EXISTENCE THEOREMS FOR COMPLETELY NON CONVEX PROBLEMS

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In this note we give some existence theorems for integral functionals with non convex integrand. We consider first the simpler case in which minimization is taken on decomposable spaces and successively we prove an existence theorem also for the minimum of an integral functional on non-decomposable space.

Introduction.

Let $f : [0,1] \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a normal proper lower semicontinuous integrand as it is defined in [1] by R.T. Rockafellar; for every function $x \in \mathcal{M}(0,1,\mathbf{R}^n)$ the space of all measurable functions from $[0,1]$ to $\mathbf{R}^n$, we can consider $f(\cdot, x(\cdot))$ and it results a measurable function in its turn.

So we can define an integral functional $I_f : \mathcal{N} \to \mathbf{R} \cup \{\pm\infty\}$, where $\mathcal{N}$ is a suitable subspace of $\mathcal{M}(0,1,\mathbf{R}^n)$, by

$$I_f(x) = \int_0^1 f(t, x(t))\, dt$$

and we can set the problem to

(1) $$Minimize\{I_f(x) : x \in \mathcal{N}\}.$$

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This problem is deeply affected by two distinct features: the convexity of f and the decomposability of $\mathcal{N}$. When at least one of these two conditions holds an existence theorem for problem (1) can be easily found, but when both conditions lack the question is a bit more complicated.

In this note we are intended to consider existence theorems for problem (1) without any convexity properties on f; first of all we study the situation on a typically decomposable space such as $L^1(0,1,\mathbf{R}^n)$ and secondly we face the less simple case which is created by the substitution of $L^1(0,1,\mathbf{R}^n)$ with a non-decomposable space such as the space $L_0^1(0,1,\mathbf{R}^n)$ of all functions in $L^1(0,1,\mathbf{R}^n)$ such that $\int_0^1 x(t)\,dt = 0$.

This last problem is obviously equivalent to a problem of calculus of variations in $H_0^{1,1}(0,1,\mathbf{R}^n)$ associated to a non convex integrand depending only on velocity.

The characterization of biconjugate integrands contained in [2] plays here a fundamental role and also allows us to prove an interesting result about the range of an integral functional.

Some Notations and Preliminaries.

In this paper we shall use the following notations: $\mathbf{R}^n$ is the usual euclidean space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ while $[0,1]$ is the unit interval on the real line equipped with the Lebesgue measure. We indicate with $\mathcal{B}$ the $\sigma - algebra$ of all Borel sets in $\mathbf{R}^n$ and with $\mathcal{L}$ the $\sigma - algebra$ of all Lebesgue sets in $[0,1]$.

$f : [0,1] \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a normal, proper, lower semicontinuous integrand in the sense of [1] i.e. f is a measurable function with respect to the $\sigma - algebra$ $\mathcal{L} \otimes \mathcal{B}$ in $\mathbf{R} \times \mathbf{R}^n$, $f(t,\cdot)$ is a lower semicontinuous function and $f(t,\cdot) \not\equiv +\infty$, $a.e. - t \in [0,1]$.

$\mathcal{M}(0,1,\mathbf{R}^n)$ is the space of all measurable functions from $[0,1]$ to $\mathbf{R}^n$ and we shall also consider the spaces

$$L^1(0,1,\mathbf{R}^n) = \{x \in \mathcal{M}(0,1,\mathbf{R}^n) \colon \int_0^1 \|x(t)\|\,dt < +\infty\}$$

$$L_0^1(0,1,\mathbf{R}^n) = \{x \in L^1(0,1,\mathbf{R}^n) : \int_0^1 x(t)\,dt = 0\}.$$

Obviously both spaces are normed by the $L^1 - norm$ defined as

$$||x||_{L^1} = \int_0^1 ||x(t)||\, dt.$$

By means of this norm $L^1(0,1,\mathbf{R}^n)$ and $L_0^1(0,1,\mathbf{R}^n)$ are Banach spaces and we can also easily prove that $L_0^1(0,1,\mathbf{R}^n)$ is a closed subspace of $L^1(0,1,\mathbf{R}^n)$ with respect to the weak convergence; moreover let us observe that $L^1(0,1,\mathbf{R}^n)$ is a decomposable space (see [1]) while $L_0^1(0,1,\mathbf{R}^n)$ has not this property.

