ON THE EVOLUTION OF ORDINARY DISCONTINUITY AND CHARACTERISTIC SHOCKS

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The evolution laws of ordinary discontinuity and characteristic shocks are derived using the kinematic and geometric of singular surfaces. Applications to contact discontinuity in gas dynamics and to Alfvén shocks are made.

1. Introduction.

The evolution of ordinary discontinuity and characteristic shocks in hyperbolic system is a well established subject [1, 2, 3, 4]. Several approaches have been used in this area according to different physical context. Here we want to investigate in detail an approach which is derived in the framework of a much more general method, the so called «Generalized Wavefront Expansion» (GWE), already used in the study of the propagation of weak shock waves [5,6] and of «step shocks» [7]. The same method has been used in the context of «corrugation stability» [8]. The advantage of the method consists in its simplicity in deriving the transport equations and in the fact that it does not require the solution of the partial differential equation for the phase.

Instead a system of ordinary differential equations has to be solved in order to find the evolution of the weak discontinuity or the characteristic shock, along the «rays». A transport equation for the second fundamental form of the characteristic surface is also derived, which closes the system

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of differential equations. As an application of the method we derive the transport equations for the contact discontinuity in gas dynamics and the Alfvén shock in magneto-fluid dynamics. The equations are then solved in the case of a contact discontinuity in an isothermal stratified athmosphere. The plan of the paper is the following: in Section 2 we introduce the formalism and derive the transport equations for the ordinary discontinuity and for the second fundamental form of the surface; in Section 3 we derive the transport equations for characteristic shocks and for one-dimensional intermediate discontinuities and make the applications.

2. General formalism and weak discontinuities.

Let us consider a hyperbolic system of conservation laws in R^4 [2]

$$\partial_t \mathbf{F}^0(\mathbf{U}) + \partial_i \mathbf{F}^i(\mathbf{U}) = \mathbf{f}(\mathbf{U}) \tag{2.1}$$

where $\mathbf{F}^0(\mathbf{U})$, $\mathbf{F}^i(\mathbf{U})$ are regular functions of the N-vector field $\mathbf{U}$ defined in an open domain $\Omega \subseteq R^N$.

The system may be written in the form:

$$\mathcal{A}^0 \partial_t \mathbf{U} + \mathcal{A}^i \partial_i \mathbf{U} = \mathbf{f} \tag{2.2}$$

where $\mathcal{A}^0 = \nabla_{\mathbf{U}} \mathbf{F}^0$, $\mathcal{A}^i = \nabla_{\mathbf{U}} F^i$, or, in the explicit form:

$$\partial_t \mathbf{U} + \mathcal{M}^i \partial_i \mathbf{U} = \mathbf{g} \tag{2.3}$$

where $\mathcal{M}^i = (\mathcal{A}^0)^{-1}$, $\mathbf{g} = (\mathcal{A}^0)^{-1}\mathbf{f}$, and the hyperbolicity of the system (2.2) ensure us that $\mathcal{A}^0$ is non-singular.

Let $\Sigma(t)$ be a surface across which there is a discontinuity in the field derivatives. The first and second order kinematic and geometric compatibility relations across the singular surface generally read:

$$[\![\partial_t \mathbf{U}]\!] = \frac{\delta [\![\mathbf{U}]\!]}{\delta t} - V_\Sigma \mathbf{Z} \tag{2.4.a}$$

$$[\![\partial_i \mathbf{U}]\!] = \tilde{\partial}_i [\![\mathbf{U}]\!] + n_i \mathbf{Z} \tag{2.4.b}$$

(2.4.c) $$[\![\partial_t \partial_j \mathbf{U}]\!] = n_j \frac{\delta \mathbf{Z}}{\partial t} + \mathbf{Z}\frac{\delta n_j}{\delta t} + \frac{\delta}{\delta t}(\tilde{\partial}_j [\![\mathbf{U}]\!]) - V_\Sigma \tilde{\partial}_j \mathbf{Z} - V_\Sigma n_j \mathbf{W} - V_\Sigma n^i \tilde{\partial}^2_{ji} [\![\mathbf{U}]\!]$$

(2.4.d) $$[\![\partial_i \partial_j \mathbf{U}]\!] = \tilde{\partial}_{ij} [\![\mathbf{U}]\!] + \mathbf{Z}\chi_{ij} + n_i \tilde{\partial}_j \mathbf{Z} + n_j \tilde{\partial}_i \mathbf{Z} + n_i n_j \mathbf{W} + n_i n^k \tilde{\partial}_{jk} [\![\mathbf{U}]\!]$$

where, for any quantity $h(\mathbf{U})$, we define $[\![h]\!] \equiv h_- - h_+$, h_- and h_+ being the values evaluated just behind or ahead of $\Sigma(t)$, and

(2.5) $$\mathbf{Z} \equiv [\![n^i \partial_i \mathbf{U}]\!], \ \mathbf{W} \equiv [\![n^i n^j \partial_{ij} \mathbf{U}]\!]$$

(2.6) $$\frac{\delta}{\delta t} \equiv \partial_t + V_\Sigma n^i \partial_i \text{ (the Thomas displacement derivative)}$$

(2.7) $$\tilde{\partial}_i \equiv h_i^j \partial_j, \ h_i^j \equiv \delta_i^j - n_i n^j \text{ (derivative along } \Sigma)$$

and $\chi_{ij} \equiv \tilde{\partial}_i n_j$ is the second fundamental form of the surface.

In the case of a weak discontinuity the relations (2.4) simplify because the field is continuous across $\Sigma(t)$.

Now let us take the jump of equation (2.3) across Σ:

(2.8) $$[\![\partial_t \mathbf{U}]\!] + \mathcal{M}^i [\![\partial_i \mathbf{U}]\!] = 0$$

(note that $\mathcal{M}^i(\mathbf{U})$ and $\mathbf{g}(\mathbf{U})$ are continuous across Σ).

