LE MATEMATICHE
Vol. LXIII (2008) – Fasc. II, pp. 197–211

NOTES ON SYMMETRIC AND EXTERIOR DEPTH AND ANNIHILATOR NUMBERS

GESA KÄMPF - MARTINA KUBITZKE

We survey and compare invariants of modules over the polynomial ring and the exterior algebra. In our considerations, we focus on the depth. The exterior analogue of depth was first introduced by Aramova, Avramov and Herzog. We state similarities between the two notion of depth and exhibit their relation in the case of squarefree modules. Work of Conca, Herzog and Hibi and Trung, respectively, shows that annihilator numbers are a meaningful generalization of depth over the polynomial ring. We introduce and study annihilator numbers over the exterior algebra. Despite some minor differences in the definition, those invariants show common behavior. In both situations a positive linear combination of the annihilator numbers can be used to bound the symmetric and exterior graded Betti numbers, respectively, from above.

1. Introduction

Throughout this paper, let K denote an infinite field. We write $S = K[x_1, \dots, x_n]$ for the polynomial ring and $E = K\langle e_1, \dots, e_n\rangle$ for the exterior algebra. Many invariants of modules can be defined over the polynomial ring as well as over the exterior algebra. Sometimes if we want to define an exterior analogue of an invariant over the polynomial ring some changes are needed. In some cases, we even have to define a notion which at first glance seems unrelated to the original

Entrato in redazione: 2 gennaio 2009

AMS 2000 Subject Classification: primary 16E05, 16E65, 13F55, secondary 13C15, 16W50.
Keywords: exterior depth, annihilator numbers, Stanley-Reiner ring.

one but later turns out to play a similar role. One such example is the projective dimension [7] defined over the polynomial ring which measures the length of the minimal free resolution of an S-module. As the minimal free resolution of an E-module is infinite except for the case of free modules the projective dimension of an E-module is infinite if the module is not free. Thus, even though it makes sense to define a projective dimension over the exterior algebra the notion is almost meaningless. Instead, there exists the notion of complexity [1] which – as a measure for the degree of the polynomial growth of the Betti numbers – turns out to have similar properties (see Section 2).

In order to get even one step further, if we have a certain construction over the polynomial ring and the exterior algebra, respectively, in many cases we can find an exterior and symmetric version, respectively, of this very construction. One of the first examples are the Koszul complex [7] and the Cartan complex [8].

In this survey article, we compare some algebraic invariants of modules over the polynomial ring and the exterior algebra. We do not only state the similarities and the differences between those invariants but we also try to explain why notion differ in order to get adapted to the particular situation. As an example, we mention the definition of regular elements [5]. In the second section we concentrate on the comparison of the traditional depth of a module over the polynomial ring [5] and the more recently introduced exterior depth of a module over the exterior algebra, originally defined by Aramova, Avramov and Herzog [1]. Starting with the definition of regular elements we develop the two notion of depth step by step. Then we present some similarities, e.g., that the invariants do not change under the passage to the generic initial ideal [4], [7]. We state the result that the symmetric depth of a squarefree S-module is always an upper bound for the exterior depth of a certain associated squarefree E-module (Theorem 2.19). The explicit relation between the two modules is expounded by Aramova, Avramov and Herzog [1] and Römer [11], respectively. To conclude this section we recall the well-known Auslander-Buchsbaum formula (Theorem 2.20) and its corresponding exterior analogue (Theorem 2.21) by Aramova, Avramov and Herzog.

In the third section we introduce the so-called annihilator numbers of a module which can be considered as an iteration of the concept of depth. Beginning with the definition of almost regular elements over the polynomial ring we explain why over the exterior algebra it is not reasonable to use the same definition since all elements would have this property. We further give the definitons of annihilator numbers with respect to a sequence over the polynomial ring, first introduced by Conca, Herzog and Hibi [6] and Trung [13], and over the exterior algebra [10], respectively, emphasizing the points in which these notion differ.

Despite the differences, it turns out that in both situations the annihilator numbers do not depend on the particular sequence when choosing from a non-empty Zariski-open set (Theorems 3.5 and 3.6). In the remaining part of this section we consider the so-called symmetric and exterior generic annihilator numbers, respectively. Over the polynomial ring as well as over the exterior algebra a positive linear combination of these numbers, even though not the same, can be used to bound the symmetric and the exterior Betti numbers, respectively, from above (Theorems 3.12 and 3.14). Although the generic annihilator numbers minimize those linear combinations under all annihilator numbers with respect to a sequence, Example 3.10 shows that the single generic annihilator numbers – the symmetric as well as the exterior ones – are not minimal under all the annihilator numbers with respect to a sequence.

2. Depth over the polynomial ring and the exterior algebra

Over the polynomial ring S as well as over the exterior algebra E there exists the notion of regular element and depth.