We define, as in [1],

$$f^*(t,y) = \sup\{\langle y,x\rangle - f(t,x) : x \in \mathbf{R}^n\}$$

and

$$f^{**}(t,x) = \left(f^*(t,\cdot)\right)^*(x).$$

Both f^* and f^{**} are normal proper lower semicontinuous and convex integrands; and for every $x \in \mathcal{M}(0,1,\mathbf{R}^n)$ we indicate

$$I_f(x) = \int_0^1 f(t,x(t))\, dt$$

$$I_{f^*}(x) = \int_0^1 f^*(t,x(t))\, dt$$

$$I_{f^{**}}(x) = \int_0^1 f^{**}(t,x(t))\, dt.$$

Moreover $R(I_f)$, $R(I_{f^*})$ and $R(I_{f^{**}})$ indicate the range of I_f, I_{f^*} and $I_{f^{**}}$ respectively.

Now, if we consider $I_f : L^1(0,1,\mathbf{R}^n) \to \mathbf{R} \cup \{\pm\infty\}$ and the duality between $L^1(0,1,\mathbf{R}^n)$ and $L^\infty(0,1,\mathbf{R}^n)$, it can be proven that

$$\left(I_f\right)^* = I_{f^*} \qquad \text{and} \qquad \left(I_f\right)^{**} = I_{f^{**}}.$$

This fact provides a useful representation of the convex regularization $I_{f^{**}}$ of I_f on $L^1(0,1,\mathbf{R}^n)$ and greatly simplify the study of the minimum problem for I_f on $L^1(0,1,\mathbf{R}^n)$.

Unfortunately analogous results cannot be stated in $L_0^1(0,1,\mathbf{R}^n)$ and this makes all things more complicated as we shall see later.

Non convex problems on decomposable spaces.

We illustrate in this brief section the simple result that can be proven in $L^1(0,1,\mathbf{R}^n)$.

We consider here a normal proper l.s.c. integrand f which satisfies the following condition:

$$(M.B.G.C.) \qquad \exists r \in \mathbf{R}+\ ,\ \exists \gamma \in L^1(0,1,\mathbf{R}) \qquad \text{such that}$$

$$f^*(t,p) \leq \gamma(t) \quad \forall p \in \mathbf{R}^n \quad ||p|| \leq r \quad a.e. - t \in [0,1]$$

(M.B.G.C.) is a milder version of the "basic growth condition" which is a classical hypothesis in calculus of variations and which will be used later for non convex problems in non decomposable spaces.

We prove in this section a simple existence result and a property of the range of the integral functional I_f on $L^1(0,1,\mathbf{R}^n)$.

THEOREM 1. *When (M.B.G.C.) holds and when there is at least* $\hat{x} \in L^1(0,1,\mathbf{R}^n)$ *such that* $I_f(\hat{x}) \in \mathbf{R}$, *then* $\exists x_0 \in L^1(0,1,\mathbf{R}^n)$ *such that*

$$I_f(x_0) \leq I_f(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n).$$

Proof. By (M.B.G.C.) and the definition of f^*, we have

$$f(t,v) \geq \langle p,v\rangle - f^*(t,p) \geq \langle p,v\rangle - \gamma(t)$$

$$\forall p \in \mathbf{R}^n \quad ||p|| \leq r$$

and

$$f(t,v) \geq r||v|| - \gamma(t).$$

So, by a standard generalization of the Weierstrass theorem, we can find

$$\alpha(t) = \min\{f(t,v) : v \in \mathbf{R}^n\}$$

and it results

$$-\gamma(t) \leq \alpha(t) \leq f\big(t,\hat{x}(t)\big);$$

hence $\alpha \in L^1(0,1,\mathbf{R}^n)$.

Let $\Gamma : [0,1] \to \mathbf{R}^n$ be the multifunction defined by

$$\Gamma(t) = \{v \in \mathbf{R}^n : f(t,v) \leq \alpha(t)\};$$

Γ has non-empty and closed values and, by measurable selection theorems [1], we can find a measurable function x_0 such that

$$x_0(t) \in \Gamma(t) \qquad a.e. - t \in [0,1].$$

Moreover, since

$$\alpha(t) \geq f\big(t, x_0(t)\big) \geq r||x_0(t)|| - \gamma(t),$$

we deduce that $x_0 \in L^1(0,1,\mathbf{R}^n)$.

