A NUMBER-THEORETICAL CONVERGENCE PROOF
OF A POINT APPROXIMATION
OF A SPACE-HOMOGENEOUS TRANSPORT EQUATION

SANTO MOTTA (Catania) - GIOVANNI RUSSO (Catania)
HARDY MOOCK (Kaiserlautern) - JOACHIM WICK (Kaiserslautern)*

We present a point approximation method of a space homogeneous
transport equation. A number-theoretical convergence proof of the method
is given.

1. Introduction.

We consider a one-dimensional, space-homogeneous transport problem
in periodic geometry. This problem arises from a strong simplification
of a model equation for semi-conductors. In the past a drift-diffusion
approach was used in the computation of semiconductors ([11], [13]).
But this model is not satisfactory for the next generation of devices,
since the assumption of thermodynamical equilibrium is no longer true.
A better description for this case can be given by a kinetic model ([14]).
Previous attempts in using kinetic theory to model carrier transport in
semiconductors were based on Monte-Carlo methods ([1], [16]). One of the
major disadvantage of these methods is that they cannot be implemented
for a vectorcomputer. The computing time of the model is then too high
for realistic applications.

* Entrato in Redazione il 20 maggio 1988.
Here we write down the electrostatic case only:

\[(1) \quad \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + E \cdot \frac{\partial f}{\partial k} = \int Q(k, k')\{f(t, x, k') - f(t, x, k)\}dk'
\]

\[v = g(k) \quad \text{div} E = \int f(t, x, k)dk - n(t, x)\]

The l.h.s. is similar to the Vlasov equation in plasmaphysics, which is treated numerically using particles ([2], [6], [9]). The r.h.s. is similar to a collision term in linear transport theory, where completely different methods are used ([5], [7]). Our aim is to extend the particle method to (1). Since the situation for the Vlasov-part is well-known, we have concentrated on handling the collision term. We consider (1) in the space homogeneous case with a more general collision term. For this purpose let \(T_j \simeq [0, 1]^j\) be the \(j\)-dimensional torus, \(P, Q : T_2 \rightarrow \mathbb{R}^+\) uniform Lipschitz-continuous functions, \(t_0 > 0\) and \(f : [0, t_0] \times T_1 \rightarrow \mathbb{R}^+\), the integrable solution of

\[(2) \quad \frac{\partial f}{\partial t}(t, k) = \int_{T_1} P(k, k')f(t, k')dk' - f(t, k)\int_{T_1} Q(k, k')dk',\]

where \(f(0, k) = f_0(k)\) with \(\int_{T_1} f_0(k)dk = 1\) is given.

For \(P \neq Q\) we get the space-homogeneous case of (1).

In addition we require

\[(3) \quad \int_{T_1} P(k, k')dk = \int_{T_1} Q(k', k)dk.\]

Then (2) is a conservation equation

\[\frac{d}{dt} \int_{T_1} f(t, k)dk = \int_{T_2} \int P(k, k')f(t, k')dk'dk - \int_{T_2} \int Q(k, k')f(t, k)dk'dk
\]

\[= \int_{T_2} \int \{P(k, k') - Q(k', k)\}f(t, k')dkdk'
\]

\[= 0.\]
The equation (2) is similar to the master equation in the theory of stochastic processes.

In this context Nanbu [14] has given a scheme, where the path of the particles is described by a stochastic difference equation. Another deterministic approach was given in [12] and [15]. For the implementation of this method on a computer see [4].

The plan of the paper is the following. In § 2 we describe the Euler scheme and we show its convergence. § 3 deals with the point approximation and its convergence. For an easier reading we divided this paragraph in two sections.

The first gives some definitions, that will be used in the proof. The second one describes the point approximation and shows its convergence. Finally § 4 gives some concluding remarks.