Definition 2.1. Let N be a graded S-module. A linear form $y \in S_1$ is called N-*regular* if $yz = 0$ for $z \in N$ implies $z = 0$, in other words, if y is not a zero-divisor on N. A sequence $y_1, \dots, y_r$ of linear forms in S_1 is called an N-*regular sequence* if y_i is an $N/(y_1, \dots, y_{i-1})N$-regular element for $1 \le i \le r$ and $N/(y_1, \dots, y_r)N \neq 0$.

In general a regular element is not required to be linear and if the field is too small there exist S-modules having no linear regular elements. However, if the field is infinite it is always possible to find regular elements of degree 1.
The following classical theorem of Rees is crucial for the definition of depth.

Theorem 2.2. ([5, Theorem 1.2.5.]) *Let N be a finitely generated graded S-module. Every N-regular sequence can be extended to a maximal N-regular sequence. Furthermore all maximal N-regular sequences have the same length.*

This gives rise to the definition of the depth of N over S.

Definition 2.3. Let N be a finitely generated graded S-module. The common length of all maximal N-regular sequences is called the *depth* of N over S and denoted by $\operatorname{depth}_S N$.

For further details concerning the classical depth see e.g., the book of Bruns and Herzog, [5, Chapters 1.1, 1.2, 1.5].

Let $\mathscr{M}$ be the category of finitely generated graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for homogeneous elements $a \in E$, $m \in M$.

For example if $J \subseteq E$ is a graded ideal, then E/J belongs to $\mathcal{M}$. Note that every left ideal is a right ideal and vice versa. The relation $am = (-1)^{\deg a \deg m} ma$ follows just by changing the order of the elements of am.

It is not reasonable to define regular elements over the exterior algebra in exactly the same way as over the polynomial ring. For every element $v \in E$ it always holds that $v^2 = 0$. So, in particular, every linear form in E_1 is a zero-divisor. Therefore we demand for a regular element v over the exterior algebra that any annihilation $vm = 0$ is a consequence of $v^2 = 0$.

Definition 2.4. Let $M \in \mathcal{M}$ be an E-module. A linear form $v \in E_1$ is called *M-regular* if $0 :_M v = vM$. A sequence $v_1, \dots, v_r$ of linear forms in E_1 is called an *M-regular sequence* if v_i is an $M/(v_1, \dots, v_{i-1})M$-regular element for $1 \leq i \leq r$ and $M/(v_1, \dots, v_r)M \neq 0$.

Note that a linear form $v \in E_1$ is M-regular if and only if the annihilator of v in M is the smallest possible.
The examples we use in this survey in order to illustrate definitions and results are mainly Stanley-Reisner rings of simplicial complexes. We now recall the definiton:

Definition 2.5. Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. The *Stanley-Reisner ring* of Δ is the quotient ring $K[\Delta] = S/I_\Delta$, where $I_\Delta = (x_F \mid F \notin \Delta)$. Analogously, the *exterior face ring* of Δ is the quotient ring $K\{\Delta\} = E/J_\Delta$, where $J_\Delta = (e_F \mid F \notin \Delta)$. Here, $e_F = e_{i_1} \wedge \dots \wedge e_{i_s}$ for $F = \{i_1, \dots, i_s\}$ denotes a monomial in E.

Example 2.6. Let Δ be the simplicial complex on $[3]$ whose maximal faces are the edges $\{1,2\}$ and $\{2,3\}$. Then $x_1 + x_3$ and x_2 are $K[\Delta]$-regular. They even constitute a $K[\Delta]$-regular sequence. On the contrary, $e_1 + e_3$ is not $K\{\Delta\}$ regular since for example $(e_1 + e_3) \wedge e_3 = e_1 \wedge e_3 \in J_\Delta = (e_1 \wedge e_3)$. But e_2 is $K\{\Delta\}$-regular.

The notion of exterior regular element was first introduced by Aramova, Avramov and Herzog in [1]. They prove the analogous result to Theorem 2.2 which is essential for the definiton of exterior depth.

Theorem 2.7. ([1]) *Let $M \in \mathcal{M}$ be an E-module. Every M-regular sequence can be extended to a maximal M-regular sequence. Furthermore all maximal M-regular sequences have the same length.*

This leads to the definition of the exterior depth of an E-module.

Definition 2.8. Let $M \in \mathcal{M}$. The common length of all maximal M-regular sequences is called the *depth* of M over E and denoted by $\operatorname{depth}_E M$.

These two notion of depth behave similarly in many cases. In the following we present some of their common behavior. Afterwards we compare these two invariants directly (if possible) and study their differences.