But

$$I_f(x_0) = \int_0^1 f\big(t, x_0(t)\big)dt \leq \int_0^1 \alpha(t)dt \leq$$

$$\leq \int_0^1 f\big(t, y(t)\big)dt = I_f(y) \quad \forall y \in L^1(0,1,\mathbf{R}^n)$$

and the theorem is proven. ■

COROLLARY 2. *Assuming the hypotheses and the notations of theorem 1*

$$I_{f^{**}}(x_0) \leq I_{f^{**}}(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n).$$

Proof. We have

$$I_{f^{**}}(x_0) \leq I_f(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n)$$

so

$$I_{f^{**}}(x_0) \leq (I_f)^{**}(x) = I_{f^{**}}(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n)$$

and the thesis. ■

The previous results may be completed using [1]; indeed we can assert that:

$$\inf\{I_f(x) : x \in L^1(0,1,\mathbf{R}^n)\} =$$

$$= -(I_f)^*(0) = -(I_f)^{***}(0) =$$

$$= \inf\{(I_f)^{**}(x) : x \in L^1(0,1,\mathbf{R}^n)\} =$$

$$= \inf\{I_{f^{**}}(x) : x \in L^1(0,1,\mathbf{R}^n)\},$$

and we deduce that

$$\lambda = \min\{I_f(x) : x \in L^1(0,1,\mathbf{R}^n)\} =$$

$$(2) \qquad = \min\{I_{f^{**}}(x) : x \in L^1(0,1,\mathbf{R}^n)\} =$$

$$= I_f(x_0) = I_{f^{**}}(x_0) = -I_{f^*}(0).$$

We conclude this section comparing the range of the integral functionals I_f and $I_{f^{**}}$, on $L^1(0,1,\mathbf{R}^n)$.

To this end let us define, limitedly to this section,

$$R(I_f) = \{I_f(x) : x \in L^1(0,1,\mathbf{R}^n)\}$$

and

$$R(I_{f^{**}}) = \{I_{f^{**}}(x) : x \in L^1(0,1,\mathbf{R}^n)\}.$$

We wish to recall that the same notation will be used in the next section with different significance.

THEOREM 3. *Let us suppose that the conditions of theorem 1 are satisfied, then*

$$R(I_{f^{**}}) \subset R(I_f) \subset [\lambda, +\infty).$$

Proof. Let $\alpha \in R(I_{f^{**}})$,then

$$\lambda = I_f(x_0) = I_{f^{**}}(x_0) \leq \alpha = I_{f^{**}}(z) \leq I_f(z).$$

For some $z \in L^1(0,1,\mathbf{R}^n)$ and for x_0 and λ defined as in (2). Moreover we can find $y \in L^1(0,1,\mathbf{R}^n)$ such that

$$\alpha \leq I_f(y) \in \mathbf{R}.$$

Indeed, if it were

$$I_f(y) < \alpha \qquad \forall y \in \text{dom } (I_f),$$

we should have

$$I_{f^{**}}(u) < \alpha \qquad \forall u \in \text{dom } (I_{f^{**}})$$

and this is false because $\alpha \in R(I_{f^{**}})$.

Let us consider $\varphi : [0,1] \to \mathbf{R}$ defined as

$$\varphi(t) = \int_0^t f\big(s, y(s)\big)ds + \int_t^1 f\big(s, x_0(s)\big)ds.$$

φ is absolutely continuos and

$$\varphi(0) = \lambda \leq \alpha \leq \varphi(1).$$

So, there is $t_0 \in [0,1]$ such that $\varphi(t_0) = \alpha$ and if we define

$$w(t) = \begin{cases} y(t), & t \in [0, t_0] \\ x_0(t), & t \in (t_0, 1] \end{cases}$$

we have

$$I_f(w) = \alpha \quad , \quad w \in L^1(0,1,\mathbf{R}^n)$$

and the theorem is proven. ■

We conclude this section with two examples which show as condition (M.B.G.C.) is not yet sufficient to work with problems defined in $L_0^1(0,1,\mathbf{R}^n)$ instead of $L^1(0,1,\mathbf{R}^n)$ neither when the integrand f is convex.