2. The Euler-scheme.

With respect to the time $t$ we discretize (2) with the timestep $\Delta t$ and indicate the time-levels by superscripts

$$f^{n+1}(k) = f^n(k)(1 - \Delta t \int_{T_1} Q(k, k')dk') + \Delta t \int_{T_1} P(k, k')f^n(k')dk'.$$

For an estimation of

$$||f^n|| := \int_{T_1} |f^n(k)|dk$$

we define

$$G(k) := \int_{T_1} Q(k, k')dk' = \int_{T_1} P(k', k)dk'.$$

Then holds

$$||f^{n+1}|| \leq \int_{T_1} |1 - \Delta t G(k)||f^n(k)|dk + \Delta t \int_{T_2} |P(k, k')f^n(k')|dk'dk$$

$$\leq ||1 - \Delta t G||_\infty ||f^n|| + \Delta t ||G||_\infty ||f^n||$$

$$\leq ||1 + 2\Delta t G||_\infty ||f^n||.$$
From the last inequality follows

$$\|f^n\| \leq \|1 + 2\Delta t G\|_\infty \|f^0\|$$

$$\leq \|f^0\| \exp(\|1 + 2t_0 G\|_\infty).$$

Now we compare the Euler-solution (4) with the true solution of (2) at $t_n = n\Delta t$.

$$f^{n+1}(k) - f(t_{n+1}, k) = f^n(k) - f(t_n, k) +$$

$$+ \int_{t_n}^{t_{n+1}} \int_{T_1} P(k, k')\{f^n(k') - f(\tau, k')\} dk' d\tau -$$

$$- \int_{t_n}^{t_{n+1}} \int_{T_1} Q(k, k')\{f^n(k) - f(\tau, k)\} dk' d\tau$$

$$\|f^{n+1} - f(t_{n+1})\| \leq \|f^n - f(t_n)\| + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^n(k) - f(\tau, k)| dk d\tau$$

$$\leq \|f^n - f(t_n)\| + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^n(k) - f^{n+1}(k)| dk d\tau +$$

$$+ 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^{n+1}(k) - f(\tau, k)| dk d\tau$$

$$\leq \|f^n - f(t_n)\| + 2\Delta t^2 \int_{T_2} G(k)|P(k, k')f^n(k') -$$

$$- Q(k, k')f^n(k)dk' dk +$$

$$+ 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^{n+1}(k) - f(\tau, k)| dk d\tau$$

$$\leq \|f^n - f(t_n)\| + 2\Delta t^2 \int_{T_1} \left\{ \int_{T_1} (G(k')P(k', k) +$$


\[ + G(k)Q(k, k')dk + \int_{t_n}^{t_{n+1}} |f^n(k)| dk + \]
\[ + 2 \int_{t_n}^{t_{n+1}} ||G||_\infty ||f^{n+1} - f(\tau)|| d\tau \]
\[ \leq ||f^n - f(t_n)|| + 2C||f^n||\Delta t^2 + 2 \int_{t_n}^{t_{n+1}} ||G||_\infty ||f^{n+1} - f(\tau)|| d\tau. \]

Here and in the sequel we use \( \tilde{C}, C, C', \ldots \) to indicate constant factors in the estimations.

From Gronwall's Lemma follows
\[ ||f^{n+1} - f(t_{n+1})|| \leq (||f^n - f(t_n)|| + \tilde{C}\Delta t^2) \exp(2||G||_\infty \Delta t) \]
and finally
\[ ||f^n - f(t_n)|| \leq ||f^0 - f(t_0)|| \exp(C'n\Delta t) + \tilde{C}\Delta t^2 \sum_{\nu=1}^{n} \exp(C'\nu\Delta t) \]
\[ \leq ||f^0 - f(t_0)|| \exp(C't_n) + C\Delta t \exp(C't_n) \]
which shows the convergence of the Euler-Scheme (4).

If \( 1 - \Delta t G(k) > 0 \) the Euler-Scheme conserves the non-negativity and the convergence transfers this to (2).

In this case from property (3) it follows
\[ ||f^n|| = ||f^0|| \]

3. The point-approximation.

Because of the non-negativity and the conservation property we can interpret \( f^n \) as a density function of a probability-measure \( \mu^n \), which can be approximated by a discrete measure. In contrast to the situation in plasma-physics a discrete measure is not preserved by (4), since the gain term \( \int_{T_1} Q(k, k')f^n(k')dk' \) creates a continuous part. Therefore after each time step we must approximate the new measure by an appropriate discrete one.
3.1 Preliminary definitions.