It is widely used that being a regular sequence is a generic condition in both situations. However, we cannot give a reference for this fact for the polynomial ring. In [12] Swartz gives a proof for the special case of Stanley-Reisner rings of simplicial complexes. Herzog provided an argument for the general caseusing almost regular sequences. A sequence $y_1, \ldots, y_n$ of linear forms is called an almost regular sequence on N if the modules $0 :_{N/(y_1,\ldots,y_{i-1})} y_i$ have finite length for $1 \le i \le n$. The set of almost regular sequences is a non-empty Zariski-open set (see e.g [8, Theorem 4.3.6]). Furthermore [8, Proposition 4.3.4] implies that the first $\operatorname{depth}_S(N)$ elements of an almost regular sequence form a regular sequence on N. Hence, the set of N-regular sequences is Zariski-open.

Proposition 2.9. *Let N be a finitely generated graded S-module and let the depth of N be t. There exists a Zariski-open set $U \subseteq GL_n(K)$ such that the sequence $\gamma_{1,1}x_1 + \ldots + \gamma_{n,1}x_n, \ldots, \gamma_{1,t}x_1 + \ldots + \gamma_{n,t}x_n$ is an N-regular sequence for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$.*

We now specify this open set in an explicit example.

Example 2.10. Let Δ be the simplicial complex on $[2]$ consisting of the two vertices 1 and 2. Then $\operatorname{depth}_S(K[\Delta]) = 1$ and $ax_1 + bx_2 \in K[x_1, x_2]$ is $K[\Delta]$-regular if and only if $ab \ne 0$, i.e.

$$\{v \in K[x_1,x_2]_1 \mid v \text{ is } K[\Delta]\text{-regular}\} = \{ax_1 + bx_2 \in K[x_1,x_2] \mid ab \ne 0\}.$$

Proposition 2.11. (*see e.g., [10, Proposition 2.11]*) *Let $M \in \mathscr{M}$ and let the depth of M be t. There exists a Zariski-open set $U \subseteq GL_n(K)$ such that the sequence $\gamma_{1,1}e_1 + \ldots + \gamma_{n,1}e_n, \ldots, \gamma_{1,t}e_1 + \ldots + \gamma_{n,t}e_n$ is an M-regular sequence for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$.*

As in the symmetric case we consider an example of the Stanley-Reisner ring of a simplicial complex.

Example 2.12. Let Δ be the simplicial complex on $[3]$ consisting of two edges which intersect in a common vertex. One can show that $ae_1 + be_2 + ce_3 \in K\langle e_1, e_2, e_3\rangle$ is $K\{\Delta\}$-regular whenever $b \ne 0$ which is an open condition.

For an ideal I in S one can consider its *generic initial ideal* $\operatorname{gin}_{\mathrm{rlex}}(I)$ with respect to the reverse lexicographic order, where $x_1 < \ldots < x_n$ (see e.g., [4]). When passing from an ideal I to its generic initial ideal $\operatorname{gin}_{\mathrm{rlex}}(I)$ the depth does not change. This can be deduced from a result of Bayer and Stillman ([4]).

Theorem 2.13. *([7, Corollary 19.11]) Let I be a graded ideal in S. Then*

$$\operatorname{depth}_S(S/I) = \operatorname{depth}_S(S/\operatorname{gin}_{\mathrm{rlex}}(I)).$$

In the same way as over the polynomial ring Aramova, Herzog and Hibi ([3, Theorem 1.6]) define the generic initial ideal $\operatorname{gin}_{\mathrm{rlex}}(J)$ of an ideal J in E with respect to the reverse lexicographic order, where $e_1 < \ldots < e_n$. Again after the passage to the generic initial ideal of J the depth remains the same.

Theorem 2.14. *([9, Proposition 2.3]) Let J be a graded ideal in E. Then*

$$\operatorname{depth}_E(E/J) = \operatorname{depth}_E(E/\operatorname{gin}_{\mathrm{rlex}}(J)).$$

In the following we compare the symmetric depth with the exterior depth of the ”same” module. This is possible if the modules in question are squarefree. Yanagawa [14] introduces squarefree modules over the polynomial ring as a generalization of squarefree monomial ideals.

Definition 2.15. A finitely generated $\mathbb{N}^n$-graded S-module $N = \oplus_{a\in\mathbb{N}^n} N_a$ is called *squarefree* if the multiplication map $N_a \to N_{a+\varepsilon_i} : \; y \mapsto x_i y$ is bijective for any $a \in \mathbb{N}^n$ and for all $i \in \operatorname{supp}(a)$, where $\operatorname{supp}(a) = \{ \, j \mid a_j \neq 0 \, \}$ and ε_i denotes the i-th unit vector in $\mathbb{R}^n$.

Römer ([11, Definition 1.4]) defines the exterior analogue of a squarefree module.

