MEASURABLE REPRESENTATION OF BICONJUGATE INTEGRANDS

OTTAVIO CALIGARIS (Genova) *

We find here a representation of convex regularization of a non convex proper function and of a non convex proper normal integrand by means of a suitable multifunction which reveals to be very useful in existence theorems for non convex problems of calculus of variations.

Introduction.

It is well known that the biconjugate function f^{**} of a non convex function f provides its convex regularization; since f^{**} is the supremum of all affine hyperplanes supporting the epigraph of f, we may wonder that the graph of f^{**}, when it does not coincide with the graph of f, is formed by pieces of affine linear subspaces, not necessarily of dimension $n-1$, supporting epi f.

Our present aim is to precise the representation of f^{**} under suitable hypotheses and, secondly, to obtain a similar representation for a biconjugate integral $f^{**}(t,x)$ relative to a normal non convex integrand.

More precisely, we show that, for every integrable function x, it is possible to find a measurable multifunction Γ such that

$$(x(t), f^{**}(t, x(t))) \in \text{ co } \Gamma(t)$$

where

$$\Gamma(t) \subset \{(x, \alpha) : \alpha = f(t, x) = f^{**}(t, x)\}.$$

* Entrato in Redazione il 10 marzo 1988

This results offers a tool to prove some new theorems for non-convex problems of calculus variations.

The subsequent material will be divided into two parts: we consider first of all the simpler case of a non-convex fucntion and, secondly, we extend our results also to the case of a non-convex finite normal integrand.

The case of a non-convex normal integrand taking infinite values is currently studied and will be the argument of subsequent works.

Convex regularization of function.

In this section we study the convex regularization of a proper lower semicontinuous function

$$f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$$

we indicate by f^* its conjugate function in the sense of Moreau-Rockafellar [1], [2], and by f^{**} its biconjugate. We limit ourselves to recall that

$$f^*(p) = \sup\{\langle p, v\rangle - f(v) : v \in \mathbf{R}^n\}$$

while

$$f^{**} = (f^*)^*$$

(a complete treatment of this concepts may be found in [1], [2] themselves).

epi f will be the epigraph of f i.e.

$$\text{epi } f = \{(x, \alpha) \in \mathbf{R}^{n+1} : f(x) \leq \alpha\}$$

while co A is the convex hull and cl A is the closure of any subset $A \subset \mathbf{R}^n$. $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are used to indicate the scalar product and the norm in $\mathbf{R}^n$.

We say that f satisfies the "basic growth condition" when

$$(B.G.C.) \qquad f^*(p) < +\infty \quad \forall p \in \mathbf{R}^n.$$

Let us briefly recall that (B.G.C.) is equivalent to the following condition

There is a function ω such that

(1) $$\lim_{||v||\to+\infty} \omega(v) = +\infty \quad \text{and} \quad f(v) = ||v||\omega(v).$$

Inded f^* is a convex finite, and hence continuous, function on $\mathbf{R}^n$ and for every $r \in \mathbf{R}_+$ there is $\gamma_r \in \mathbf{R}$ such that

$$\gamma_r \geq \sup\{f^*(p) : ||p|| \leq r\} =$$

$$= \sup\{\sup\{\langle p, v\rangle - f(v) : v \in \mathbf{R}^n\} : ||p|| \leq r\}.$$

So we can deduce that

$$f(v) \geq \langle p, v\rangle - \gamma_r \quad \forall v \in \mathbf{R}^n \quad \forall p \in \mathbf{R}^n,\ ||p|| \leq r \quad \forall r \in \mathbf{R}_+$$

and

$$f(v) \geq r||v|| - \gamma_r \quad \forall v \in \mathbf{R}^n \quad \forall r \in \mathbf{R}_+$$

Dividing by $||v||$, and taking the lim inf over $||v|| \to +\infty$ if we define

$$\omega(v) = \frac{f(v)}{||v||}$$

we obtain

$$\liminf_{||v||\to+\infty} \omega(v) \geq \liminf_{||v||\to+\infty} r - \frac{\gamma_r}{||v||} = r \quad \forall r \in \mathbf{R}_+$$

Hence

$$\lim_{||v||\to+\infty} \omega(v) = \liminf_{||v||\to+\infty} \omega(v) = +\infty$$

and

$$f(v) = ||v||\omega(v)$$

Moreover, since

$$f(v) \geq -f^*(0) = K$$

We can always suppose that

$$f(v) \geq 0$$

if we agree to make a translation, when necessary.