Using (2.4) we have:

(2.9) $$(\mathcal{M}^i n_i - V_\Sigma I)\mathbf{Z} = 0$$

This means that V_Σ is an eigenvalue and $\mathbf{Z}$ an eingenvector of $\mathcal{M}^i n_i$.

Let $\lambda = \lambda(\mathbf{U}, \vec{n}) = V_\Sigma$ be the corresponding eigenvalue and let m be its multiplicity.

Then equation (2.9) gives:

(2.10) $$\mathbf{Z} = \sum_{I=1}^{m} \Phi^I \mathbf{R}^I$$

where $\mathbf{R}^I$ are the m (independent) eigenvectors corresponding to λ.

Now if we take the jump in the space derivative of the system (2.3) across Σ, use the compatibility relations (2.4) and multiply by n^j, we get:

$$(2.11)\quad \frac{\delta \mathbf{Z}}{\delta t} + (\mathcal{M}^i n_i - \lambda I)\mathbf{W} + \mathcal{N}^i_a n_i Z^a \mathbf{Z} + \mathcal{N}^i_a n_i n^j \partial_j U^a_0 \mathbf{Z} + \mathcal{N}^i_a Z^a \partial_i \mathbf{U}_0$$

$$+ \mathcal{M}^i \tilde{\partial}_i \mathbf{Z} - \nabla_a \mathbf{g} Z^a = 0$$

where $\mathcal{N}^i_a \equiv \partial \mathcal{M}^i / \partial U^a$.

Now multiplying (2.11) by ${}^T\mathbf{L}_J$, the J-th left eigenvector of $\mathcal{M}^i n_i$ we have:

$$(2.12)\quad \sum_I ({}^T\mathbf{L}_J \mathbf{R}^I)\frac{\delta \Phi^I}{\partial t} + \sum_I \Phi^I\, {}^T\mathbf{L}_J \frac{\delta \mathbf{R}^I}{\delta t} + \sum_I {}^T\mathbf{L}_J \mathcal{M}^i \mathbf{R}^I \tilde{\partial}_i \Phi^I$$

$$+ \sum_I \Phi^I\, {}^T\mathbf{L}_J \mathcal{M}^i \tilde{\partial}_i \mathbf{R}^I + \sum_{IK} \Phi^I \Phi^K\, {}^T\mathbf{L}_J \mathcal{N}^i_a n_i R^{Ia} \mathbf{R}^K$$

$$+ \sum_I \Phi^I ({}^T\mathbf{L}_J \mathcal{N}^i_a n_i n^j \partial_j U^a_0 \mathbf{R}^I + {}^T\mathbf{L}_J\, \mathcal{N}^i_a R^{Ia} \partial_i \mathbf{U}_0 - {}^T\mathbf{L}_J \nabla_a \mathbf{g} R^{Ia}) = 0$$

Now, using the identities [1]:

$$(2.13.a)\quad {}^T\mathbf{L}_J \mathcal{M}^i \mathbf{R}^I = \frac{\partial \lambda}{\partial n_i}\, {}^T\mathbf{L}_J \mathbf{R}^I$$

$$(2.13.b)\quad {}^T\mathbf{L}_J \mathcal{N}^i_a n_i \mathbf{R}^I = \nabla_a \lambda\, {}^T\mathbf{L}_J \mathbf{R}^I$$

equation (2.12) may be written, after some easy manipulation,

$$(2.14)\quad \sum_I {}^T\mathbf{L}_J \mathbf{R}^I \frac{d\Phi^I}{dt} + \sum_{JK} (\Phi^I \phi^K \nabla_a \lambda R^{Ia})\, {}^T\mathbf{L}_J \mathbf{R}^K +$$

$$\sum_I \Phi^I \Bigg({}^T\mathbf{L}_j \nabla_a \mathbf{R}^I \frac{\delta U^a_0}{\delta t} + {}^T\mathbf{L}_J \frac{\partial \mathbf{R}^I}{\delta n_j} \frac{\delta n^J}{\delta t} + {}^T\mathbf{L}_J \mathcal{M}^i \nabla_a \mathbf{R}^I \tilde{\partial}_i U^a_0 +$$

$$\frac{1}{2} \frac{\partial^2 \lambda}{\partial n_i \partial n_j} \chi_{ij}\, {}^T\mathbf{L}_J \mathbf{R}^I + \chi_{ij} \frac{\partial \lambda}{\partial n_i}\, {}^T\mathbf{L}_J \frac{\partial \mathbf{R}^I}{\partial n_j}$$

$$+ {}^T\mathbf{L}_J \mathbf{R}^I \nabla_a \lambda n^J \partial_j U^a_0 + {}^T\mathbf{L}_J \mathcal{N}^i_a R^{Ia} \partial_i \mathbf{U}_0 - {}^T\mathbf{L}_j \nabla_a \mathbf{g} R^{Ia} \Bigg) = 0$$

where

(2.15)
$$\frac{d}{dt} \equiv \frac{\delta}{\delta t} + \frac{\partial \lambda}{\partial n_i}\tilde{\partial}_i$$

is the total derivative along the «rays» defined by

(2.16)
$$\frac{dx^i}{dt} = \lambda n^i + h^i_j \frac{\partial \lambda}{\partial n_j}$$

and $\mathbf{U}_0$ is the unperturbed field ahead of Σ and is supposed to be a known function of x^i.

Note that in (2.14) the derivatives $(d\Phi^i/dt)$ are well defined, because $det({}^T\mathbf{L}_J\mathbf{R}^I) \neq 0$ [1].

In the case of propagation into a constant state these equations have already been derived by Giambó [9].

This system must be supplemented with transport equations for n^i and χ_{ij}.

The transport equation for n^i is given by [10]:

(2.17)
$$\frac{\delta n_i}{\delta t} + \tilde{\partial}_i V_\Sigma = 0$$

which can be written:

(2.18)
$$\frac{dn_i}{dt} + \nabla_a \lambda \tilde{\partial}_i U_0^a = 0$$

We use the dependence of $\mathbf{U}_0$ on x^i to write

(2.19)
$$\lambda = \lambda(\mathbf{U}_0, \vec{n}) = \lambda(x^i, \vec{n})$$

We derive now a transport equation for the second fundamental form in cartesian coordinates.