Definition 2.16. A finitely generated $\mathbb{N}^n$-graded E-module $M = \oplus_{a\in\mathbb{N}^n} M_a$ is *squarefree* if $M_a = 0$ for $a \notin \{0,1\}^n$.

Aramova, Avramov and Herzog [1] and Römer [11] construct a minimal free resolution of a squarefree E-module N_E starting from the minimal free resolution of a squarefree S-module N. The assignment $N \mapsto N_E$ induces an equivalence between the category of squarefree S-modules and the category of squarefree E-modules in $\mathscr{M}$, where the morphisms are the $\mathbb{N}^n$-graded homomorphisms. This equivalence enables us to compare symmetric and exterior invariants directly. Although we do not give the explicit construction of Aramova, Aramov, Herzog and Römer we hope that the following example of Stanley-Reisner rings helps to get an impression how the assignment $N \mapsto N_E$ works.

Example 2.17. Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex on the vertex set $[n]$. Then the Stanley-Reisner ring $K[\Delta]$ is an example of a squarefree S-module and the exterior face ring $K\{\Delta\}$ is an example of a squarefree E-module. Under the equivalence of categories $K[\Delta]$ corresponds to $K\{\Delta\}$.

We are interested in the relation between the depth of an S-module N and the exterior depth of the corresponding E-module N_E. Using the construction of the assignment $N \mapsto N_E$ one can express the graded exterior Betti numbers as a linear combination of the graded Betti numbers over S.

Corollary 2.18. *([11, Corollary 1.3]) Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Let β^S and β^E denote the graded Betti numbers over S and E, respectively. Then*

$$\beta^E_{i,i+j}(N_E) = \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N).$$

The just stated connection between the graded Betti numbers over S and E, respectively, is crucial in the proof of the following theorem.

Theorem 2.19. *([10, Theorem 2.6, Remark 2.7]) Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Then*

$$\operatorname{depth}_E(N_E) \leq \operatorname{depth}_S(N) \tag{1}$$

and

$$\operatorname{depth}_S(N) - \operatorname{depth}_E(N_E) \leq \operatorname{reg}_S(N).$$

Here $\operatorname{reg}_S(N) = \max\{j \mid \beta_{i,i+j}(N) \neq 0\}$ *denotes the regularity of N.*

A natural question to ask is how much the symmetric and the exterior depth can differ. It turns out that independently of the particular numbers each difference can occur, i.e. for each pair of numbers $(s \leq t)$ there exists a squarefree S-module N with the property that $\operatorname{depth}_S N = t$ and $\operatorname{depth}_E N_E = s$. Indeed, let $(s,t) \in \mathbb{N}^2$ with $s \leq t$. Set $\Delta = 2^{[s]} * \partial(2^{[t-s+2]})$, where $\partial(2^{[t-s+2]})$ denotes the boundary of the $(t-s+2)$-simplex. Then $e_1, \ldots, e_s$ is a $K\{\Delta\}$-regular sequence and therefore $\operatorname{depth}_E(K\{\Delta\}) = s$ ([10, Theorem 2.12, Example 2.14]). Since Δ is Cohen-Macaulay $\operatorname{depth}_S(K[\Delta]) = t$, as required. This in particular shows that inequality (1) cannot be improved and that the symmetric depth cannot be bounded from above in terms of the exterior depth.

Over the polynomial ring there is the well-known Auslander-Buchsbaum formula which relates the depth of an S-module to its projective dimension. The *projective dimension* $\operatorname{proj\,dim}_S N$ of a finitely generated graded S-module N is the length of its minimal free graded resolution.

Theorem 2.20. *(Auslander-Buchsbaum, [7, Theorem 19.9]) Let N be a finitely generated graded S-module. Then*

$$\operatorname{proj\,dim}_S N + \operatorname{depth}_S N = n.$$

The minimal free graded resolution of an E-module $M \in \mathscr{M}$ is always infinite (unless M is free). Whence, instead of the projective dimension, one needs another number in order to prove a result analogous to the Auslander-Buchsbaum formula. To this end one considers the degree of the polynomial growth of the Betti numbers of M over E,

$$\mathrm{cx}_E M = \inf\{\ c \in \mathbb{Z} \mid \beta_i^E(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\ \},$$

and calls this number the *complexity* of M [1]. The complexity can be considered as the exterior analogue of the projective dimension. The complexity and the exterior depth of an E-module are related in the same way as the projective dimension and the symmetric depth of an S-module.

Theorem 2.21. *([1, Theorem 3.2]) Let $M \in \mathscr{M}$. Then*

$$\mathrm{depth}_E M + \mathrm{cx}_E M = n.$$

If N is a squarefree S-module and if N_E is the associated squarefree E-module the complexity of N_E can be computed just from the knowledge of the symmetric Betti numbers of N. From Corollary 2.18 it can be deduced that the i-th exterior Betti number $\beta_i^E(N_E)$ is a polynomial in i of degree $m^{(i)}$, where $m^{(i)} := \max\{k + j \mid \beta^S_{k,k+j}(N) \neq 0,\ 0 \leq k \leq i, j \geq 0\}$.