Conversely when condition (1) holds, since

$$f^*(p) = \sup\{\langle p, v\rangle - f(v) : v \in \mathbf{R}^n\},$$

we can easily deduce, by a standard generalization of Weierstrass theorem, that $f^*(p) \in \mathbf{R}$.

It is well known [1], [2] that

$$\text{epi } f^{**} = \text{cl co epi } f$$

let us prove that, under (B.G.C.),

$$\text{cl co epi } f = \text{co epi } f$$

whence it results

$$\text{epi } f^{**} = \text{co epi } f.$$

Let us remark that (B.G.C.) is fundamental because if it does not holds co epi f can be a non-closed set.

Indeed if we choose $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = |x| + \frac{|x|}{x^2+1}$$

we have that

$$f^{**}(x) = |x|$$

while

$$\text{epi } f^{**} = \{(x,\alpha) : \alpha \geq |x|\} \neq \text{co epi } f = \{(x,\alpha) : \alpha > |x|\} \cup \{(0,0)\}.$$

THEOREM 1. *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a l.s.c. function satisfying (B.G.C.); then*

$$\text{epi } f^{**} = \text{cl co epi } f = \text{co epi } f.$$

Proof. What precedes allow us to prove only the second equality; moreover it will be evidently sufficient to prove that

$$\text{cl co epi } f \subset \text{co epi } f.$$

We also recall that we can always suppose that $f \geq 0$.

Now, let

$$(x_k, \alpha_k) \in \text{co epi } f, \quad (x_k, \alpha_k) \to (x, \alpha).$$

By Caratheodory's lemma we have

$$x_k = \sum_{i=1}^{n+2} \lambda_i^k x_i^k, \;\; \alpha_k = \sum_{i=1}^{n+2} \lambda_i^k \alpha_i^k$$

where

$$(x_i^k, \alpha_i^k) \in \operatorname{epi} f, \;\; \sum_{i=1}^{n+2} \lambda_i^k = 1, \;\; 0 \leq \lambda_i^k \leq 1.$$

Clearly we have

$$0 \leq f(x_i^k) \leq \alpha_i^k$$

and, since $(x_k, \alpha_k) \to (x, \alpha)$ we may suppose that

$$||x_k|| \leq M$$

(1) $$0 \leq \lambda_i^k \alpha_i^k \leq \alpha_k \leq M.$$

So, if we take a suitable subsequence, we have that

(2) $$\lambda_i^k \alpha_i^k \to \mu_i \in \bar{\mathbf{R}}_+ = \{x \in \mathbf{R} : x \geq 0\}$$

(3) $$\lambda_i^k \to \lambda_i \in [0, 1]$$

$$\alpha_i^k \to \alpha_i \in \bar{\mathbf{R}}_+ \cup \{+\infty\}.$$

Let $I \subset \{1, \ldots, n+2\}$ such that

$$i \in I \Rightarrow ||x_i^k|| \leq N.$$

We can always suppose that

$$x_i^k \to x_i \in \mathbf{R}^n \quad \forall i \in I$$

and that

$$||x_i^k|| \to +\infty \quad \forall i \notin I.$$

Let's moreover indicate by J the set of all $i \in \{1, \ldots, n+2\}$ such that

$$\alpha_i^k \to \alpha_i \in \bar{\mathbf{R}}_+.$$

By (1), when $i \notin J$ we have

$$0 \leq \lambda_i^k \leq \frac{M}{\alpha_i^k}$$

and

$$\lambda_i^k \to 0 = \lambda_i$$

moreover, since by (B.G.C.)

$$||x_i^k||\omega(x_i^k) \leq f(x_i^k) \leq \alpha_i^k,$$

we have

$$J \subset I.$$

On the other side when $i \notin I$, we have $i \notin J$, $\lambda_i^k \to 0$ and moreover, by (B.G.C.),

$$0 \leq \lambda_i^k ||x_i^k|| \leq \frac{M||x_i^k||}{\alpha_i^k} \leq \frac{M||x_i^k||}{f(x_i^k)} \to 0.$$