We shall make use of the following properties, which are proven in the appendix:

i) Symmetry of the second fundamental form:

(2.20)
$$\chi_{ij} = \chi_{ji}$$

ii) First commutation relation:

$$\frac{\delta}{\delta t}\tilde{\partial}_j - \tilde{\partial}_j\frac{\delta}{\delta t} = \left[\left(\frac{\partial\lambda}{\partial x_k} + \frac{\partial\lambda}{\partial n_s}\chi_s^k\right)n_j - \lambda\chi_j^k\right]\tilde{\partial}_k \tag{2.21}$$

iii) second commutation relation:

$$\tilde{\partial}_{ij} - \tilde{\partial}_{ji} = (n_j\chi_i^s - n_i\chi_j^s)\tilde{\partial}_s \tag{2.22}$$

Now let us take the derivative of relation (2.17):

$$\tilde{\partial}_j\frac{\delta n_i}{\delta t} + (\tilde{\partial}_j h_i^k)\frac{\partial\lambda}{\partial x^k} + h_i^k\left(\tilde{\partial}_j\frac{\partial\lambda}{\partial x^k}\right) + \left(\tilde{\partial}_j\frac{\partial\lambda}{\partial n^k}\right)\chi_i^k + \frac{\partial\lambda}{\partial n^k}\tilde{\partial}_j\chi_i^k = 0 \tag{2.23}$$

or, developing the derivatives:

$$\tilde{\partial}_j\frac{\delta n_j}{dt} - (\chi_{ji}n^k + n_i\chi_j^k)\frac{\partial\lambda}{\partial x^k} + h_{ik}\left(h_{js}\frac{\partial^2\lambda}{\partial x_k\partial x_s} + \frac{\partial^2\lambda}{\partial x_k\partial n_s}\tilde{\partial}_j n_s\right)$$

$$+\left(h_{js}\frac{\partial^2\lambda}{\partial x_s\partial n_k} + \frac{\partial^2\lambda}{\partial n_k\partial n_s}\chi_{js}\right)\chi_{ki} + \frac{\partial\lambda}{\partial n_k}\tilde{\partial}_j\tilde{\partial}_i n_k = 0$$

Now, making use of (2.20), (2.21) and (2.22) we get, after some easy manipulations,

$$\frac{d\chi_{ij}}{dt} + h_{jk}h_{is}\frac{\partial^2\lambda}{\partial x_s\partial x_k} + h_{ik}\frac{\partial^2\lambda}{\partial x_k\partial n_s}\chi_{js} + h_{jk}\frac{\partial^2\lambda}{\partial x_k\partial n_s}\chi_{is} \tag{2.25}$$

$$+\frac{\partial^2\lambda}{\partial n_k\partial n_s}\chi_{js}\chi_{ik} - \frac{\partial\lambda}{\partial x_k}(\chi_{ik}n_j + \chi_{ji}n_k + \chi_{kj}n_i) + \left(\lambda - \frac{\partial\lambda}{\partial n_k}n_k\right)\chi_j^s\chi_{si}$$

The essential advantage of this method is that we obtain a closed set of O.D.E.'s. Instead, with other methods we must solve the eikonal equation for the phase, which is a partial differential equation for the phase. It can be checked that the transport equations obtained are equivalent to those obtained by Boillat [1].

3. The evolution of characteristic shocks and intermediate discontinuities.

For a characteristic shock the shock surface Σ coincides with a characteristic surface and its velocity V_Σ with an eigenvalue of the system, both ahead and behind the shock [11].

$$V_\Sigma = \lambda(\mathbf{U}_0, \vec{n}) = \lambda(\mathbf{U}, \vec{n})$$

where we denote by a subscript 0 the field ahead.

The corresponding eigenvalue may be single (in this case the shock is characteristic iff the so called «exceptionality condition», $(\nabla_a \lambda R^a) = 0$, is satisfied, or have multiplicity $m > 1$ (the shock corresponding to multiple eigenvalue is always *exceptional* [11]).

We shall consider the general case $m \geq 1$.

The Rankine-Hugoniot conditions at the shock surface:

$$-V_\Sigma [\![\mathbf{F}^0]\!] + n_i [\![\mathbf{F}^i]\!] = 0 \tag{3.2}$$

constitute a set of N relations, $N - m$ independent, and the system may be solved giving the field the shock as a function of the field behind ahead and of m parameter [11].

$$[\![\mathbf{U}]\!] = \mathbf{Y}(\mathbf{U}_0, \vec{n}, u^I), \quad I = 1, \ldots, m \tag{3.3}$$

By taking the jump of the field equations (3.3) and using first order compatibility relations we get:

$$\frac{\delta \mathbf{Y}}{\delta t} + (\mathcal{M}^i_- n_i - V_\Sigma I)\mathbf{Z} + \mathcal{M}^i_- \tilde{\partial}_i \mathbf{Y} + [\![\mathcal{M}^i]\!] \partial_i \mathbf{U}_0 - [\![\mathbf{g}]\!] = 0 \tag{3.4}$$

By developing the derivatives of $\mathbf{Y}$ as

$$\frac{\delta \mathbf{Y}}{\delta t} = \nabla_a \mathbf{Y} \frac{\delta U_0^a}{\delta t} + \frac{\partial \mathbf{Y}}{\partial u^I}\frac{\delta u^I}{\delta t} + \frac{\delta \mathbf{Y}}{\delta n^i}\frac{\delta n^i}{\partial t}$$