Corollary 2.22. *Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Then*

$$\mathrm{cx}_E(N_E) = \max\{l \mid \beta^S_{i,l}(N) \neq 0,\ i \geq 0,\ l \geq 0\}.$$

Using this description of the complexity of a squarefree E-module, one immediately gets the following inequality between the projective dimension and the complexity of the ”same” module.

Corollary 2.23. *Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Then*

$$\mathrm{cx}_E(N_E) \geq \mathrm{proj\,dim}_S(N). \tag{2}$$

Example 2.24. Consider the simplicial complex Δ from Example 2.6, i.e. Δ consists of two edges intersecting in a common vertex. Since Δ is Cohen-Macaulay it holds that $\mathrm{depth}_S(K[\Delta]) = 2$. The only non-zero graded Betti numbers of $K[\Delta]$ are $\beta^S_{0,0}(K[\Delta]) = 1$ and $\beta^S_{1,2}(K[\Delta]) = 1$. Furthermore, the minimal free resolution of $K[\Delta]$ has length 1, i.e. $\mathrm{proj\,dim}_S(K[\Delta]) = 1$. Thus,

$$\mathrm{proj\,dim}_S(K[\Delta]) + \mathrm{depth}_S(K[\Delta]) = 1 + 2 = 3,$$

which is exactly the equality we can deduce from Auslander-Buchsbaum.
On the other hand we have $\operatorname{depth}_E(K\{\Delta\}) = 1$ and from Corollary 2.22 it follows that $\operatorname{cx}_E(K\{\Delta\}) = \max\{l \mid \beta_{i,l} \neq 0,\ i \geq 0,\ l \geq 0\} = 2$. Hence,

$$\operatorname{cx}_E(K\{\Delta\}) + \operatorname{depth}_E(K\{\Delta\}) = 2 + 1,$$

which also follows from Theorem 2.21.
Comparing the projective dimension of $K[\Delta]$ with the complexity of $K\{\Delta\}$ we obtain

$$\operatorname{cx}_E(K\{\Delta\}) = 2 \geq 1 = \operatorname{proj\,dim}_S(K[\Delta]).$$

This inequality already follows from Corollary 2.23.

3. Annihilator numbers over the polynomial ring and the exterior algebra

Annihilator numbers are an iteration of the notion of depth. In a certain sense they measure how strongly a sequence of linear forms fails to be regular. Furthermore, they are closely related to Betti numbers. If $y \in S_1$ is a regular element on a finitely generated graded S-module N, then the multiplication by y is injective, hence $0 :_N y = 0$. This property is weakened if one only requires the annihilator $0 :_N y$ to have finite length (i.e. to be of finite K-vectorspace dimension).

Definition 3.1. Let N be a finitely generated graded S-module. A linear form $y \in S_1$ is called *almost regular* on N if $0 :_N y$, i.e. the kernel of the multiplication with y, has finite length. A sequence of $y_1, \dots, y_r$ of linear forms is called an *almost regular sequence* on N if y_i is an almost regular element on $N/(y_1, \dots, y_{i-1})N$ for $1 \leq i \leq r$.

The exterior algebra E has only finitely many graded components. As a consequence every finitely generated graded E-module has finite K-vectorspace dimension. Therefore for our purposes we do not need an extra notion of almost regular elements over the exterior algebra. In [2] Aramova and Herzog give a proof that the regularity of a submodule of a free module over the polynomial ring and that of its generic initial module coincide using almost regular sequences. They modify this proof to give a similar statement over the exterior algebra. However, in this setting there is the need to define special sequences over the exterior algebra corresponding in some sense to the almost regular sequences over the polynomial ring.

For the class of Borel fixed ideals in the polynomial ring an explicit almost regular sequence is known. Recall that a monomial ideal $I \subseteq S$ is called *Borel fixed* if it is fixed under the action of the Borel subgroup $\mathscr{B} \subset GL_n(K)$, i.e. $\alpha(I) = I$ for all $\alpha \in \mathscr{B}$. Here, the Borel subgroup $\mathscr{B} \subset GL_n(K)$ is the subgroup of all nonsingular upper triangular matrices.