This led us to obtain that

$$x_k = \sum_{i \in I} \lambda_i^k x_i^k + \sum_{i \notin I} \lambda_i^k x_i^k \to \sum_{i \in I} \lambda_i x_i = \sum_{i \in J} \lambda_i x_i = x$$

$$\alpha_k = \sum_{i \in J} \lambda_i^k \alpha_i^k + \sum_{i \notin J} \lambda_i^k \alpha_i^k \to \sum_{i \in J} \lambda_i \alpha_i + \mu = \alpha$$

where

$$\mu = \sum_{i \neq J} \mu_i \geq 0.$$

We also have that

$$1 = \sum_{i=1}^{n+2} \lambda_i^k \to \sum_{i \in J} \lambda_i,$$

so that $\sum_{i \in J} \lambda_i = 1$.

Moreover, by l.s.c. of f we can assert that

$$f(x_i) \leq \liminf f(x_i^k) \leq \liminf \alpha_i^k = \alpha_i$$

whence

$$(x, \alpha) = \sum_{i \in J} \lambda_i (x_i, \alpha_i + \mu)$$

where

$$f(x_i) \leq \alpha_i + \mu, \quad \sum_{i \in J} \lambda_i = 1.$$

This fact allow us to conclude that

$$(x, \alpha) \in \text{co epi } f. \qquad \blacksquare$$

The preceding theorem 1 allow us to represent the biconjugate of a non convex function with pieces of affine planes which support the graph of f.

More precisely we can state that

THEOREM 2. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a l.s.c. function satisfying (B.G.C.), then*

$$\forall x \in \mathbf{R}^n \ \exists x_i \in \mathbf{R}^n \ \exists \lambda_i \in \mathbf{R}, \ 0 < \lambda_i \leq 1, \ \sum_{i=1}^{m} \lambda_i = 1$$

such that $f(x_i) = f^{**}(x_i)$, $m \leq n+2$ and

$$(x, f^{**}(x)) = \sum_{i=1}^{m} \lambda_i (x_i, f(x_i)) = \sum_{i=1}^{m} \lambda_i (x_i, f^{**}(x_i)).$$

Proof. Since (B.G.C.) holds, by the preceding theorem 1, we can assert that $\forall x \in \mathbf{R}^n$

$$(x, f^{**}(x)) \in \text{epi } f^{**} = \text{co epi } f$$

and we can find $x_i \in \mathbf{R}^n$, $\lambda_i \in \mathbf{R}$, $0 < \lambda_i \leq 1$, $\sum_{i=1}^{m} \lambda_i = 1$, such that

$$(x, f^{**}(x)) = \sum_{i=1}^{m} \lambda_i (x_i, f(x_i) + \alpha_i)$$

where $\alpha_i \geq 0$.

To conclude we only have to prove that

$$\alpha_i = 0 \quad \forall_i = 1, 2, \ldots, m.$$

Let us suppose that $\alpha_j > 0$ for some $j \in \{1, 2, \cdots, m\}$ then

$$f^{**}(x) = \sum_{i=1}^{m} \lambda_i f(x_i) + \sum_{i=1}^{m} \lambda_i \alpha_i \geq \sum_{i=1}^{m} \lambda_i f(x_i) + \lambda_j \alpha_j > \sum_{i=1}^{m} \lambda_i f(x_i).$$

So we obtain that

$$\left(x, \sum_{i=1}^{m} \lambda_i f(x_i) \right) \in \text{co epi } f = \text{epi } f^{**}$$

and

$$f^{**}(x) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

which is impossible.

So we may assert that $\alpha_i = 0$, $\forall i \in \{1, 2, \ldots, m\}$ and

$$(x, f^{**}(x)) = \sum_{i=1}^{m} \lambda_i (x_i, f(x_i)).$$

Moreover

$$\sum_{i=1}^{m} \lambda_i f(x_i) = f^{**}(x) = f^{**} \left(\sum_{i=1}^{m} \lambda_i x_i \right) \leq \sum_{i=1}^{m} \lambda_i f^{**}(x_i)$$

whence

$$\sum_{i=1}^{m} \lambda_i (f^{**}(x_i) - f(x_i)) \geq 0.$$

We deduce that

$$\sum_{i=1}^{m} \lambda_i (f^{**}(x_i) - f(x_i)) = 0$$

and, since every term in the sum is non-positive,

$$f^{**}(x_i) = f(x_i) \quad \forall i \in \{1, 2, \ldots, m\}.$$

■

Convex regularization of normal integrands.