(3.5)

$$\tilde{\partial}_i \mathbf{Y} = \nabla_a \mathbf{Y} \tilde{\partial}_i U^a + \frac{\partial \mathbf{Y}}{\partial u^I} \tilde{\partial}_i u^I + \frac{\partial \mathbf{Y}}{\partial n_j} \tilde{\partial}_i n_j$$

and using definition (2.15) and transport equation (2.14) we get:

$$\sum_I {}^T\mathbf{L}_J \frac{\partial \mathbf{Y}}{\partial u^I}\frac{du^I}{dt} +^T \mathbf{L}_J \nabla_a \mathbf{Y}\frac{\delta U_0^a}{\delta t} +^T \mathbf{L}_J \mathcal{M}_-^i \nabla_a \mathbf{Y} \partial_i U_0^a$$
$$-\frac{\partial\, {}^T\mathbf{L}_J}{\partial n_i}(\mathcal{M}_-^k n_k - \lambda I)\frac{\partial \mathbf{Y}}{\partial n_j}\chi_{ij}$$
$$-{}^T\mathbf{L}_J \frac{\partial \mathbf{Y}}{\partial n_i}\nabla_a \lambda \tilde{\partial}_i U_0^a +^T \mathbf{L}_J [\![\mathcal{M}^i]\!] \partial_i \mathbf{U}_0 -^T \mathbf{L}_J [\![g]\!] = 0 \tag{3.6}$$

where the fact that $(\partial \mathbf{Y}/\partial u^I)$ is a right eigenvector of $\mathcal{M}^i n_i$ [11] and relation (2.13.a) have been used.

System (3.6), together with equations (2.16), (2.18) and (2.25) constitute a set of $m + 12$ equations for the unknown quantities u^I, x^i, n^i, χ_{ij}.

In the case of an intermediate discontinuity, that is when $V_\Sigma = \lambda(\mathbf{U}_-)$ [2], the method is not directly applicable for multidimensional propagation because the spatial derivatives of the field just behind the shock are not known. But in the one dimensional case the extension is straightforward.

Let us consider such a situation, described by the system

$$\partial_t \mathbf{U} + \mathcal{M}\partial_x \mathbf{U} = \mathbf{g} \tag{3.7}$$

By using the usual technique we get

$$\frac{\delta [\![\mathbf{U}]\!]}{\delta t} + (\mathcal{M}_- - V_\Sigma I)\mathbf{Z} + [\![\mathcal{M}]\!]\partial_x \mathbf{U}_0 - [\![\mathbf{g}]\!] = 0 \tag{3.8}$$

Let u be a parameter describing the jump of the field across Σ, that is let us solve (3.2) in the form

$$[\![\mathbf{U}]\!] = \mathbf{Y}(\mathbf{U}_0, u) \tag{3.9}$$

By multiplying (3.8) by the left eigenvector of $\mathcal{M}_-$ corresponding to the eigenvalue $\lambda_i = V_\Sigma$ we get the transport equation:

$${}^T\mathbf{L}\frac{\partial \mathbf{Y}}{\partial u}\frac{\delta u}{\delta t} +^T \mathbf{L}\nabla_a \mathbf{Y}\frac{\delta U^a}{\delta t^0} +^T \mathbf{L}[\![\mathcal{M}]\!]\partial_x \mathbf{U}_0 -^T \mathbf{L}[\![\mathbf{g}]\!] = 0 \tag{3.10}$$

Contact discontinuity in gas dynamics

As an application of the theory developed we study the evolution of a contact discontinuity in classical gas dynamics [12, 13].

In this case the field $\mathbf{U}$ is given by

$$ {}^T\mathbf{U} = (\rho, v^1, v^2, v^3, p) \tag{3.11} $$

and

$$ \mathbf{F}^0 = \begin{pmatrix} \rho \\ \rho\vec{v} \\ w \end{pmatrix}, \quad \mathbf{F}^i = \begin{pmatrix} \rho v^i \\ p\hat{e}^i + \rho v^i \vec{v} \\ (w+p)v^i \end{pmatrix} \tag{3.12} $$

where ρ, v, p are, respectively, density, velocity and pressure of the gas, $w = \rho(|\vec{v}|^2/2 + \varepsilon)$, ε is the internal energy and is related to p and ρ by a given equation of state $\varepsilon = \varepsilon(p, \rho)$ and $\hat{e}^i$ is the unit vector of the x^i axis.

In the case of a contact discontinuity the jump relations across Σ read:

$$ [\![\vec{v}]\!] \cdot \vec{n} = 0, \ [\![p]\!] = 0 \tag{3.13} $$

The matrices $\mathcal{M}^k$ are given by

$$ \mathcal{M} = \begin{pmatrix} v^k & \rho^T\hat{e}^k & 0 \\ 0 & v^k I & \hat{e}^k/\rho \\ 0 & \rho a^2\, T\hat{e}^k & v^k \end{pmatrix} \tag{3.14} $$

where $a^2 = (\partial p/\partial \rho)_s$.

The eigenvales are $\lambda_1 = v_n - a$, $\lambda_2 = v_n$, $\lambda_3 = v_n + a$.

The multiplicity of the characteristic eigenvalue $\lambda = \lambda_2$ is $m = 3$. Instead of using three independent parameters for describing the shock we use four parameters linked by a relation. We obtain a set of four equations, not all independent.

The eigenvectors are given by:

$$ \mathbf{L}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1/a^2 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 0 \\ 0 \\ n_2 \\ -n_1 \\ 0 \end{pmatrix}, \tag{3.15} $$

$$ \mathbf{L}_2 = \begin{pmatrix} 0 \\ -n_3 \\ 0 \\ n_1 \\ 0 \end{pmatrix}, \quad \mathbf{L}_3 = \begin{pmatrix} 0 \\ 0 \\ n_2 \\ -n_1 \\ 0 \end{pmatrix}, $$

$${}^T\mathbf{R}_0 = (1,0,0,0,0),\ \mathbf{R}_i = \mathbf{L}_i\ i = 1,2,3.$$

These are not independent because $n^i\mathbf{L}_i$ is the null vector.

