A NUMBER-THEORETICAL CONVERGENCE PROOF OF A POINT APPROXIMATION OF A SPACE-HOMOGENEOUS TRANSPORT EQUATION

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We present a point approximation method of a space homogeneous transport equation. A number-theoretical convergence proof of the method is given.

1. Introduction.

We consider a one-dimensional, space-homogeneous transport problem in periodic geometry. This problem arises from a strong simplification of a model equation for semi-conductors. In the past a drift-diffusion approach was used in the computation of semiconductors ([11], [13]). But this model is not satisfactory for the next generation of devices, since the assumption of thermodynamical equilibrium is no longer true. A better description for this case can be given by a kinetic model ([14]). Previous attempts in using kinetic theory to model carrier transport in semiconductors were based on Monte-Carlo methods ([1], [16]). One of the major disadvantage of these methods is that they cannot be implemented for a vectorcomputer. The computing time of the model is then too high for realistic applications.

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Here we write down the electrostatic case only:

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + E \cdot \frac{\partial f}{\partial k} = \int Q(k,k')\{f(t,x,k') - f(t,x,k)\}dk' \tag{1}$$

$$v = g(k) \qquad \operatorname{div} E = \int f(t,x,k)dk - n(t,x)$$

The l.h.s. is similar to the Vlasov equation in plasmaphysics, which is treated numerically using particles ([2], [6], [9]). The r.h.s. is similar to a collision term in linear transport theory, where completely different methods are used ([5], [7]). Our aim is to extend the particle method to (1). Since the situation for the Vlasov-part is well-known, we have concentrated on handling the collision term. We consider (1) in the space homogeneous case with a more general collision term. For this purpose let $T_j \simeq [0,1]^j$ be the j-dimensional torus, $P, Q : T_2 \longrightarrow \mathbb{R}^+$ uniform Lipschitz-continuous functions, $t_0 > 0$ and $f : [0,t_0] \times T_1 \longrightarrow \mathbb{R}^+$, the integrable solution of

$$\frac{\partial f}{\partial t}(t,k) = \int_{T_1} P(k,k')f(t,k')dk' - f(t,k)\int_{T_1} Q(k,k')dk', \tag{2}$$

where $f(0,k) = f_0(k)$ with $\int_{T_1} f_0(k)dk = 1$ is given.

For $P = Q$ we get the space-homogeneous case of (1).

In addition we require

$$\int_{T_1} P(k,k')dk = \int_{T_1} Q(k',k)dk. \tag{3}$$

Then (2) is a conservation equation

$$\begin{aligned}\frac{d}{dt}\int_{T_1} f(t,k)dk &= \int_{T_2}\int P(k,k')f(t,k')dk'dk - \int_{T_2}\int Q(k,k')f(t,k)dk'dk\\ &= \int_{T_2}\int \{P(k,k') - Q(k',k)\}f(t,k')dkdk'\\ &= 0.\end{aligned}$$

The equation (2) is similar to the master equation in the theory of stochastic processes.

In this context Nanbu [14] has given a scheme, where the path of the particles is described by a stochastic difference equation. Another deterministic approach was given in [12] and [15]. For the implementation of this method on a computer see [4].

The plan of the paper is the following. In § 2 we describe the Euler scheme and we show its convergence. § 3 deals with the point approximation and its convergence. For an easier reading we divided this paragraph in two sections.

The first gives some definitions, that will be used in the proof. The second one describes the point approximation and shows its convergence. Finally § 4 gives some concluding remarks.

2. The Euler-scheme.

With respect to the time t we discretize (2) with the timestep Δt and indicate the time-levels by superscripts

$$f^{n+1}(k) = f^n(k)(1 - \Delta t \int_{T_1} Q(k,k')dk') + \Delta t \int_{T_1} P(k,k') f^n(k')dk'. \tag{4}$$

For an estimation of

$$||f^n|| := \int_{T_1} |f^n(k)|dk$$

we define

$$G(k) := \int_{T_1} Q(k,k')dk' = \int_{T_1} P(k',k)dk'. \tag{5}$$

Then holds

$$\begin{aligned}
||f^{n+1}|| &\leq \int_{T_1} |1 - \Delta t G(k)||f^n(k)|dk + \Delta t \int_{T_2} |P(k,k')f^n(k')|dk'dk \\
&\leq ||1 - \Delta t G||_\infty ||f^n|| + \Delta t ||G||_\infty ||f^n|| \\
&\leq ||1 + 2\Delta t G||_\infty \, ||f^n||.
\end{aligned}$$