Indeed, let us consider

$$f(t,v) = (1+t)|v-1|.$$

If we choose

$$x_k(t) = \begin{cases} 1-k, & t \in [0, \frac{1}{k}] \\ 1, & t \in (\frac{1}{k}, 1]. \end{cases}$$

We have $x_k \in L_0^1(0,1,\mathbf{R})$,

$$I_f(x_k) = 1 + \frac{1}{2k} \to 1$$

and moreover, $\forall x \in L_0^1(0,1,\mathbf{R})$

$$I_f(x) > \int_0^1 |x(t)-1|dt \geq \Big|\int_0^1 \big(x(t)-1\big)dt\Big| = 1.$$

So

$$\inf\{I_f(x) : x \in L_0^1(0,1,\mathbf{R})\} = 1 < I_f(x) \qquad \forall x \in L_0^1(0,1,\mathbf{R}).$$

Moreover it could be true that $I_{f^{**}}$ attains its minimun on the space $L_0^1(0,1,\mathbf{R})$, while I_f has not this property.

Let

$$f(t,v) = |v-1|\frac{v^2+2}{v^2+1}$$

whence

$$f^{**}(t,v) = |v-1|.$$

It can be proven, as in preceding example that

$$I_{f^{**}}(x) \geq 1 = I_{f^{**}}(0) \qquad \forall x \in L_0^1(0,1,\mathbf{R})$$

but

$$I_f(x_k) = \frac{k^2-2k+3}{k^2-2k+2} \to 1$$

and

$$I_f(x) > 1 \qquad \forall x \in L_0^1(0,1,\mathbf{R}).$$

Non convex problems in non decomposable spaces.

Let us now consider a problem of calculus of variations on the space $L_0^1(0,1,\mathbf{R})$ which we introduced in earlier pages.

As we saw, (M.B.G.C.) is no longer sufficient to estabilish an existence theorem in this space, even in convex case; so we need to introduce a condition which is typical in existence theorems for calculus of variations: the basic growth condition, which we simply call (B.G.C.).

Here we prove that (B.G.C.) is still sufficient to assure the existence of a minimum, also in non convex case, for a problem of calculus of variations in $L_0^1(0,1,\mathbf{R}^n)$.

We recall that f satisfies the basic growth condition when

$$(B.G.C.) \qquad \forall p \in \mathbf{R}^n \quad \exists \gamma_p \in L^1(0,1,\mathbf{R}) \quad \text{such that}$$

$$f^*(t,p) \leq \gamma_p(t)$$

We also need, in this section, a condition which we used in [2] to prove a representation theorem for f^{**}.

We say that it is satisfied a Caratheodory condition on f^{**} when

$$(C.C.^{**}) \qquad f^{**}(t,v) < +\infty \qquad \forall v \in \mathbf{R}^n.$$

As a comment we remark that this condition assures that $f^{**}(t,\cdot)$ is convex and finite; so it is continuous and f^{**} is a Caratheodory integrand.

For functions satisfying property $(C.C.^{**})$ we proved in [2] the following representation theorem.

THEOREM 4. - [2] - *Let f be a normal proper integrand satisfying (B.G.C.) and $(C.C.^{**})$, then for every $x \in L^1(0,1,\mathbf{R}^n)$ there is a measurable multifunction*

$$\Gamma : [0,1] \to \mathbf{R}^n \times \mathbf{R}$$

which has compact values and which satisfies the following properties:

$$\Gamma(t) \subset \{(v,\alpha) \in \mathbf{R}^n \times \mathbf{R} : \alpha = f(t,v) = f^{**}(t,v)\}$$

$$(x(t), f^{**}(t,x(t))) \in \text{co } \Gamma(t) \qquad a.e. - t \in [0,1].$$

We remark that

$$\Gamma(t) \neq \emptyset \qquad a.e. - t \in [0,1]$$

because

$$(x(t), f^{**}(t,x(t))) \in \text{co } \Gamma(t).$$

Using this result we prove a lemma which is essential for non-convex existence theorems; in this lemma and throughout the remaining part of this section we use the notation $R(I_f)$ and $R(I_{f^{**}})$ to indicate

$$R(I_f) = \{I_f(x) : x \in L_0^1(0,1,\mathbf{R}^n)\}$$

and

$$R(I_{f^{**}}) = \{I_{f^{**}}(x) : x \in L_0^1(0,1,\mathbf{R}^n)\}$$

LEMMA 5. *Let f be a normal proper integrand satisfying (B.G.C.) and $(C.C.^{**})$; let us moreover suppose that there is $\bar{x} \in L_0^1(0,1,\mathbf{R}^n)$ such that $I_{f^{**}}(\bar{x}) \in \mathbf{R}$. Then*

$$R(I_{f^{**}}) \subset R(I_f).$$

Proof. Let $x \in L^1_0(0,1,\mathbf{R}^n)$ and let Γ be the multifunction defined in theorem 4.

