x_j}(u_i)_{x_k}dx = \int_{u_i>t} f_j(u_i)_{x_j}dx + \int_{u_i>t} h(u_i - t)dt, \quad i = 1, 2. \tag{8}$$

Now, with a standard procedure (see [12], [9],...), in which we use, between other things, the ellipticity condition (2') and Schwartz's inequality, we have:

$$-\frac{d}{dt}\int_{u_i>t} |Du_i|^2dx \leq \left(-\frac{d}{dt}\int_{u_i>t} g_i^2dx\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{u_i>t} |Du_i|^2dx\right)^{\frac{1}{2}} + \tag{9}$$

$$+ (-1)^{i+1}\int_{u_i>t} h(x)dx,$$

where we have put:

$$g_i(x) = \begin{cases} |f| & x \in G : u_i(x) \neq 0 \\ 0 & x \in G : u_i(x) = 0 \end{cases}$$

with $|f| = \left(\sum_i f_i^2\right)^{1/2}$.

Before we combine (7) and (9), it is convenient to remind a few notions about the rearrangements of a function. Taking into account the definition of distribution function $\mu(t)$ of a function ψ, which we have given before, the decreasing rearrangement ψ^* of ψ is defined as

$$\psi^*(s) = \inf\{t \geq 0 : \mu(t) < s\},$$

and the spherically symmetric rearrangement $\psi^\#$ of ψ is defined as:

$$\psi^\#(x) = \psi^*(C_n|x|^n)$$

where C_n is the measure of the n-dimensional unit ball. We have the following property:

$$||\psi||_{L^p(G)} = ||\psi^*||_{L^p[0,|G|[} = ||\psi^\#||_{L^p(G^\#)}$$

where $G^\#$ is the ball with center in the origin and measure $|G|$. We have also:

$$\int_G \psi(x)g(x)dx \leq \int_{G^\#} \psi^\#(x)g^\#(x)dx = \int_0^\infty \psi^*(s)g^*(s)ds$$

and

$$A \subseteq G \Rightarrow \int_A \psi(x)dx \leq \int_0^{|A|} \psi^*(s)ds.$$

For more properties about rearrangements see, for example, [1] and [4]. The last result we remind is an approximation lemma which can be found in [2]. For all $\psi \in L^p(G)$ we put

$$K(\psi) = \{F \in L^p(0,|G|) \text{ for which } \exists\{\psi_k\} \subset L^p(0,|G|) :$$

$$\psi_k^* = \psi^* \text{ and } \psi_k \rightharpoonup F \text{ in } L^p(0,|G|)\};$$

if $\mu_i(t)$ is the distribution function of u_i, $\forall s \in [0,|G|[$ we can find a measurable subset $D_i(s)$ of G such that:

$$|D_i(s)| = s$$

$$s_1 < s_2 \Rightarrow D_i(s_1) \subset D_i(s_2)$$

$$D_i(s) = \{x \in G : |u_i| > t\} \quad \text{if } s = \mu_i(t).$$

LEMMA 1. *Some functions $F_i \in K(\psi)$, $i = 1,2$, exist such that:*

$$\int_{D_i(s)} \psi(x)dx = \int_0^s F_i(t)dt \quad s \in [0,|G|]. \tag{10}$$

Now, if, in the previous lemma, we put $\psi = g_i^2$, we have:

$$\int_{u_i>t} g_i^2 dx = \int_0^{\mu_i(t)} F_i^2(s)ds \quad i = 1,2$$

where $F_i^2 \in K(g_i^2)$. It is easy to verify that, by construction, $F_i(s) = 0$ for $s \geq |G|/2$. Then we have:

$$\left(-\frac{d}{dt}\int_{u_i>t} g_i^2 dx\right)^{\frac{1}{2}} = (-\mu_i'(t))^{1/2} F_i(\mu_i(t)). \tag{11}$$

Like in [9], the last term in (9) can be estimated in the following way

$$(-1)^{i+1}\int_{u_i>t} h(x)dx \leq \int_0^{\mu_i(t)} h_i^*(s)ds, \tag{12}$$

where $h_1^*(s)$ and $h_2^*(s)$ are the decreasing rearrangements of the positive and the negative part of $h(x)$ respectively. Using (11) and (12), from (9) we get:

$$(13)\quad \left(-\frac{d}{dt}\int\limits_{u_i>t}|Du_i|^2dx\right)^{\frac{1}{2}} \leq (-\mu_i'(t))^{1/2}F_i(\mu_i(t))+$$

$$+\int\limits_0^{\mu_i(t)} h_i^*(s)ds\left(-\frac{d}{dt}\int\limits_{u_i>t}|Du_i|^2dx\right)^{-\frac{1}{2}}$$