Lemma 3.2. ([8, Proposition 4.3.3]) *Let I be a Borel fixed ideal in S. Then $x_1,\dots,x_n$ is an almost regular sequence on S/I.*

Definition 3.3. (see e.g., [6], [13]) Let $y_1,\dots,y_n$ be a sequence of linear forms in S_1 and N a finitely generated graded S-module. We denote by $A_i(y_1,\dots,y_n;N)$ the graded module

$$0 :_{N/(y_1,\dots,y_{i-1})N} y_i.$$

We write A_i instead of $A_i(y_1,\dots,y_n;N)$ if it is clear from the context which sequence is used. The vectorspace dimension of $(A_i)_j$ is denoted by

$$\alpha_{i,j}(y_1,\dots,y_n;N) = \dim_K(A_i)_j$$

for $1 \leq i \leq n$. The numbers $\alpha_{i,j}(y_1,\dots,y_n;N)$ are called the *annihilator numbers* of N with respect to the sequence $y_1,\dots,y_n$.

The exterior annihilator numbers are defined in almost the same way. Analogously to the definition of a regular element over the exterior algebra one has to bear in mind that every linear form in E_1 is a zero-divisor. This issue is accounted for by dividing out the image of the multiplication with the particular linear form.

Definition 3.4. Let $\mathbf{v} = v_1,\dots,v_n$ be a sequence of linear forms in E_1 and let $M \in \mathscr{M}$. The numbers

$$\alpha_{i,j}(\mathbf{v};M) = \dim_K\left(\left(0 :_{M/(v_1,\dots,v_{i-1})M} v_i\right) / \left((v_1,\dots,v_i)M/(v_1,\dots,v_{i-1})M\right)\right)_j$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ are called the *exterior annihilator numbers* of M with respect to the sequence $v_1,\dots,v_n$.

The annihilator numbers might depend on the chosen sequence of linear forms but they are stable on a Zariski-open set.

Theorem 3.5. ([8, Theorem 4.3.6]) *Let $I \subseteq S$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq \mathrm{GL}_n(K)$ such that*

$$\alpha_{i,j}(\gamma(x_1,\dots,x_n);S/I) = \alpha_{i,j}(x_1,\dots,x_n;S/\operatorname{gin}_{<_{\mathrm{rlex}}}(I))$$

for all $\gamma = (\gamma_{i,j})_{1\leq i,j\leq n} \in U$, where $\gamma(x_1,\dots,x_n)$ is the sequence $(\gamma_{1,1}x_1+\ldots+\gamma_{n,1}x_n,\dots,\gamma_{1,n}x_1+\ldots+\gamma_{n,n}x_n)$.

The same is true over the exterior algebra and the proof follows exactly the same steps.

Theorem 3.6. *([10, Theorem 3.5]) Let $J \subseteq E$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq \mathrm{GL}_n(K)$ such that*

$$\alpha_{i,j}(\gamma(e_1,\ldots,e_n);E/J) = \alpha_{i,j}(e_1,\ldots,e_n;E/\operatorname{gin}_{<_{\text{rlex}}}(J))$$

for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$, where $\gamma(e_1,\ldots,e_n)$ is the sequence $(\gamma_{1,1}e_1+\ldots+\gamma_{n,1}e_n,\ldots,\gamma_{1,n}e_1+\ldots+\gamma_{n,n}e_n)$.

This fact justifies the definition of generic annihilator numbers.

Definition 3.7. Let $I \subseteq S$ be a graded ideal in S. We set

$$\alpha_{i,j}(S/I) = \alpha_{i,j}(x_1,\ldots,x_n;S/\operatorname{gin}_{<_{\text{rlex}}}(I))$$

for $j \in \mathbb{Z}$ and $1 \le i \le n$ and call these numbers the *generic annihilator numbers* of S/I over S.

Definition 3.8. Let $J \subseteq E$ be a graded ideal. We set

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_1,\ldots,e_n;E/\operatorname{gin}_{<_{\text{rlex}}}(J))$$

for $j \in \mathbb{Z}$ and $1 \le i \le n$ and call these numbers the *generic annihilator numbers* of E/J over E.

Remark 3.9. By definition, it holds that $\alpha_{i,j}(S/I) = \alpha_{i,j}(S/\operatorname{gin}_{<_{\text{rlex}}}(I))$ and $\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\operatorname{gin}_{<_{\text{rlex}}}(J))$.

It is an interesting question if the generic annihilator numbers play a special role among the annihilator numbers with respect to an arbitrary sequence of linear forms. One may think that they are minimal in the sense that $\alpha_{i,j}(S/I) \le \alpha_{i,j}(y_1,\ldots,y_n;S/I)$ for all $1 \le i \le n$, $j \in \mathbb{Z}$ and all sequences $y_1,\ldots,y_n$ of linear forms. Nevertheless, during the attempt of proving this minimality we found a counterexample.