By choosing $[\![\rho]\!]$ and $[\![\vec{v}]\!]$ as parameters for describing the shock and performing all the calculations one obtains the following transport equations:

$$\frac{d[\![\rho]\!]}{dt} + [\![\vec{v}]\!] \cdot \left(\nabla\rho_0 - \frac{1}{a^2}\nabla p_0\right) - \rho_0\left(1 - \frac{a_0^2}{a^2}\right)\nabla\cdot\vec{v}_0 = 0 \tag{3.16}$$

$$\left(\frac{d[\![\vec{v}]\!]}{dt} + [\![\frac{1}{\rho}]\!]\nabla p_0\right) \times \vec{n} = 0 \tag{3.17}$$

$$\frac{dn_i}{dt} + n_j\tilde{\partial}_i v_0^j = 0 \tag{3.18}$$

$$\frac{dx^i}{dt} = v_0^i + [\![v^i]\!] \tag{3.19}$$

where $\times$ denotes the vector product.

These equations are not in a useful form because it is not possible to obtain expressions for the single jump $[\![v^i]\!]$.

By performing a change of variables let

$$\vec{\Pi} = [\![\vec{v}]\!] \times \vec{n} \tag{3.20}$$

Then we have:

$$\vec{n} \times \vec{\Pi} = \vec{n} \times ([\![\vec{v}]\!] \times \vec{n}) = [\![\vec{v}]\!], \quad \vec{\Pi}\cdot\vec{n} = 0 \tag{3.21}$$

In terms of $\vec{\Pi}$ equations (3.17) are written in the form:

$$\frac{d\vec{\Pi}}{dt} + \left(\vec{\Pi}\cdot\frac{d\vec{n}}{dt}\right)\vec{n} + [\![\frac{1}{\rho}]\!]\nabla p_0 \times \vec{n} = 0 \tag{3.22}$$

and the constraint $\vec{\Pi}\cdot\vec{n} = 0$ is automatically satisfied by the equations.

Note that in this case the second form of the surface does not appear in the equations.

Let us solve these equation in the case of a contact discontinuity in a stratified isothermal polytropic gas. We consider a two dimensional problem with x and y denoting, respectively, the horizontal and vertical coordinates. At time $t = 0$ the contact line is a straight line forming an angle θ with the direction of the x-axes. Above the contact line the unperturbed state is at rest with an exponential decreasing density:

$$\rho(y)_0 = \bar{\rho}_0 e^{-y/L} \tag{3.23}$$

and the pressure is given by $p_0/\rho_0 = \bar{p}_0/\bar{\rho}_0$. Here $L = \bar{p}_0/(g\bar{\rho}_0)$ and g is the acceleration of gravity.

From equations (3.18) and (3.19) it follows that the position of the contact line remains unchanged.

Equations (3.16), (3.17), (3.19) become:

$$\frac{d\Pi}{dt} = \frac{[\![\rho]\!]}{\rho_0 + [\![\rho]\!]} g \sin\theta \tag{3.24}$$

$$\frac{d[\![\rho]\!]}{dt} = \frac{\Pi \rho_0 \sin\theta}{L} \left(\frac{\rho_0 + [\![\rho]\!]}{\gamma \rho_0} - 1 \right) \tag{3.25}$$

$$\frac{dy}{dt} = -\Pi \sin\theta \tag{3.26}$$

where $\vec{\Pi} = \Pi \hat{e}_z$ and γ is the polytropic constant.

The equations can be put in a non-dimensional form. Let

$$\xi = y/L, \ V \equiv \Pi\sqrt{\bar{\rho}_0/\bar{p}_0}, \ r \equiv \rho/\rho_0, \ \tau \equiv \sqrt{\bar{\rho}_0/\bar{p}_0} g t \sin\theta \tag{3.27}$$

then the system (3.24)-(3.26) becomes:

$$\dot{\xi} = V \tag{3.28}$$

$$\dot{V} = -\frac{r-1}{r} \tag{3.29}$$

$$\dot{r} = kVr \tag{3.30}$$

where $k = 1 - 1/\gamma$ and the dot denotes the differentiation with respect to τ. From the last two equations we get the relation between V and r:

$$V = \pm\sqrt{\frac{2}{k}\left(\frac{1}{\bar{r}} - \frac{1}{r} - \log\frac{r}{\bar{r}}\right)} \tag{3.31}$$

where $\bar{r}$ is the value of r for which $V = 0$.

The system (3.28-30) has periodic solutions which represent oscillating gravity waves in the fluid: in the second equation the restoring force for the velocity of the fluid is proportional to the jump in the density.

The system is solved numerically and the amplitude and the period of the oscillations are evaluated as a function of the initial density ratio $\bar{r}$, for various value of γ. The results are shown in Figure 1 and 2