In this section we consider a normal proper integrand

$$f : [0,1] \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$$

as it is defined by R.T. Rockafellar in [3].

We recall that f is called a normal integrand on $[0,1] \times \mathbf{R}^n$, where $[0,1]$ is equipped with the Lebesgue measure and $\mathbf{R}^n$ is topologized by means of the euclidean norm, when the multifunction

$$t \mapsto \text{epi } f(t,\cdot) = \{(v,\alpha) \in \mathbf{R}^{n+1} : f(t,v) \leq \alpha\}$$

is measurable and closed valued; moreover f is said to be a proper integrand when

$$f(t,\cdot) \not\equiv +\infty.$$

An important feature of a normal proper integrand is that for every measurable function $x : [0,1] \to \mathbf{R}^n$, $f(t,x(t))$ results to be measurable too; this fact leads to an extensive use of normal integrands in calculus of variations.

We also indicate

$$f^*(t,p) = (f(t,\cdot))^*(p)$$

while f^{**} is defined in similar way.

Both $f^*(t,\cdot)$ and $f^{**}(t,\cdot)$ are obviously convex functions and the standard theory of normal integrands, which is completely developed in [3], shows that f^* and f^{**} are normal integrands too.

In this section we shall always assume true some assumptions.

First of all we shall assume that the following Caratheodory condition on f does hold

$$(C.C.^{**}) \qquad f^{**}(t,x) < +\infty \quad \forall x \in \mathbf{R}^n \quad a.e.\, t \in [0,1].$$

As a comment we note that, since f^{**} is a convex proper normal integrand, when $(C.C.^{**})$ holds, it happens to be finite and hence continuous; consequently f^{**} is a Caratheodory integrand and this motivates the name of our condition.

secondly we shall use the following basic growth condition

$$\forall p \in \mathbf{R}^n, \quad \exists \gamma_p \in L^1(0,1,\mathbf{R}) \quad \text{such that}$$

$$(B.G.C.) \qquad f^*(t,p) \leq \gamma_p(t) \quad a.e. - t \in [0,1].$$

As a consequence of the discussion contained in the preceding section we see that $(B.G.C.)$ holds only if there is a fucntion

$$\omega : [0,1] \times \mathbf{R}^n \to \mathbf{R}$$

such that

$$f(t,v) = ||v||\omega(t,v) \quad \text{and} \quad \lim_{||v|| \to +\infty} \omega(t,v) = +\infty \quad a.e. - t \in [0,1].$$

Moreover we always suppose that there is at least a function $\bar{x} \in L^1(0,1,\mathbf{R}^n)$ such that $f(\cdot, \bar{x}(\cdot)) \in L^1(0,1,\mathbf{R})$ and we have

$$f(t,v) \geq -f^*(t,0) = K(t) \quad K \in L^1(0,1,\mathbf{R}).$$

Under $(B.G.C.)$ and $(C.C.^{**})$ we can prove an interesting characterization of the biconjugate $f^{**}(t,x)$ of a normal proper integrand which can be of some utility in studying existence theorems for non convex problems in calculus of variations.

THEOREM 3. *Let f be a normal proper integrand on $[0,1] \times \mathbf{R}^n$ satisfying $(B.G.C.)$ and $(C.C.^{**})$. Then for every $x \in L^1(0,1,\mathbf{R}^n)$ there is a compact valued measurable multifunction*

$$\Gamma : [0,1] \to \mathbf{R}^n \times \mathbf{R}$$

such that

$$\Gamma(t) \subset \{(v,\alpha) : \alpha = f(t,v) = f^{**}(t,v)\}$$

and

$$(x(t), f^{**}(t,x(t))) \in \text{co}\, \Gamma(t) \quad a.e. t \in [0,1].$$

Proof. Since $t \mapsto \partial f^{**}(t,x(t))$ is a closed convex not empty valued measurable multifucntion [3] ($\partial f^{**}(t,x(t))$ indicates the subdifferential of the convex function $f^{**}(t,\cdot)$ at $x(t)$ which is not empty because $f^{**}(t,\cdot)$ is everywhere finite), we can find a measurable selection

$$-\beta(t) \in \partial f^{**}(t,x(t)).$$

So we can assert that

$$(4) \qquad \langle \beta(t), x(t) \rangle + f^{**}(t, x(t)) \leq \langle \beta(t), v \rangle + f^{**}(t, v) \quad \forall v \in \mathbf{R}^n.$$