Example 3.10. ([10, Example 4.4]) Let $1 \le i \le j \le n$ and let $I = (x_{l_1}\cdot\ldots\cdot x_{l_{j+1}} \mid i \le l_1 \le \ldots \le l_{j+1}) \subseteq S$ be a graded ideal. By construction I is strongly stable and it therefore holds that $\operatorname{gin}_{<_{\text{rlex}}}(I) = I$. Thus we can use the sequence $x_1,\ldots,x_n$ to compute the symmetric generic annihilator numbers of S/I. Computing the annihilator numbers with respect to the (possibly not generic) sequence $x_1,\ldots,x_{i-1},x_i,x_{i-1},\ldots,x_n$ where the positions of x_{i-1} and x_i are switched does not necessarily give the generic annihilator numbers. Indeed, we show in [10, Example 4.4] that

$$\alpha_{i,j}(S/I) > \alpha_{i,j}(x_1,\ldots,x_{i-1},x_i,x_{i-1},\ldots,x_n;S/I).$$

Thus the generic annihilator numbers are not minimal in the sense aforementioned. In particular this example shows that the order of the sequence matters for the computation of the generic annihilator numbers.

The same example works over the exterior algebra, as we show in the following example for a concrete number n.

Example 3.11. We consider the above example for $n = 5$ and $i = 3$ over the exterior algebra. Then the ideal in question is $J = (e_3 \wedge e_4, e_3 \wedge e_5, e_4 \wedge e_5)$ in the exterior algebra $E = K\langle e_1, \ldots, e_5\rangle$. This is a strongly stable ideal and therefore it holds that $\mathrm{gin}_{<_{\mathrm{rlex}}}(J) = J$. We use the sequence e_1, e_2, e_3, e_4, e_5 to compute the generic annihilator numbers $\alpha_{i,j} = \alpha_{i,j}(E/J)$. Let $\alpha_i = \sum_{j\in\mathbb{Z}} \alpha_{i,j}$ denote the sum over all annihilator numbers computed in step i. Since e_1 and e_2 do not appear among the generators of J, these two form a regular sequence on E/J. This implies that the corresponding annihilator numbers are zero, i.e. $\alpha_1 = \alpha_2 = 0$. To compute $\alpha_{3,j}$ we have to compute the vectorspace dimension of

$$((\bar{J} : e_3)/(\bar{J} + (e_3)))_j = ((e_1, e_2, e_3, e_4, e_5)/(e_1, e_2, e_3, e_4 \wedge e_5))_j,$$

where $\bar{J} = J + (e_1, e_2)$. Therefore $\alpha_{3,1} = 2$ and in all other degrees the annihilator number is zero. In the next step we look at

$$((\bar{J} : e_4)/(\bar{J} + (e_4)))_j = ((e_1, e_2, e_3, e_4, e_5)/(e_1, e_2, e_3, e_4))_j,$$

where now $\bar{J} = J + (e_1, e_2, e_3)$. Thus $\alpha_{4,1} = 1$ and zero in all other degrees. In the last step we obtain $\alpha_5 = 0$. Note that the last generic annihilator number is always zero, as it is the dimension of a quotient over $J + (e_1, \ldots, e_n) = (e_1, \ldots, e_n)$ whence it must be zero. In particular we see that $\mathrm{depth}_E E/J = 2$.

Now we compute the annihilator numbers $\alpha'_{i,j}$ of E/J with respect to the sequence e_1, e_3, e_2, e_4, e_5. Again e_1 is an E/J-regular element and thus $\alpha'_1 = 0$. But $\alpha'_2 \neq 0$ in contrary to α_2 since

$$((J + (e_1)) : e_3) / (J + (e_1, e_3)) = (e_1, e_3, e_4, e_5)/(e_1, e_3, e_4 \wedge e_5).$$

Therefore $\alpha'_{2,1} = 2$, $\alpha'_{2,2} = 2$ and zero otherwise. Now e_2 is still regular on $E/(J + (e_1, e_3))$ whence $\alpha'_3 = 0 < \alpha_3$. So here the generic annihilator number is greater. The last two steps are the same as above, i.e. $\alpha'_{4,j} = \alpha_{4,j}$ which is 1 for $j = 1$ and zero otherwise and $\alpha'_5 = \alpha_5 = 0$.

If one compares the respective sets of non-zero annihilator numbers in the above example, one has in the first case (i.e. the generic numbers) $\{2, 1\}$ and in the second $\{2, 2, 1\}$. Therefore the generic annihilator numbers seem to be minimal in a certain global sense. The next result implies this, at least if the ideal is componentwise linear. The annihilator numbers give rise to an upper bound for the Betti numbers. The following two results were first proven by Conca, Herzog and Hibi ([6]) for the total Betti numbers β_i^S. In the forthcoming book of Herzog and Hibi the graded version can be found.