Taking into account the fact that (7) can be written as

$$\left(-\frac{d}{dt}\int\limits_{u_i>t}|Du_i|^2dx\right)^{-\frac{1}{2}} \leq Q\mu_i(t)^{-1+1/n}(-\mu_i'(t))^{1/2},$$

from (7) and (13) we obtain:

$$\frac{\mu_i(t)^{1-1/n}}{Q(-\mu_i'(t))^{1/2}} \leq (-\mu_i'(t))^{1/2}F_i(\mu_i(t)) + Q\mu_i(t)^{-1+1/n}(-\mu_i'(t))^{1/2}\int\limits_0^{\mu_i(t)} h_i^*(s)ds,$$

that is:

$$1 \leq [-\mu_i'(t)]\left[Q\mu_i(t)^{-1+1/n}F_i(\mu_i(t)) + Q^2\mu_i(t)^{-2+2/n}\int\limits_0^{\mu_i(t)} h_i^*(s)ds\right].$$

In a standard way we have

$$u_i^*(C_n|x|^n) \leq QnC_n^{1/n}v_i(x) + Q^2n^2C_n^{2/n}V_i(x),$$

where we have put:

$$v_i(x) = \frac{1}{nC_n^{1/n}}\int\limits_{C_n|x|^n}^{\frac{|G|}{2}} F_i(t)t^{-1+1/n}dt$$

$$V_i(x) = \frac{1}{n^2C_n^{2/n}}\int\limits_{C_n|x|^n}^{\frac{|G|}{2}} r^{-2+2/n}\left(\int\limits_0^r h_i^*(t)dt\right)dr.$$

It is easy to recognize that the functions we have just defined are, for $i = 1, 2$, solutions of the following Dirichlet problems:

$$(14) \qquad \begin{cases} -\Delta v = \left(F_i(C_n|x|^n)\dfrac{x_j}{|x|}\right)_{x_j} & \text{in } G_2^\# \\ v = 0 & \text{on } \partial G_2^\#. \end{cases}$$

and

$$(15) \qquad \begin{cases} -\Delta V = h_i^\# & \text{in } G_2^\# \\ V = 0 & \text{on } \partial G_2^\# \end{cases}$$

where $G_2^\#$ is the ball with center in the origin and measure $|G|/2$. We can summarize the above arguments in the following theorem:

THEOREM 1. *Let us consider the problem (1) with the assumptions:*
i) $a_{ij}(x) \in L^\infty(G)$ *with:*

$$a_{ij}\xi_i\xi_j \geq |\xi|^2 \quad \textit{a.e. in } G, \quad \forall \xi \in R^n;$$

ii) $f_i \in L^p(G)$, $p \geq 2$, $i = 1, \ldots, n$;
iii) $h \in L^s(G)$, $s \geq \dfrac{2n}{n+2}$;
iv) G *is an open bounded subset of* R^n *and satisfies (4).*

Let w *be a solution of (1) and let* $u = w - k$ *be such that (5) is satisfied, then, if* $v_i (i = 1, 2)$ *and* $V_i (i = 1, 2)$ *are the weak solutions of (14) and (15) respectively, we have:*

$$(16) \qquad u_i^\#(x) \leq QnC_n^{1/n} v_i(x) + Q^2 n^2 C_n^{2/n} V_i(x)$$

where $u_1^\#(x)$ *and* $u_2^\#(x)$ *are the rearrangements of the positive and the negative part of* $u(x)$ *respectively.*