For this construction and to show convergence we introduce the following definitions.

**DEFINITION:** Let

\[ U := \{ \phi : T_1 \to [0, 1] : |\phi(x) - \phi(y)| \leq \min_{j \in \mathbb{Z}} |j + x - y| \} \]

and \( M_1 \) be the set of all probability measures on \( T_1 \).

Then for \( \mu, \nu \in M_1 \) we define

\[ \rho(\mu, \nu) := \sup_{\phi \in U} \left| \int \phi d\mu - \int \phi d\nu \right|. \]

We call a sequence \( \{\nu_N\}_{N \in \mathbb{N}} \subset M_1 \) (weakly) convergent to \( \nu \), if

\[ \lim_{N \to \infty} \rho(\nu_N, \nu) = 0. \]

**Remarks:**

1) \( \rho \) is the well-known bounded Lipschitz distance ([3], [8]) on a torus geometry.

2) Our definition of (weak) convergence coincides with the normal one defined by

\[ \lim_{N \to \infty} \int \phi d\nu_N = \int \phi d\nu \quad \text{for all } \phi \in C^0(T_1). \]

**DEFINITION:** Let be \( \mu, \nu \in M_1 \), then the discrepancy \( D(\mu, \nu) \) is given by

\[ D(\mu, \nu) := \sup_{x \in [0, 1]} \left| \int_0^x d\mu - \int_0^x d\nu \right|. \]

**Remarks:** A.) This is an extension of the concept of discrepancy used in number-theory (compare [9]).

B.) \( D \) does not create the weak convergence in \( M \).

But it holds
THEOREM. [Koksma]: Let \( h : T_1 \to \mathbb{R} \) be of bounded variation \( V(h) \), 
\[
\mu = \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n) \in M_1 \text{ a discrete and } \nu \in M_1 \text{ an absolutely continuous measure, then}
\]
\[
\left| \int_{T_1} h d\mu - \int_{T_1} h d\nu \right| \leq V(h)D(\mu, \nu).
\]

Proof: In [10] this is shown for the equidistribution. The proof can be easily extend. Let be \( P(x) := \int_{0}^{x} \nu(dy) \) and \( x_0 = 0, x_{N+1} = 1 \), then we compute the Stieltjes-integral
\[
\left( \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \left( P(y) - \frac{n}{N} \right) dh(y) \right) = \int_{0}^{1} P(y)dh(y) - \sum_{n=0}^{N} \frac{n}{N} (h(x_{n+1}) - h(x_n))
\]
\[
= P(y)h(y)|_{0}^{1} - \int_{0}^{1} h(y)\nu(dy) + \frac{1}{N} \sum_{n=1}^{N} h(x_n) - h(1)
\]
\[
= \int_{T_1} h d\mu - \int_{T_1} h d\nu.
\]

With a similar modification of the theorem 1.4 in Chapter 2 of [10], we get
\[
D(\mu, \nu) = \max_{1 \leq n \leq N} \max \left( \left| P(x_n) - \frac{n}{N} \right|, \left| P(x_n) - \frac{n-1}{N} \right| \right)
\]
and (6) follow.

COROLLARY: (6) is also true, if \( \nu = \frac{1}{M} \sum_{m=1}^{M} \delta(y - y_m) \). Analogous to (7) we get
\[
\int_{T_1} h d\mu - \int_{T_1} h d\nu = \sum_{m=0}^{M} \sum_{y_m}^{y_{m+1}} \frac{m}{M} dh(y) - \sum_{n=0}^{N} \sum_{x_n}^{x_{n+1}} \frac{n}{N} dh(y)
\]
and from the direct computation of the discrepancy the desired assertion follows.

Since for all $\phi \in U$ the variation $V(\phi) \leq 1$, we get in this case

$$\rho(\mu, \nu) \leq D(\mu, \nu).$$

3.2 The method and its convergence.

Now we can proceed to describe our discretization method. We define with the initial measure $\mu^0$ an approximation

$$\mu^0_N = \frac{1}{N} \sum_{i=1}^{N} \delta(k - k^0_i)$$

by

$$\frac{2i - 1}{2N} = \int_{0}^{k^0_i} d\mu^0 \quad i = 1(1)N.$$  

This guarantees [10]: $D(\mu^0, \mu^0_N) = \frac{1}{2N}$.