Let's define the multifunction $\Gamma : [0,1] \to \mathbf{R}^n \times \mathbf{R}$ by

$$\Gamma(t) = \{(v, \alpha) \in \mathbf{R}^n \times \mathbf{R} : \alpha = f(t,v) = f^{**}(t,v),$$

$$\langle \beta(t), x(t) \rangle + f^{**}(t, x(t)) = \langle \beta(t), v \rangle + f^{**}(t,v) =$$

$$= \langle \beta(t), v) \rangle + \alpha\} =$$

$$= \{(v, \alpha) \in \mathbf{R}^n \times \mathbf{R} : \alpha = f(t,v) = f^{**}(t,v),$$

$$\langle \beta(t), x(t) \rangle + f^{**}(t, x(t)) \geq \langle \beta(t), v \rangle + \alpha\}.$$

Since f and f^{**} are normal proper integrands and since f^{**} is, by $(C.C.^{**})$, continuous Γ is a closed valued measurable multifunction. Moreover we can prove that Γ has bounded values so that it is a compact valued multifunction. Let's prove the last assertion.

Let $(v, \alpha) \in \Gamma(t)$, we have

$$\langle \beta(t), v \rangle + f^{**}(t,v) \leq \langle \beta(t), x(t) \rangle + f^{**}(t, x(t)) = \xi(t)$$

and, by $(B.G.C.)$, it results

$$||v|| \leq r(t).$$

Moreover we have

$$-f^*(t,0) \leq f(t,v) = \alpha \leq \xi(t) - \langle \beta(t), v \rangle \leq \xi(t) + ||\beta(t)|| r(t)$$

whence

$$||(v, \alpha)|| \leq R(t).$$

Finally let's prove that

$$(x(t), f^{**}(t, x(t))) \in \text{co}\, \Gamma(t).$$

By theorem 2 we have

$$(x(t), f^{**}(t, x(t))) = \sum_{i=1}^{m} \lambda_i (x_i(t), f^{**}(t, x_i(t)))$$

where

$$0 < \lambda_i \leq 1, \quad \sum_{i=1}^{m} \lambda_i = 1$$

$$f^{**}(t, x_i(t)) = f(t, x_i(t)).$$

By (4) we have

$$\langle \beta(t), x(t) \rangle + f^{**}(t, x(t)) = \sum_{i=1}^{m} \lambda_i (\langle \beta(t), x(t) \rangle + f^{**}(t, x(t))) \leq$$

$$\leq \sum_{i=1}^{m} \lambda_i (\langle \beta(t), x_i(t) \rangle + f^{**}(t, x_i(t))) = \langle \beta(t), x(t) \rangle + f^{**}(t, x(t)).$$

So we can assert that

$$\sum_{i=1}^{m} \lambda_i (\langle \beta(t), x_i(t) - x(t) \rangle - f^{**}(t, x(t)) + f^{**}(t, x_i(t)) = 0$$

and every term, being non-negative, must be equal to zero. So

$$\langle \beta(t), x_i(t) \rangle + f^{**}(t, x_i(t)) = \langle \beta(t), x(t) \rangle + f^{**}(t, x(t))$$

and

$$(x_i(t), f^{**}(t, x_i(t))) \in \Gamma(t)$$

This allows us to conclude. ■

REFERENCES

[1] Moreau J.J., *Fonctionnelles convexes,* Lecture Notes - Séminaire sur les équations aux derieés partielles. Collège de France 1966/67.

[2] Rockafellar R.T., *Convex Analysis,* Princeton Univ. Press. Princeton 1969.

[3] Rockafellar R.T., *Integral functionals, Normal integrands and Measurable selections,* Nonlinear Operators and the Calculus of Variations, L. Waelbroeck ed., Springer Verlag Series, Lecture Notes in Mathematics. Berlin 1976.

Istituto Matematico di Ingegneria
P.le Kennedy - Pad. D
16129 Genova