From a certain point of view, the theorem we have just proved is not «sharp» in an absolute sense, because the functions F_i we have used in it are related in some way to the solution of the problem itself. Nevertheless, using Lemma 1, Theorem 1 gives the possibility to estimate the norm of w. Indeed, if we put $x = 0$ in (16), we get:

$$\text{ess. sup}\,.w - \text{ess. inf}\,.w = \text{ess. sup}\,.u - \text{ess. inf}\,.u = \text{ess. sup}\,.u_1 + \text{ess. sup}\,.u_2 \leq$$

$$\leq QnC_n^{1/n}(v_1(0)+v_2(0)) + Q^2n^2C_n^{2/n}(V_1(0)+V_2(0)).$$

We can deal with the last term in the previous inequality like in [9], obtaining

$$V_1(0)+V_2(0) \leq C(||h_1||_{L^s(G)} + ||h_2||_{L^s(G)})$$

where $s > n/2$, $s' = s/(s-1)$ and

$$C = \left(\frac{|G|}{2}\right)^{\frac{2}{n}-\frac{1}{s}} \frac{\Gamma(1+n/2)^{2/n}}{n(n-2)\pi} \left\{\frac{\Gamma(1+s')\Gamma((n/(n-2))-s')}{\Gamma(n/(n-2))}\right\}^{\frac{1}{s'}}.$$

We have still to estimate the term in which the quantities $v_i(0)$, $i = 1,2$, are present. Using Hölder's inequality, we get, for $p > n$:

$$v_i(0) = \frac{1}{nC_n^{1/n}} \int_0^{\frac{|G|}{2}} F_i(r) r^{-1+1/n} dr \leq$$

$$\leq \frac{1}{nC_n^{1/n}} \left[\frac{n(p-1)}{p-n}\right]^{\frac{p-1}{p}} \left(\frac{|G|}{2}\right)^{\frac{p-n}{np}} ||F_i||_{L^p}.$$

Because of Lemma 1, we have:

$$||F_1||_{L^p}^p = \int_0^{\frac{|G|}{2}} (F_1^2)^{p/2} ds \leq \int_{u_1>0} g_1^p dx = \int_{u>0} |f|^p dx,$$

$$||F_2||_{L^p}^p = \int_0^{\frac{|G|}{2}} (F_2^2)^{p/2} ds \leq \int_{u_2>0} g_2^p dx = \int_{u<0} |f|^p dx,$$

which give:

$$v_1(0)+v_2(0) \leq \frac{1}{nC_n^{1/n}} \left[\frac{n(p-1)}{p-n}\right]^{\frac{p-1}{p}} \left(\frac{|G|}{2}\right)^{\frac{p-n}{np}} \times$$

$$\times \left\{ \left(\int_{u>0} |f|^p dx\right)^{\frac{1}{p}} + \left(\int_{u<0} |f|^p dx\right)^{\frac{1}{p}} \right\}.$$

Taking into account the inequality

$$a^{1/p} + b^{1/p} \leq 2^{\frac{p-1}{p}} (a+b)^{1/p}, \tag{17}$$

we have

$$v_1(0) + v_2(0) \leq \frac{A}{nC_n^{1/n}} ||f||_{L^p},$$

where

$$A = \left[\frac{2n(p-1)}{p-n}\right]^{\frac{p-1}{p}} \left(\frac{|G|}{2}\right)^{\frac{p-n}{np}}.$$

In the case $p < n$ we can estimate the quantity $(\inf\{||\bar{w}||_{L^q}; \bar{w} = w + \text{const}\})$, with $q = p^* = \dfrac{np}{n-p}$. We get:

$$\inf\{||\bar{w}||_{L^q}; \bar{w} = w + \text{const}\} \leq ||u||_{L^q} \leq ||u_1||_{L^q} + ||u_2||_{L^q} \leq$$

$$\leq QnC_n^{1/n}(||v_1||_{L^q} + ||v_2||_{L^q}) + Q^2 n^2 C_n^{2/n}(||V_1||_{L^q} + ||V_2||_{L^q}).$$

As before we will limit ourselves to estimate the norm $||v_i||_{L^q}$:

$$||v_i||_{L^q} = \frac{1}{nC_n^{1/n}} \left[\int_{G_2^\#} \left(\int_{C_n|x|^n}^{\frac{|G|}{2}} F_i(s) s^{-1+1/n} ds\right)^q dx\right]^{\frac{1}{q}} =$$

$$= \frac{1}{nC_n^{1/n}} \left[\int_0^\infty \left(\int_r^{\frac{|G|}{2}} F_i(s) s^{-1+1/n} ds\right)^q dr\right]^{\frac{1}{q}}.$$