From the last inequality follows

$$||f^n|| \leq ||1 + 2\Delta t G||_\infty^n \, ||f^0||$$

$$\leq ||f^0|| \exp(||1 + 2t_0 G||_\infty).$$

Now we compare the Euler-solution (4) with the true solution of (2) at $t_n = n\Delta t$.

$$\begin{aligned} f^{n+1}(k) - f(t_{n+1}, k) = & f^n(k) - f(t_n, k) + \\ & + \int_{t_n}^{t_{n+1}} \int_{T_1} P(k, k')\{f^n(k') - f(\tau, k')\} dk' d\tau - \\ & - \int_{t_n}^{t_{n+1}} \int_{T_1} Q(k, k')\{f^n(k) - f(\tau, k)\} dk' d\tau \end{aligned}$$

$$\begin{aligned} ||f^{n+1} - f(t_{n+1})|| \leq & ||f^n - f(t_n)|| + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^n(k) - f(\tau, k)| dk d\tau \\ \leq & ||f^n - f(t_n)|| + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^n(k) - f^{n+1}(k)| dk d\tau + \\ & + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^{n+1}(k) - f(\tau, k)| dk d\tau \\ \leq & ||f^n - f(t_n)|| + 2\Delta t^2 \int_{T_2} G(k)|P(k, k')f^n(k') - \\ & - Q(k, k')f^n(k)| dk' dk + \\ & + 2\int_{t_n}^{t_{n+1}} \int_{T_1} G(k)|f^{n+1}(k) - f(\tau, k)| dk d\tau \\ \leq & ||f^n - f(t_n)|| + 2\Delta t^2 \int_{T_1} \left\{ \int_{T_1} (G(k')P(k', k) + \right. \end{aligned}$$

$$+ G(k)Q(k,k'))dk' \Big\} |f^n(k)|dk +$$

$$+ 2\int_{t_n}^{t_{n+1}} ||G||_\infty \, ||f^{n+1} - f(\tau)||d\tau$$

$$\leq ||f^n - f(t_n)|| + 2C||f^n||\Delta t^2 + 2\int_{t_n}^{t_{n+1}} ||G||_\infty \, ||f^{n+1} - f(\tau)||d\tau.$$

Here and in the sequal we use $\tilde{C}$, C, C', ... to indicate constant factors in the estimations.

From Gronwall's Lemma follows

$$||f^{n+1} - f(t_{n+1})|| \leq (||f^n - f(t_n)|| + \tilde{C}\Delta t^2) \exp(2||G||_\infty \Delta t)$$

and finally

$$||f^n - f(t_n)|| \leq ||f^0 - f(t_0)|| \exp(C' n\Delta t) + \tilde{C}\Delta t^2 \sum_{\nu=1}^{n} \exp(C'\nu\Delta t)$$

$$\leq ||f^0 - f(t_0)|| \exp(C' t_n) + C\Delta t \exp(C' t_n)$$

which shows the convergence of the Euler-Scheme (4).

If $1 - \Delta t G(k) > 0$ the Euler-Scheme conserves the non-negativity and the convergence transfers this to (2).

In this case from property (3) it follows

$$||f^n|| = ||f^0||$$

3. The point-approximation.

Because of the non-negativity and the conservation property we can interpret f^n as a density function of a probability-measure μ^n, which can be approximated by a discrete measure. In contrast to the situation in plasma-physics a discrete measure is not preserved by (4), since the gain term $\int_{T_1} Q(k,k')f^n(k')dk'$ creates a continuous part. Therefore after each time step we must approximate the new measure by an appropriate discrete one.

3.1 Preliminary definitions.

For this construction and to show convergence we introduce the following definitions.

DEFINITION: *Let*

$$U := \{\phi : T_1 \to [0,1] : |\phi(x) - \phi(y)| \leq \min_{j \in \mathbb{Z}}(|j + x - y|)\}$$

and M_1 be the set of all probability measures on T_1.