Then we define by application of (4)

$$\nu^0_N := \left( 1 - \Delta t \int_{T_1} Q(k, k') dk' \right) \mu^0_N + \Delta t \int_{T_1} P(k, k') \mu^0_N (dk'),$$

which belongs to $M_1$ if $\Delta t$ is small enough, and contains a discrete and an absolutely continuous part. In order to build the approximation of $\nu^0_N$ we define

$$q^0 := \int_{T_2} Q(k, k') dk' \mu^0_N (dk) = \frac{1}{N} \sum_{i=1}^{N} G(k^0_i).$$

From (3) follows

$$\int_{T_2} Q(k, k') dk' \mu^0_N (dk) = \int_{T_2} P(k, k') dk \mu^0_N (dk').$$
Now let

\[(q^0 \Delta t + \xi^0)N =: N_c \in \mathbb{N}, \quad \text{where } |\xi^0| \leq \frac{1}{2N}.\]

Then we choose

\[(P1) \quad \tilde{k}_i^1 = k_j^0 \quad \text{with } j := \left\lfloor \frac{i - N_c}{1 - q^0 \Delta t} \right\rfloor, \quad i = N_c + 1(1)N\]

to form the approximation of the discrete part. $[x]$ indicates here the Gaussian bracket.

For the continuous part we put

\[(P2) \quad \frac{2i - 1}{2N} = \Delta t \int_0^{\tilde{k}_i^1} \int_{T_1} P(k, k') \mu_n^0( dk') dk, \quad i = 1(1)N_c\]

Finally we sort $\tilde{k}_i^1$ in increasing order and call the values $k_i^1$, $i = 1(1)N$. This defines

$$\mu_n^1 := \frac{1}{N} \sum_{i=1}^{N} \delta(k - k_i^1)$$

\textit{Remark:} The $k_i^1$ can be also obtained by a similar order-preserving procedure which is used in the numerical scheme. But in this form we avoid too many indices.

With this algorithm we can proceed and obtain successively

$$\{\mu_n^1\}_{n=0(1)n_0}.$$

Now we wish to estimate $\rho(\mu_n^{n+1}, \mu_n^{n+1})$. From the triangle inequality follows

\[(13) \quad \rho(\mu_n^{n+1}, \mu_n^{n+1}) \leq \rho(\mu_n^{n+1}, \nu_n^n) + \rho(\nu_n^n, \mu_n^{n+1}).\]

Here and in the sequel we extend the definitions (9), (10) and (12) to all time - levels and notice from (11)

\[(14) \quad q^n = \int_{T_2} Q(k, k') dk' \mu_n^0( dk) = \int_{T_2} P(k, k') dk \nu_n^0( dk').\]
We consider both terms of (13) separately

\[ \rho(\mu_{n+1}, \nu^n_N) = \sup_{\phi \in U} \left| \int_{T_1} \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k') dk' \right) (\mu^n(dk) - \mu^n_N(dk)) + \Delta t \int_{T_2} \phi(k) P(k, k')(\mu^n(dk') - \mu^n_N(dk')) dk' \right| \]

\[ = \sup_{\phi \in U} \left| \int_{T_1} \left\{ \phi(k)(1 - \Delta t \int_{T_1} Q(k, k') dk') + \Delta t \int_{T_1} \phi(k') P(k', k) dk' \right\} (\mu^n(dk) - \mu^n_N(dk)) \right|. \]

We define

\[ \psi(k) := \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k') dk' \right) + \Delta t \int_{T_1} \phi(k') P(k', k) dk' \]

\[ A := \max\{\Psi(k), k \in T_1\} \]

\[ B := \max \left( 1, \text{ the smallest global Lipschitz-constant of } \frac{1}{A} \Psi \text{ in } T_1 \right) \]