Using the Bliss'inequality ([3], [12])

$$\int_0^\infty \left(\int_r^\infty \psi(s) ds\right)^q dr \leq \tilde{B} \left(\int_0^\infty \psi(r)^p r^{-1+p+p/q} dr\right)^{\frac{q}{p}}, \quad q > p > 1$$

where

$$\tilde{B} = \left[q\left(1 - \frac{1}{p}\right)\right]^{q-\frac{q}{p}} \left\{\frac{\Gamma(qp/(q-p))}{\Gamma(q/(q-p))\Gamma(p(q-1)/(q-p))}\right\}^{\frac{q}{p}-1},$$

we obtain

$$||v_i||_{L^q} \leq \frac{\hat{B}}{nC_n^{1/n}} \left(\int_0^{\frac{|G|}{2}} F_i^p(s)ds \right)^{\frac{1}{p}}$$

where

$$\hat{B} = \left[\frac{n(p-1)}{n-p}\right]^{\frac{p-1}{p}} \left\{ \frac{\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right\}^{\frac{1}{n}}.$$

Finally, by (17), we have:

$$||v_1||_{L^q} + ||v_2||_{L^q} \leq \frac{2^{\frac{p-1}{p}}\hat{B}}{nC_n^{1/n}} ||f||_{L^p}.$$

If we remind that (see [9]), for $s = np/(n+p)$, we have:

$$||V_1||_{L^q} + ||V_2||_{L^q} \leq D(||h_1||_{s,s} + ||h_2||_{s,s})$$

where $1 < s < \frac{n}{2}$,

$$D = \frac{1}{n^2\pi} q^{1-1/q} \left(\frac{s-1}{s}\right)^{1-\frac{1}{s}} \left\{ \frac{\Gamma(n/2)^2}{\Gamma(n/(2q))\Gamma((n-n/q)/2)} \right\}^{\frac{2}{n}},$$

and

$$||h_i||_{s,s} = \left[\int_0^\infty \left(\frac{1}{t} \int_0^t h_i^*(t')dt' \right)^s dt \right]^{\frac{1}{s}}$$

is a norm equivalent to the usual norm in L^s, we can state the following theorem:

THEOREM 2. *With the assumptions of Theorem 1, let w be a solution of (1); then w satisfies the following inequalities:*

i) $\text{ess.}\sup .w - \text{ess.}\inf .w \leq QA||f||_{L^p} + Q^2n^2C_n^{2/n}C(||h_1||_{L^s} + ||h_2||_{L^s})$

where $p > n$, $s > n/2$, $s' = s/(s-1)$,

$$A = \left[\frac{2n(p-1)}{p-n}\right]^{\frac{p-1}{p}} \left(\frac{|G|}{2}\right)^{\frac{p-n}{np}}$$

and

$$C = \left(\frac{|G|}{2}\right)^{\frac{2}{n}-\frac{1}{s}} \frac{\Gamma(1+n/2)^{2/n}}{n(n-2)\pi} \left\{\frac{\Gamma(1+s')\Gamma((n/(n-2))-s')}{\Gamma(n/(n-2))}\right\}^{1/s'}.$$

ii) $\inf\{||\bar{w}||_{L^q}; \bar{w} = w + const\} \leq QB||f||_{L^p} + Q^2 n^2 C_n^{2/n} D(||h_1||_{s,s} + ||h_2||_{s,s})$

where $2 < p < n$, $q = \dfrac{np}{n-p}$, $s = \dfrac{np}{n+p} \in]1, n/2[$,

$$B = \left[\frac{2n(p-1)}{n-p}\right]^{\frac{p-1}{p}} \left\{\frac{\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{\frac{1}{n}}$$

and

$$D = \frac{1}{n^2\pi} q^{1-1/q} \left(\frac{s-1}{s}\right)^{1-\frac{1}{s}} \left\{\frac{\Gamma(n/2)^2}{\Gamma(n/(2q))\Gamma((n-n/q)/2)}\right\}^{\frac{2}{n}}.$$

Remark If we substitute (2') with the weaker assumption:

$$a_{ij}(x)\xi_i\xi_j \geq \nu(x)|\xi|^2$$

where $\nu(x) \geq 0$, $\nu \in L^1$, $\dfrac{1}{\nu} \in L^t$ with a suitable value for t, following, for example, the arguments in [2], [7] and [13], we get results similar to Theorem 1 and 2.

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