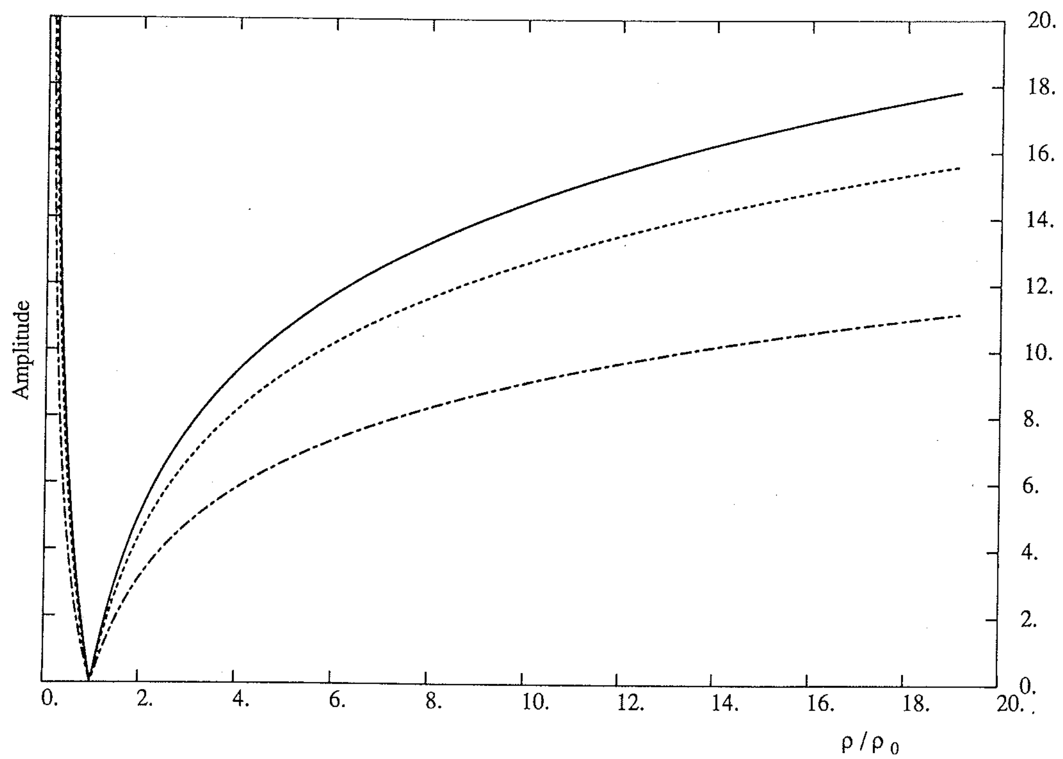

Fig. 1 - Amplitude $\xi_{max} - \xi_{min}$ of the oscillation described by equations (3.28-30) as a function of the maximum (or minimum) compression ratio ρ/ρ_0. The value of γ are:

$\gamma = 4/3$ (comtinuous line), $\gamma = 7/5$ (dotted line) and $\gamma = 5/3$ (dot-dash).

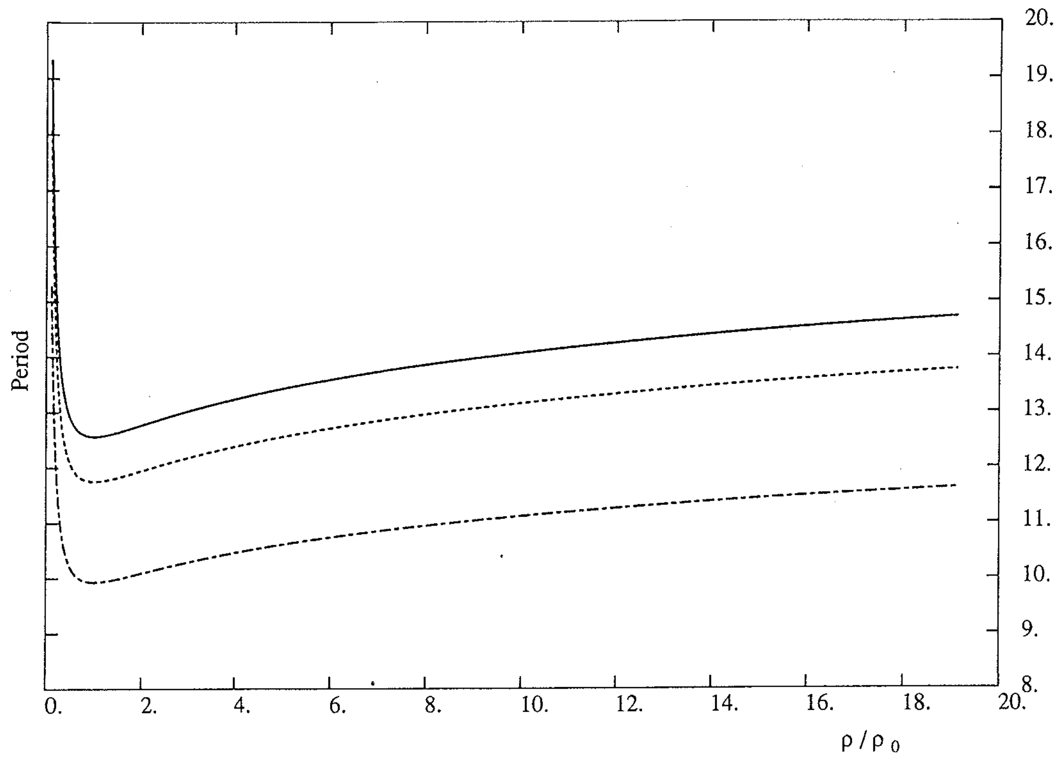

Fig. 2 - Perid of the oscillation expressed in terms of the non dimensional time $\tau = \sqrt{\bar{\rho}_0/\bar{p}_0}\, g t \sin\theta$

We remark that this is a rather idealized case in which the interaction of acoustic waves with the discontinuity surface have been neglected. These interactions, generally, cause an instability of the surface and a perturbation of the flow on both sides.

The Alfvén shock.

As another application we consider the propagation of an Alfvén shock in magneto-fluid dynamics [14, 15].

In this case the field is given by

$$\mathbf{U} = \begin{pmatrix} \vec{H} \\ \vec{v} \\ \rho \\ s \end{pmatrix} \tag{3.32}$$

where $\vec{H}$ is the magnetic field vector, $\vec{v}$ the velocity of the fluid, ρ the density and s the specific entropy.

The eigenvalue corresponding to the Alfvén shock is $\lambda = v_n \pm \omega H_n$ where for any vector $\vec{f}$ we put $f_n \equiv \vec{f} \cdot \vec{n}$ and $\omega^2 \equiv \mu/\rho$, μ being the magnetic permeability of the fluid. The matrices governing the system may be written:

$$(3.33) \quad \mathcal{M}^k = \begin{pmatrix} v^k I - \vec{v}\,{}^T\hat{e}^k & \vec{H}\,{}^T\hat{e}^k - H^k I & 0 & 0 \\ \omega^2({}^T\vec{H}\hat{e}^k - H^k I) & v^k I & (p'_\rho/\rho)\hat{e}^k & (p'_s/\rho)\hat{e}^k \\ 0 & \rho\,{}^T\hat{e}^k & v^k & 0 \\ 0 & 0 & 0 & v^k \end{pmatrix}$$