Theorem 3.12. *([6, Corollary 1.2], [8]) Let $I \subseteq S$ be a graded ideal and let $y_1, \dots, y_n$ be a K-basis of S_1 which is almost regular on S/I. Then*

$$\beta^S_{i,i+j}(S/I) \leq \sum_{k=1}^{n-i+1} \binom{n-k}{i-1} \alpha_{k,j}(y_1, \dots, y_n; S/I).$$

For a special class of ideals and for generic sequences of linear forms the bound is tight. A graded ideal I is called *componentwise linear*, if for all $j \in \mathbb{Z}$ the ideal $I_{<j>}$ has a j-linear resolution. Here $I_{<j>}$ denotes the ideal which is generated by the degree j elements of I, i.e. $I_{<j>} = (I_j)$.

Corollary 3.13. *([6, Theorem 1.5], [8]) Let $I \subseteq S$ be a componentwise linear ideal in S. Then*

$$\beta^S_{i,i+j}(S/I) = \sum_{k=1}^{n-i+1} \binom{n-k}{i-1} \alpha_{k,j}(S/I).$$

A similar result can be obtained over the exterior algebra except that the binomial coefficients appearing in the linear combination are slightly different.

Theorem 3.14. *([10, Theorem 3.15]) Let $J \subseteq E$ be a graded ideal and $v_1, \dots, v_n$ be a K-basis of E_1. Then*

$$\beta^E_{i,i+j}(E/J) \leq \sum_{k=1}^{n} \binom{n+i-k-1}{i-1} \alpha_{k,j}(v_1, \dots, v_n; E/J).$$

Corollary 3.15. *([10, Theorem 3.15]) Let $J \subseteq E$ be a componentwise linear ideal in E. Then*

$$\beta^E_{i,i+j}(E/J) = \sum_{k=1}^{n} \binom{n+i-k-1}{i-1} \alpha_{k,j}(E/J).$$

Remark 3.16. In particular the above results imply that if I is a componentwise linear ideal, its graded Betti numbers are bounded above by a linear combination of the annihilator numbers $\alpha_{k,j}(y_1, \dots, y_n; S/I)$ for any K-basis $y_1, \dots, y_n$ which is almost regular on S/I. If in addition the sequence is generic (in the sense that it computes the generic annihilator numbers), the graded Betti numbers of S/I are equal to this linear combination. This shows that, although the single generic annihilator numbers $\alpha_{k,j}(S/I)$ are not minimal, at least this linear combination is minimal. The same is true for a componentwise linear ideal in the exterior algebra.

Acknowledgement

The main work for this article was performed while the authors were participating in the summer school PRAGMATIC 2008 in Catania, Sicily. The authors thank the organizers of PRAGMATIC 2008 for support and hospitality. We are grateful to Jürgen Herzog and Volkmar Welker for their suggestion to study this topic and for sharing their knowledge, insights and experience in helpful discussions. Further thanks go to Eran Nevo for contributing combinatorial ideas to many discussions.

REFERENCES

[1] A. Aramova - L. L. Avramov - J. Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Am. Math. Soc. **352** (2000), 579–594.

[2] A. Aramova - J. Herzog, *Almost regular sequences and Betti numbers*, Am. J. Math. **122** n.4 (2000), 689–719.

[3] A. Aramova - J. Herzog - T. Hibi, *Gotzmann theorems for exterior algebras and combinatorics*, J. Algebra **191** (1997), 174–211.

[4] D. Bayer - M. Stillman, *A criterion or detecting m-regularity*, Invent. Math. **87** (1987), 1–11.

[5] W. Bruns - J. Herzog, *Cohen-Macaulay rings. Rev. ed.*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.

[6] A. Conca - J. Herzog - T. Hibi, *Rigid resolutions and big Betti numbers*, Comment. Math. Helv. **79** (2004), 826–839.

[7] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, New York, 1995.

[8] J. Herzog - T. Hibi, *Monomials*. Springer, to appear.

[9] J. Herzog - N. Terai, *Stable properties of algebraic shifting*, Result in Math. **35** (1999), 260–265.

[10] G. Kämpf - M. Kubitzke, *Exterior depth and exterior generic annihilator numbers*, Preprint, 2008.

[11] T. Römer, *Generalized Alexander duality and applications*, Osaka J. Math. **38** (2001), 469–485.

[12] E. Swartz, *g-elements, finite buildings and higher Cohen-Macaulay connectivity*, J. Combin. Theory Ser. A **113** n. 7 (2006), 1305–1320.

[13] N. V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc. **101** (1987), 229–236.

[14] K. Yanagawa, *Alexander duality for Stanley-Reisner rings and squarefree* $\mathbb{N}^n$*-graded modules*, J. Algebra **225** (2000), 630–645.

GESA KÄMPF
Fachbereich Mathematik und Informatik
Universität Osnabrück
49069 Osnabrück, Germany
e-mail: gkaempf@mathematik.uni-osnabrueck.de

MARTINA KUBITZKE
Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
35032 Marburg, Germany
e-mail: kubitzke@mathematik.uni-marburg.de