Then for $\mu, \nu \in M_1$ we define

$$\rho(\mu, \nu) := \sup_{\phi \in U} \left| \int \phi d\mu - \int \phi d\nu \right| .$$

We call a sequence $\{\nu_N\}_{N \in \mathbb{N}} \subset M_1$ (weakly) convergent to ν, if $\lim_{N \to \infty} \rho(\nu_N, \nu) = 0$.

Remarks:

1) ρ is the well-known bounded Lipschitz distance ([3], [8]) on a torus geometry.
2) Our definition of (weak) converge coincides with the normal one defined by

$$\lim_{N \to \infty} \int \phi d\nu_N = \int \phi d\nu \qquad \text{for all } \phi \in C^0(T_1).$$

DEFINITION: *Let be μ, $\nu \in M_1$, then the discrepancy $D(\mu, \nu)$ is given by*

$$D(\mu, \nu) := \sup_{x \in [0,1]} \left| \int_0^x d\mu - \int_0^x d\nu \right| .$$

Remarks: A.) This is an extension of the concept of discrepancy used in number-theory (compare [9]).

B.) D does not create the weak convergence in M.

But it holds

THEOREM. [Koksma]: *Let* $h : T_1 \to \mathbb{R}$ *be of bounded variation* $V(h)$, $\mu = \frac{1}{N}\sum_{n=1}^{N} \delta(x - x_n) \in M_1$ *a discrete and* $\nu \in M_1$ *an absolutely continuous measure, then*

$$\left| \int_{T_1} h d\mu - \int_{T_1} h d\nu \right| \leq V(h) D(\mu, \nu). \tag{6}$$

Proof: In [10] this is shown for the equidistribution. The proof can be easily extend. Let be $P(x) := \int_0^x \nu(dy)$ and $x_0 = 0$, $x_{N+1} = 1$, then we compute the Stieltjes-integral

$$\begin{aligned}
(7)\ \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \left(P(y) - \frac{n}{N} \right) dh(y) &= \int_0^1 P(y) dh(y) - \sum_{n=0}^{N} \frac{n}{N} (h(x_{n+1}) - h(x_n)) \\
&= P(y)h(y)|_0^1 - \int_0^1 h(y)\nu(dy) + \\
&\quad + \frac{1}{N} \sum_{n=1}^{N} h(x_n) - h(1) \\
&= \int_{T_1} h d\mu - \int_{T_1} h d\nu.
\end{aligned}$$

With a similar modification of the theorem 1.4 in Chapiter 2 of [10], we get

$$D(\mu, \nu) = \max_{1 \leq n \leq N} \max \left(\left| P(x_n) - \frac{n}{N} \right|, \left| P(x_n) - \frac{n-1}{N} \right| \right)$$

and (6) foolows.

COROLLARY: *(6) is also true, if* $\nu = \frac{1}{M} \sum_{m=1}^{M} \delta(y - y_m)$. *Analogous to (7) we get*

$$\int_{T_1} h d\mu - \int_{T_1} h d\nu = \sum_{m=0}^{M} \int_{y_m}^{y_{m+1}} \frac{m}{M} dh(y) - \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \frac{n}{N} dh(y)$$

and from the direct computation of the discrepancy the desired assertion follows.

Since for all $\phi \in U$ the variation $V(\phi) \leq 1$, we get in this case

$$\rho(\mu, \nu) \leq D(\mu, \nu).$$

3.2 The method and its convergence.

Now we can proceed to describe our discretization method.
We define with the initial measure μ^0 an approximation

$$\mu_N^0 = \frac{1}{N} \sum_{i=1}^{N} \delta(k - k_i^0)$$

by

$$\frac{2i-1}{2N} = \int_0^{k_i^0} d\mu^0 \qquad i = 1(1)N.$$

This guarantees [10]: $D(\mu^0, \mu_N^0) = \dfrac{1}{2N}$.
Then we define by application of (4)

$$\text{(9)} \qquad \nu_N^0 := \left(1 - \Delta t \int_{T_1} Q(k, k')dk'\right) \mu_N^0 + \Delta t \int_{T_1} P(k, k')\mu_N^0(dk'),$$