Let us consider the + eigenvalue: $\lambda = v_n + \omega H_n$. Then the left and right eigenvectors may be written:

$$(3.34) \qquad \mathbf{R} = \begin{pmatrix} \vec{n} \times \vec{H} \\ -\omega\vec{n} \times \vec{H} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \omega[\vec{n} \times \vec{H} - (\vec{n} \times \vec{H} \cdot \vec{v})\vec{n}/\lambda] \\ -\vec{n} \times \vec{H} \\ 0 \\ 0 \end{pmatrix}$$

The Rankine-Hugoniot relations across the shock give [14, 15].

$$(3.35) \qquad [\![H_n]\!] = 0,\ [\![\rho]\!] = 0,\ [\![p]\!] = 0,\ [\![v_n]\!] = 0, [\![\vec{v}]\!] = -\omega[\![\vec{H}]\!]$$

Furthermore, if we call $\vec{h} \equiv \vec{H} - \vec{n}H_n$, then we have:

$$(3.36) \qquad [\![h^2]\!] = 0$$

and hence, if the field ahead of the shock is known, the only unknown quantity is a rotational angle θ for the tangential component of the magnetic field. We choose this parameter for describing the evolution of the shock.

In terms of the parameter θ we have:

$$(3.37) \qquad h^i = M^i_j(\theta, \vec{n})h^j_0$$

where $M^i_j(\theta, \vec{n})$ is a matrix which rotates a vector by an angle θ around the direction $\vec{n}$. It is easy to show that:

$$(3.38) \qquad M^i_j = (\delta^i_j - n^i n_j)\cos\theta + n^i n_j - \varepsilon^i_{jk} n^k \sin\theta$$

where ε^i_{jk} is the pseudo-tensor completely antisymmetric and $\varepsilon^1_{23} = 1$. The magnetic field behind the shock is given by

$$H^i = (M^i_j - n^i n_j)H^j_0 \tag{3.39}$$

By evaluating all the terms of the transport system (3.6) one obtains the following transport equation:

$$\frac{d\theta}{dt} + N_i \frac{dn^i}{dt} - Q^k_i \chi^i_k + \frac{1}{|h|^2}\vec{P} \cdot \frac{\delta \vec{H}_0}{\delta t} + B = 0 \tag{3.40}$$

where

$$\vec{P} \equiv \vec{n} \times \vec{h} = \vec{n} \times \vec{H}$$

$$N_i \equiv \frac{1}{2|h|^2}\left(P_j - \frac{\vec{P} \cdot \vec{v}}{2\lambda} n_j\right) S^j_i$$

$$S^j_i \equiv \frac{\partial [\![H^j]\!]}{\partial n^i} = -(n^j H_{0i} + H_n \delta^j_i)\cos\theta - \varepsilon^j_{ki} H^k_0 \sin\theta$$

$$Q^k_i \equiv \frac{1}{2|h|^2} P^k H_j S^j_i \tag{3.41}$$

$$B \equiv \frac{1}{2|h|^2}\left[\frac{\vec{P} \cdot \vec{v}}{\lambda} [\![H^k]\!] n_j \tilde{\partial}_k \Lambda^j_0 + 2P_{0j} \Lambda^k_0 \tilde{\partial}_k H^j_0 \right.$$
$$\left. + \omega P^k h_j \tilde{\partial}_k H^j_0 - \frac{\vec{P} \cdot \vec{v}}{\omega}(\Lambda^k_0 \tilde{\partial}_k \omega + \omega \partial_k v^k_0)\right]$$

$$\vec{\Lambda} \equiv \omega \vec{H} + \vec{v}$$

The rays equations are given by:

$$\frac{dx^i}{dt} = \Lambda^i \tag{3.42}$$

The equation for n^i:

$$\frac{dn^i}{dt} + n_k \tilde{\partial}_i \Lambda^k_0 = 0 \tag{3.43}$$

and the equations for χ_{ij} become:

$$(3.44)\qquad \frac{d\chi_{ij}}{dt} = h_{jk}h_i^s n_r \frac{\partial^2 \Lambda_0^r}{\partial x_s \partial x_k} + \frac{\partial \Lambda_0^s}{\partial x_k}(h_{ki}\chi_{sj} + h_{kj}\chi_{si} - \chi_{ki}n_s n_j$$

$$\chi_{ji}n_s n_k - \chi_{kj}n_i n_s) = 0$$

Here $\vec{\Lambda}_0 = \omega_0 \vec{H}_0 + \vec{v}_0$ is supposed to be a known function of x^i.

Note that using cartesian orthogonal coordinates in R^3 there is no difference between covariant and contravariant tensors.

Equations (3.39)-(3.43) constitute a closed system of therteen ordinary differential equatins which describes the evolution of an Alfvén shock propagating into a known state.

4. Conclusions.

The method of singular surfaces has been investigated in depth in order to derive transport equations, along the «rays», for the parameters defining a weak discontinuity or a characteristic shock in hyperbolic systems of conservation laws.

The method is part of a more general method (Generalized Wavefront Expansion) [5], which has been already developed in the context of propagation of weak shock waves [6], of step shocks [7] and in the study of corrugation stability of plane shocks [8].

A transport equation for the second fundamental form of the singular surface is also derived and this yields to a closed set of ordinary differential equations for the parameters of the discontinuity, the position of the point on the ray, the unit normal and the second form of the surface. This is the main advantage of the method which escapes the solution of the eikonal equation for the phase.

The method is easily extended to the propagation of intermediate discontinuities in one dimensional.

We applied the theory developed to the contact discontinuity in fluid dynamics and to the Alfvén shocks in magneto-fluid dynamics. In the first case we obtained an explicit system of O.E.D.'s for the jump in the density and a vector defining the jump in the velocity field, the position and the unit normal, while the second fundamental form does not appear in the equations. We solved the equations in the case of a plane contact discontinuity in a stratified isothermal athmosphere, evaluating the period and the amplitude of the oscillations of the waves.