It follows directly

\[ A \leq (1 + 2\Delta t ||G'||_{\infty}) \]

and form the Lipschitz-continuity of \( P, Q \) and \( \phi \in U \)

\[ |\psi(k_1) - \psi(k_2)| \leq |\phi(k_1) - \phi(k_2)| + \Delta t \phi(k_1) \int_{T_1} |Q(k_1, k') - Q(k_2, k')| dk' + \Delta t |\phi(k_1) - \phi(k_2)| \int_{T_1} Q(k_2, k') dk' + \Delta t \int_{T_1} |\phi(k')| P(k', k_1) - P(k', k_2) dk' \]

\[ \leq |k_1 - k_2| + \Delta t L |k_1 - k_2| + \Delta t ||G'||_{\infty} |k_1 - k_2| + \Delta t L |k_1 - k_2|, \]
where $L$ is the Lipschitz constant of $P$ and $Q$.

Therefore there exists a constant $C$, such that

$$AB \leq 1 + C\Delta t.$$ 

Now, $\frac{1}{AB}\psi \in U$ and

$$\rho(\mu^{n+1}, \nu^{n}_N) \leq \sup_{\phi \in U} AB \left| \int_{T_1} \frac{1}{AB} \psi(k)(\mu^n(dk) - \mu^n_N(dk)) \right|$$

$$\leq (1 + C\Delta t)\rho(\mu^n, \mu^n_N).$$

For the second term of (13) holds

$$\rho(\nu^n_N, \mu^{n+1}_N) = \sup_{\phi \in U} \left| \int_{T_1} \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k')dk' \right) \mu^n_N(dk) +$$

$$+ \Delta t \int_{T_2} \phi(k)P(k, k')\mu^n_N(dk')dk - \frac{1}{N} \sum_{i=1}^{N} \phi(\bar{k}^{n+1}_i) \right|$$

$$\leq \sup_{\phi \in U} \left| \int_{T_1} \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k')dk' \right) \mu^n_N(dk) -$$

$$- \frac{1}{N} \sum_{i=N_c+1}^{N} \phi(\bar{k}^{n+1}_i) \right|$$

$$+ \sup_{\phi \in U} \left| \Delta t \int_{T_2} \phi(k)P(k, k')\mu^n_N(dk')dk -$$

$$- \frac{1}{N} \sum_{i=1}^{N_c} \phi(\bar{k}^{n+1}_i) \right| =: S_1 + S_2.$$

The second term is easy to handle:

$$S_2 = \sup_{\phi \in U} \left| \Delta t \int_{T_2} \phi(k)P(k, k')\mu^n_N(dk')dk - \frac{q^n\Delta t + \xi^n}{N_c} \sum_{i=1}^{N_c} \phi(\bar{k}^{n+1}_i) \right|$$

$$\leq \sup_{\phi \in U} \left| q^n\Delta t \left\{ \int_{T_1} \phi(k) \left( \int_{T_1} \frac{1}{q^n} P(k, k')\mu^n_N(dk') \right) dk - \frac{1}{N_c} \sum_{i=1}^{N_c} \phi(\bar{k}^{n+1}_i) \right\} \right|$$

$$+ \sup_{\phi \in U} \left| \frac{\xi^n}{N_c} \sum_{i=1}^{N_c} \phi(\bar{k}^{n+1}_i) \right|. $$
Let \( \tilde{\mu}^{n+1} := \frac{1}{N_c} \sum_{i=1}^{N_c} \delta(k - \tilde{k}_i^{n+1}) \) and \( \int_{T_1} \frac{1}{q^n} P(k, k') \mu^n_N(dk') \) be the density function of the measure \( \gamma^n \in M_1 \), then it follows from Koksma’s theorem and the explicit calculation of the discrepancy

\[
S_2 \leq \sup_{\phi \in U} |q^n \Delta t V(\phi) D(\tilde{\mu}^{n+1}, \gamma^n)| + \frac{1}{2N} \leq q^n \Delta t \sup_{k \in [0,1]} \left| \int_{k}^{0} \tilde{\mu}^{n+1}(dk') \right| + \frac{1}{2N}
\]