Since Γ is measurable and has non-empty compact values, we can find a Castaing representation for Γ, [1].

So there exist $\{u_i : i \in \mathbf{N}\}$, $u_i : [0,1] \to \mathbf{R}^n \times \mathbf{R}$ such that

$$\Gamma(t) = cl\{u_i(t) : i \in \mathbf{N}\}.$$

Let us define

$$\vartheta(t) = \sup\{||u_i(t)|| : i \in \mathbf{N}\}$$

ϑ is a measurable function and we have

$$\Gamma(t) \subset \vartheta(t)\ B$$

where $B = \{x \in \mathbf{R}^n \times \mathbf{R} : ||x|| \leq 1\}$.

Let us moreover define

$$A_m = \{t \in [0,1] : m-1 < \vartheta(t) \leq m\} \quad ;$$

A_m is a measurable set $\forall m \in \mathbf{N}$ and we have:

$$\begin{aligned} &A_m \cap A_p = \emptyset \quad m \neq p \\ &\bigcup_{m \in \mathbf{N}} A_m = [0,1]. \end{aligned} \tag{3}$$

Finally let $\Gamma_m : A_m \to \mathbf{R}^n \times \mathbf{R}$ defined as $\Gamma_m(t) = \mathrm{co}\left(\Gamma|_{A_m}\right)$. where $\Gamma|_{A_m}$ is the restriction of Γ to A_m.

It results

$$\Gamma_m(t) \subset m\ B$$

and, by theorem IV-17 in [3],we can find

$$(y_m, \alpha_m) : A_m \to \mathbf{R}^n \times \mathbf{R}$$

such that if we indicate with ext K the set of all extremal points of a closed convex set $K \subset \mathbf{R}^n$,

$$(y_m(t), \alpha_m(t)) \in \text{ext } (\Gamma_m(t)) = \text{ext } \left(\text{co}\left(\Gamma|_{A_m}(t)\right)\right) \quad a.e. - t \in A_m$$

$$(4) \qquad \int_{A_m} y_m(t)dt = \int_{A_m} x(t)dt$$

$$\int_{A_m} \alpha_m(t)dt = \int_{A_m} f^{**}(t, x(t))dt.$$

By theorem 11.2.2 in [4], since $\Gamma(t)$ is a compact set, we have

$$\text{ext } (\text{co}\,(\Gamma(t))) \subset \Gamma(t)$$

so that

$$(y_m(t), \alpha_m(t)) \in \Gamma(t) \qquad a.e. - t \in A_m$$

and, by theorem 4, we have

$$(5) \qquad \alpha_m(t) = f(t, y_m(t)) = f^{**}(t, y_m(t)) \qquad a.e. - t \in A_m$$

Now let us define a function $y : [0, 1] \to \mathbf{R}^n$ which coincides with y_m on A_m , $\forall m \in \mathbf{N}$. In other words let y be the measurable function defined by

$$y(t) = y_m(t) \qquad \forall t \in A_m \qquad \forall m \in \mathbf{N}$$

and let us consider the sequence of functions

$$z_k : [0, 1] \to \mathbf{R}^n$$

defined as

$$z_k(t) = \begin{cases} y_m(t), & t \in A_m \quad 1 \leq m \leq k \\ x(t), & \text{elsewhere}. \end{cases}$$

It results $z_k \in L^1(0, 1, \mathbf{R}^n)$ and moreover, by (3), if we set

$$B_k = \bigcup_{m \leq k} A_m \qquad \text{and} \qquad C_k = [0, 1] \backslash B_k$$

we have by (4)

$$(6) \qquad \int_0^1 z_k(t)dt = \int_{B_k} y_m(t) + \int_{C_k} x(t)dt = \int_0^1 x(t)dt = 0$$

$$\int_0^1 f^{**}(t, z_k(t))dt = \int_0^1 f^{**}(t, x(t))dt.$$

So by (6) we can assert $z_k \in L^1_0(0, 1, \mathbf{R}^n)$, while it is obvious that

$$z_k(t) \to y(t) \qquad a.e. - t \in [0, 1].$$

Again, by (6), we obtain that $I_{f^{**}}(z_k) = I_{f^{**}}(x)$, so that z_k belong to a level set of $I_{f^{**}}$, which is, by (B.G.C.), a weakly compact set and we can find a subsequence of z_k, which we call z_k again, such that

$$z_k \to z \qquad \text{weakly in } L^1(0, 1, \mathbf{R}^n).$$

Since $L^1_0(0, 1, \mathbf{R}^n)$ is closed under weak convergence in $L^1(0, 1, \mathbf{R}^n)$ we have $z \in L^1_0(0, 1, \mathbf{R}^n)$. Moreover

$$\int_0^1 ||z_k(t) - y(t)||dt = \int_{C_k} ||z_k(t) - y(t)|| \to 0$$

since z_k is a weakly convergent sequence, $||z_k(\cdot) - y(\cdot)||$ is an equiabsolutely continuous family of functions and meas $(C_k) \to 0$.