In the second case the jump conditions are explicitly solved in terms of a rotation angle and a transport equation for this angle is evaluated, together with equations for position, normal and second form of the surface.

5. Appendix.

Proof. (2.20). First note that $h_j^s\partial_k n_s \equiv \partial_k n_j - n_j n^s \partial_k n_s = \partial_k n_j$ because $\partial_k(n^s n_s) = 0$, and recall that [1] $n_i = a\phi_i$, $(\phi_i \equiv \partial\phi/\partial x^i)$, where $\phi(x^i, t)$ is the phase which obeys the characteristic equation and define the surface of discontinuity; $a \equiv |\nabla\phi|^{-1}$.

Then

$$\chi_{ij} \equiv \tilde{\partial}_i n_j = h_i^k \partial_k n_j = h_i^k h_j^s \partial_k(a\phi_s) = h_i^k h_j^s(\partial_k a)\phi_s + a h_i^k h_j^s \phi_{ks}$$

Now the first term in the last expression is zero and the second is symmetric in i and j thus $\chi_{ij} = \chi_{ji}$.

Proof. (2.21). For a generic function $\zeta(x^i, t)$ we have:

$$\frac{\delta}{\delta t}\tilde{\partial}_j\zeta = h_j^k(\partial_t + \lambda n^s\partial_s)\partial_k\zeta + \frac{\delta h_j^k}{\delta t}\partial_k\zeta$$

$$= h_j^k\partial_k(\partial_t + \lambda n^s\partial_s)\zeta - h_j^k\partial_k(\lambda n^s)\partial_s\zeta - \left(n^k\frac{\delta n_j}{\delta t} + n_j\frac{\delta n^k}{\delta t}\right)\partial_k\zeta$$

$$= \tilde{\partial}_j\frac{\delta\zeta}{\delta t} - (\tilde{\partial}_j\lambda)n^s\partial_s\zeta - \lambda\tilde{\partial}_j n^s\partial_s\zeta - \left(n^k\frac{\delta n_j}{\delta t} + n_j\frac{\delta n^k}{\delta t}\right)\partial_k\zeta$$

$$= \tilde{\partial}_j\frac{\delta\zeta}{\delta t} - \lambda\chi_j^s\partial_s\zeta + n_j\left(h_s^k\frac{\partial\lambda}{\partial x_s} + n_j\frac{\partial\lambda}{\partial n_s}\tilde{\partial}_s n^k\right)\partial_k\zeta$$

$$= \tilde{\partial}_j\frac{\delta\zeta}{\delta t} + \left[\left(\frac{\partial\lambda}{\partial\chi_k} + \frac{\partial\lambda}{\partial n_s}\chi_s^k\right)n_j - \lambda\chi_j^k\right]\tilde{\partial}_k\zeta \qquad \text{Q.E.D.}$$

Proof. (2.22). For a generic function $\zeta(x^i)$ we have:

$$\tilde{\partial}_j \tilde{\partial}_i \zeta = h^s_i h^k_j \partial_k \zeta + (\tilde{\partial}_j h^s_i)\partial_s \zeta$$

$$= h^s_i \partial_s (h^k_j \partial_k \zeta) - h^s_i (\partial_k \zeta)(\partial_s h^k_i) + (\tilde{\partial}_j h^s_i)\partial_s \zeta$$

$$= \tilde{\partial}_i \tilde{\partial}_j \zeta + (\tilde{\partial}_j h^s_i - \tilde{\partial}_i h^s_j)\partial_s \zeta$$

$$= \tilde{\partial}_i \tilde{\partial}_j \zeta + (\chi_{ij} n^s + n_j \chi^s_i)\partial_s \zeta - (\chi_{ij} n^s + n_i \chi^s_j)\partial_s \zeta$$

$$= \tilde{\partial}_i \tilde{\partial}_j \zeta + (n_j \chi^s_j - n_i \chi^s_j)\partial_s \zeta$$

$$= \tilde{\partial}_i \tilde{\partial}_j \zeta + (n_j \chi^s_i - n_i \chi^s_j)\tilde{\partial}_s \zeta \qquad \text{Q.E.D.}$$

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REFERENCES

[1] Boillat G., *La propagation des ondes,* Gauthier-Villars, 1965.
[2] Jeffrey A., *Quasilinear Hyperbolic Systems and Waves,* (Pitman, London, 1976).
[3] Boillat G., *Comptes Rendus Acad. Sc. Paris,* t. 274, Série A (1972), 1018-1021.
[4] Boillat G., *J. Math. Pures et Appl.,* **56** (1977), 137-147.
[5] Anile A.M., *Wave Motion,* **6** (1984), 571-578.
[6] Anile A.M., Russo G., *Wave Motion,* **8** (1986), 243-258.
[7] Anile A.M., Russo G., *Wave Motion,* **10** (1988), 3-18.
[8] Anile A.M., Russo G., *Phys. Fluids,* **29** N. 9 (1986), 2847-2852.

[9] Giambó S., *Proceedings of the Workshop on Waves and Stability in Continuous Media,* Cantania, November 3-6, 1981, University of Catania Press, 1982.

[10] Truesdel C., Toupin R., *The Classical Field Theories,* 6 in *Handbuch der Physick,* Vol. III/1, berlin, Springer-Verlag (1960).

[11] Boillat G., *Article in Wave Propagation,* International Mathematical Summer Center, Bressanone 8-17 June 1980 (G. Ferrarese Editor), Liguori, Naples (1982).

[12] Boillat G., *Comptes Rendus Acad. Sc. Paris,* t. 284 (1977), Série A, 1481-1484.

[13] Fusco D., *Atti Sem. Mat. Fis.,* Univ. Modena XXVIII (1979).

[14] Cabannes H., *Theoretical Magneto-Fluid Dynamics,* Applied Math. and Mech. Series, F.N. Frenkiel and G. Temple Editors, Academic Press, London (1970).

[15] Giambó S., Greco A., *Comptes Rendus Acad. Sc. Paris,* t. **285** (1977), Série A, 301-304.

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