\[
\leq q^n \Delta t \sup_{1 \leq i \leq N_c} \left| \frac{i}{N_c} - \frac{2i - 1}{2Nq^n \Delta t} \right| + \frac{1}{2N}
\]

\[
\leq \frac{q^n \Delta t}{2Nq^n \Delta t} + q^n \Delta t \sup_{1 \leq i \leq N_c} \left| \frac{i}{N_cNq^n \Delta t} (Nq^n \Delta t - N_c) \right| + \frac{1}{2N}
\]

\[
\leq \frac{1}{N} + q^n \Delta t \cdot \frac{|\xi^n|N}{Nq^n \Delta t} \leq \frac{3}{2N}.
\]

With (12) follows

\[ (16) \quad S_2 \leq \frac{3q^n \Delta t}{2N_c - 1}. \]

We call again \( J := \{ j \in N : j = \left[ \frac{i - N_c}{1 - q^n \Delta t} \right], i = N_c + 1(1)N \} \). Then we know

\[ \tilde{k}_i^{n+1} = k_j^n \quad j \in J, i = N_c + 1(1)N. \]

Therefore

\[ (17) \quad S_1 := \sup_{\phi \in U} \left| \int_{T_1} \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k') dk' \right) \mu^n_N(dk') - \frac{1}{N} \sum_{j \in J} \phi(k_j^n) \right| \]

\[ = \sup_{\phi \in U} \left| \frac{1}{N} \sum_{j \notin J} \phi(k_j^n) \left( 1 - \Delta t \int_{T_1} Q(k_j^n, k') dk' \right) - \frac{1}{N} \sum_{j \in J} \phi(k_j^n) \Delta t \int_{T_1} Q(k_j^n, k') dk' \right| \]

\[ \leq \sup_{\phi \in U} \left| \frac{q^n \Delta t + |\xi^n|}{N_c} \sum_{j \notin J} \phi(k_j^n) \left( 1 - \Delta t \int_{T_1} Q(k_j^n, k') dk' \right) - \right| \]
\[ - \frac{1 - q^n \Delta t - \xi^n}{N - N_c} \sum_{j \in J} \phi(k_j^n) \Delta t \int_{T_1} Q(k_j^n, k') dk' \]

\[ \leq \sup_{\phi \in U} \left| \frac{q^n \Delta t}{N_c} \sum_{j \not\in J} \phi(k_j^n) \left( 1 - \Delta t \int_{T_1} Q(k_j^n, k') dk' \right) - \right. \]

\[ - \frac{1 - q^n \Delta t}{N - N_c} \Delta t \sum_{j \in U} \phi(k_j^n) \int_{T_1} Q(k_j^n, k') dk' \]

\[ + \sup_{\phi \in U} |\xi^n| \left| \frac{1}{N_c} \sum_{j \not\in J} \phi(k_j^n) \left( 1 - \Delta t \int_{T_1} Q(k_j^n, k') dk' \right) + \right. \]

\[ + \frac{\Delta t}{N - N_c} \sum_{j \in J} \phi(k_j^n) \int_{T_1} Q(k_j^n, k') dk' \]

\[ \leq R_1 + R_2. \]

For the first term we get

\[ R_1 = \sup_{\phi \in U} \Delta t \left| \int_{T_2} Q(k, k') \mu^n_N(dk') dk. \right. \]

\[ \cdot \frac{1}{N_c} \sum_{j \not\in J} \phi(k_j^n) \left( 1 - \Delta t \int_{T_1} Q(k_j^n, k') dk' \right) - \]

\[ - \left( 1 - \Delta t \int_{T_2} Q(k, k') \mu^n_N(dk') dk \right). \]

\[ \cdot \frac{1}{N - N_c} \sum_{j \in J} \phi(k_j^n) \int_{T_1} Q(k_j^n, k') dk'. \]