So $z_k \to y$ strongly in $L^1(0, 1, \mathbf{R}^n)$ and we have $z = y$.

In similar way, when k is sufficiently large, for some $m \in \mathbf{N}$, we have

$$f(t, y(t)) = f(t, z_k(t)) = f(t, y_m(t)) =$$
$$= f^{**}(t, y_m(t)) = f^{**}(t, z_k(t)) = f^{**}(t, y(t))$$

and $f^{**}(t, z_k(t)) \to f^{**}(t, y(t))$, $a.e. - t \in [0, 1]$.

Moreover, by (B.G.C.) and (6)

$$-\int_0^1 f^*(t, 0)dt \le \int_0^1 f^{**}(t, y(t))dt \le$$

$$\le \liminf \int_0^1 f^{**}(t, z_k(t))dt = \int_0^1 f^{**}(t, x(t))dt$$

and $f^{**}(\cdot, y(\cdot)) \in L^1(0, 1, \mathbf{R})$.

So, since we have

$$-f^*(t, 0) \le f^{**}(t, z_k(t)) \le \max\{f^{**}(t, y(t)), f^{**}(t, x(t))\},$$

and since the latter function is in $L^1(0, 1, \mathbf{R})$, by dominated convergence theorem and by (5) we deduce

$$I_{f^{**}}(y) = \lim I_{f^{**}}(z_k) = I_{f^{**}}(x).$$

Since, by (5),

$$f(t, y(t)) = f^{**}(t, y(t))$$

we have

$$I_f(y) = I_{f^{**}}(y) = I_{f^{**}}(x)$$

and the theorem is proved. ■

We are now able to prove an existence theorem.

THEOREM 6. *Let us suppose that f is a normal proper integrand satisfying (B.G.C.) and (C.C.**); let us moreover suppose that there is $\bar{x} \in L_0^1(0,1,\mathbf{R}^n)$ such that $I_{f^{**}}(\bar{x}) \in \mathbf{R}$.*

Then there is $x_0 \in L_0^1(0,1,\mathbf{R}^n)$ such that

$$I_f(x_0) = \min\{I_f(x) : x \in L_0^1(0,1,\mathbf{R}^n)\}.$$

Proof. Since f^{**} is a convex normal integrand satisfying (B.G.C.), we can find $y_0 \in L_0^1(0,1,\mathbf{R}^n)$ such that

$$I_{f^{**}}(y_0) \leq I_{f^{**}}(x) \qquad \forall x \in L_0^1(0,1,\mathbf{R}^n)$$

By lemma 5 we can find $x_0 \in L_0^1(0,1,\mathbf{R}^n)$ such that

$$I_{f^{**}}(y_0) = I_f(x_0)$$

so that

$$I_f(x_0) \leq I_{f^{**}}(x) \leq I_f(x) \qquad \forall x \in L_0^1(0,1,\mathbf{R}^n)$$ ■

Some final considerations.

A problem of calculus of variations associated to a non convex integrand has been considered in [5], [6], [7], [8], [9], [10], [11] and [12] by various authors.

There are two features common to the majority of the previous cited works: the first one is that integrands are requested to satisfy a polinomial growth condition of degree strictly greater than 1, the second one is that integrands must be upper bounded by suitable polinomial functions.

The remaining works allow the integrand to satisfy only a superlinar growth condition but it must be essentially defined on the real line.

In the present work we only use a "basic growth condition" i.e. a superlinear growth condition which is not necessarily of polinomial type with degree greater than one, moreover a very weak upper boundedness condition is required: i.e. f^{**} must be finite.

Finally we wish to remark that our integrands can also assume infinite values and this fact allow us to modify the integrand itself in order to take also into account of a certain type of constraints.

Infinite values for f^{**} can be allowed but much more work is necessary, this is the theme of further research.

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