Now it holds

\[ N_c \geq N q^n \Delta t - \frac{1}{2} = -\frac{1}{2} + N \Delta t \int_{T_2} Q(k, k') \mu^n_N(dk') dk \]

\[ \geq -\frac{1}{2} + N \Delta t \min\{Q(k, k') : (k, k') \in T_2\}. \]
Since \( \min Q(k, k') > 0 \) we can choose \( N \uparrow \infty \) and \( \Delta t \downarrow 0 \), such that \( N_c \uparrow \infty \). Then
\[
\frac{1}{N} \sum_{j \notin J} \delta(k - k^n_j) \quad \text{and} \quad \frac{1}{N - N_c} \sum_{j \in J} \delta(k - k^n_j)
\]
have the same limit \( \sigma^n \) by construction and from (18) follows in the limit
\[
\begin{align*}
\Delta t \left| \int_{T_2} Q(k, k') \sigma^n dk' \right| \phi(k) \left( 1 - \Delta t \int_{T_1} Q(k, k') dk' \right) \sigma^n dk - \\
\left( 1 - \Delta t \int_{T_2} Q(k, k') \sigma^n dk' \right) \cdot \int_{T_2} \phi(k) Q(k, k') \sigma^n dk' dk' \\
= \Delta t \left| \int_{T_2} Q(k, k') \sigma^n dk' \right| \phi(k) \sigma^n dk - \\
\int_{T_2} \phi(k) Q(k, k') \sigma^n dk dk' \right|
\end{align*}
\]

\( \bar{C} \Delta t \)

and, in general, \( \bar{C} > 0 \). This is not enough to show convergence. However, under the additional assumption

(19) \[
\int_{T_1} Q(k, k') dk' = \int_{T_1} P(k', k) dk' = \text{constant} =: q
\]

we can proceed. Then (18) takes the form
\[
R_1 = \sup_{\phi \in \mathcal{U}} \Delta t q (1 - q \Delta t) \left| \frac{1}{N_c} \sum_{j \notin J} \phi(k^n_j) - \frac{1}{N - N_c} \sum_{j \in J} \phi(k^n_j) \right|
\]

and we estimate the discrepancy \( D \) of these two point sets.

\[
D \leq \max_i \left| \frac{j - i}{N_c} - \frac{i}{N - N_c} \right| + \frac{1}{N_c}
\]

where \( j = \left[ \frac{i}{1 - q \Delta t} \right] \), \( i = 1(1)N - N_c \)
Remark: \( i \) counts the points in \( J \), \( j - i \) the points in \( \{1, \ldots, N\} \setminus J \) until \( i \) and \( \max_i \frac{j - i}{N_c - N} \) calculates the "discrepancy-function" in \( i \in J \). \( \frac{1}{N_c} \) take into account the "discrepancy-function" in the other points.

\[
\frac{i}{N - N_c} - \frac{j - i}{N_c} = i \left( \frac{1}{N - N_c} - \frac{q \Delta t}{1 - q \Delta t} \cdot \frac{1}{N_c} \right) + \frac{\eta}{N_c} \quad \eta \in [0, 1]
\]

\[
= i \left( N^{-1} (1 - q \Delta t - \xi)^{-1} - N^{-1} (1 - q \Delta t)^{-1} \left( 1 + \frac{\xi}{q \Delta t} \right)^{-1} + \frac{\eta}{N_c} \right)
\]

\[
= \frac{i}{N} \cdot \frac{\xi}{q \Delta t} (1 - q \Delta t - \xi)^{-1} (1 - q \Delta t)^{-1} \left( 1 + \frac{\xi}{q \Delta t} \right)^{-1} + \frac{\eta}{N_c}.
\]

If \( N_c \) is large enough, then \((1 - q \Delta t) \left( 1 + \frac{\xi}{q \Delta t} \right) < 1 \) since \(|\xi| \leq \frac{1}{2N_c}\)

\[
\frac{i}{N - N_c} - \frac{j - i}{N_c} < \frac{2i|\xi|}{N q \Delta t} + \frac{\eta}{N_c} < \frac{2i|\xi|}{q \Delta t} \cdot \frac{q \Delta t}{N_c - \frac{1}{2}} + \frac{\eta}{N_c}
\]

\[
= \frac{1}{N} \frac{N - N_c}{N_c - \frac{1}{2}} + \frac{\eta}{N_c} < \frac{1}{N_c - \frac{1}{2}} + \frac{1}{N_c - \frac{1}{2}} < \frac{4}{2N_c - 1}.
\]

And therefore

\[
D < \frac{6}{2N_c - 1}
\]

which gives

\[
R_1 \leq \Delta t q (1 - \Delta t q) \cdot \frac{6}{2N_c - 1}.
\]

For \( R_2 \) we have

\[
R_2 \leq \sup_{\phi \in U} \frac{1}{2N} \left| \frac{1}{N_c} \sum_{j \not\in U} (1 - q \Delta t) \phi(k_j) + \frac{q \Delta t}{N - N_c} \sum_{j \in J} \phi(k_j) \right| \leq \frac{1}{2N}
\]

which leads to

\[
S_1 \leq q \Delta t (1 - q \Delta t) \frac{6}{2N_c - 1} + \frac{q \Delta t}{4N_c - 2} < q \Delta t \frac{13}{4N_c - 2}
\]
and finally

\[(20) \quad \rho(\nu_N^n, \mu_N^{n+1}) \leq S_1 + S_2 < \frac{10q \Delta t}{2N_c - 1}.\]

Together with (15) we have now

\[\rho(\mu^{n+1}, \mu_N^{n+1}) \leq (1 + C\Delta t)\rho(\mu^n, \mu_N^n) + \frac{10q \Delta t}{2N_c - 1} \sum_{\nu=0}^n (1 + C\Delta t)^\nu.\]

and by iteration

\[\rho(\mu^{n+1}, \mu_N^{n+1}) \leq (1 + C\Delta t)^n \rho(\mu^0, \mu_N^0) + \frac{10q \Delta t}{2N_c - 1} \frac{(1 + C\Delta t)^{n+1} - 1}{C\Delta t} \]

Let be \(t = n\Delta t, \quad t \in [0, t_0],\) then

\[\rho(\mu^n, \mu_N^n) \leq (1 + C\Delta t)^n \rho(\mu^0, \mu_N^0) + \frac{10q \Delta t}{2N_c - 1} \frac{(1 + C\Delta t)^{n+1} - 1}{C\Delta t}.\]

Since \((1 + C\Delta t)^n \leq \exp(Ct) \leq \exp(Ct_0),\) we showed convergence, if \(N_c \to \infty.\) We summarize this in the

**THEOREM:** Let be \(T_j \simeq [0, 1]^j\) the \(j\)-dimensional torus, \(P, Q : T_2 \to \mathbb{R}^+\) uniform Lipschitz-continuous functions with

\[\int_{T_1} P(k, k') dk = \int_{T_1} Q(k, k') dk' = q.\]

Then the approximation \(\mu_N^n\) defined by (P1), (P2) converges to \(\mu^n,\) where

\[\mu^n(dk) = f(t, k)dk\]

and \(f\) is the solution of

\[\frac{\partial f}{\partial t} = \int_{T_1} P(k, k') f(t, k') dk' - f(t, k) \int_{T_1} Q(k, k') dk'\]

\[f(0, k) = f_0(k) \geq 0, \quad \int_{T_1} f_0(k) dk = 1.\]
for every finite time-interval, if $N\Delta t \to \infty$ and $\Delta t \to 0$, provided
\[
\lim_{N \to 0} \mu_{N}^{0} = \mu^{0}.
\]

4. Conclusion.

We presented a number-theoretical convergence proof of a point approximation of a space homogeneous transport equation. Computer implementation seems promising but still needs some refinements. The extension of the method to the space inhomogeneous case is still under investigation and results will be published in due course.

Acknowledgements.

This paper has been done under the auspices of a cooperation agreement between the University of Catania and the University of Kaiserslautern.

We acknowledge also the financial support to the project given by the Italian M.P.I. and the CNR-GNFM.

REFERENCES


Santo Motta and Giovanni Russo
Dipartimento di Matematica
Università di Catania
V.le A. Doria, 6 - 95125 Catania (Italy)

Hardy Moock and Joachim Wick
Universität Kaiserslautern
Fachbereich Mathematik
Erwin Schrödinger Straße
D-6750 Kaiserslautern (W-Germany)