Which belongs to M_1 if Δt is small enough, and contains a discrete and an absolutely continuous part. In order to build the approximation of ν_N^0 we define

$$\text{(10)} \qquad q^0 := \int_{T_2} Q(k, k')dk'\mu_N^0(dk) = \frac{1}{N} \sum_{i=1}^{N} G(k_i^0).$$

From (3) follows

$$\text{(11)} \qquad \int_{T_2} Q(k, k')dk'\mu_N^0(dk) = \int_{T_2} P(k, k')dk\mu_N^0(dk').$$

Now let

$$(12) \qquad (q^0\Delta t + \xi^0)N =: N_c \in \mathbb{N}, \qquad \text{where } |\xi^0| \le \frac{1}{2N}.$$

Then we choose

$$(P1) \qquad \tilde{k}_i^1 = k_j^0 \quad \text{with} \quad j := \left[\frac{i - N_c}{1 - q^0\Delta t}\right], \qquad i = N_c + 1(1)N$$

to form the approximation of the discrete part. $[x]$ indicates here the Gaussian bracket.

For the continuous part we put

$$(P2) \qquad \frac{2i-1}{2N} = \Delta t \int_0^{\tilde{k}_i^1} \int_{T_1} P(k,k')\mu_N^0(dk')dk, \qquad i = 1(1)N_c$$

Finally we sort $\tilde{k}_i^1$ in increasing order and call the values k_i^1, $i = 1(1)N$. This defines

$$\mu_N^1 := \frac{1}{N}\sum_{i=1}^{N} \delta(k - k_i^1)$$

Remark: The k_i^1 can be also obtained by a similar order-preserving procedure which is used in the numerical scheme. But in this form we avoid too many indices.

With this algorithm we can proceed and obtain successively

$$\{\mu_N^n\}_{n=0(1)n_0}.$$

Now we wish to estimate $\rho(\mu^{n+1}, \mu_N^{n+1})$. From the triangle inequality follows

$$(13) \qquad \rho(\mu^{n+1}, \mu_N^{n+1}) \le \rho(\mu^{n+1}, \nu_N^n) + \rho(\nu_N^n, \mu_N^{n+1}).$$

Here and in the sequal we extend the definitions (9), (10) and (12) to all time - levels and notice from (11)

$$(14) \qquad q^n = \int_{T_2} Q(k,k')dk'\mu_N^n(dk) = \int_{T_2} P(k,k')dk\mu_N^n(dk').$$

We consider both terms of (13) separately

$$\rho(\mu^{n+1}, \nu_N^n) = \sup_{\phi\in U} \left| \int_{T_1} \phi(k) \left(1 - \Delta t \int_{T_1} Q(k,k')dk\right) \left(\mu^n(dk) - \mu_N^n(dk)\right) \right.$$
$$\left. + \Delta t \int_{T_2} \phi(k) P(k,k')(\mu^n(dk') - \mu_N^n(dk'))dk \right|$$
$$= \sup_{\phi\in U} \left| \int_{T_1} \left\{ \phi(k)(1 - \Delta t \int_{T_1} Q(k,k')dk') + \right.\right.$$
$$\left.\left. + \Delta t \int_{T_1} \phi(k') P(k',k) dk' \right\} (\mu^n(dk) - \mu_N^n(dk)) \right|.$$

We define

$$\psi(k) := \phi(k) \left(1 - \Delta t \int_{T_1} Q(k,k')dk'\right) + \Delta t \int_{T_1} \phi(k')P(k',k)dk'$$

$$A := \max\{\Psi(k), k \in T_1\}$$

$$B := \max\left(1, \text{ the smallest global Lipschitz-constant of } \frac{1}{A}\Psi \text{ in } T_1\right)$$

It follows directly

$$A \leq (1 + 2\Delta t ||G||_\infty)$$

and form the Lipschitz-continuity of P, Q and $\phi \in U$

$$|\psi(k_1) - \psi(k_2)| \leq |\phi(k_1) - \phi(k_2)| + \Delta t \phi(k_1) \int_{T_1} |Q(k_1,k') -$$
$$- Q(k_2,k')|dk' + \Delta t |\phi(k_1) - \phi(k_2)| \int_{T_1} Q(k_2,k')dk' +$$
$$+ \Delta t \int_{T_1} \phi(k') |P(k',k_1) - P(k',k_2)| dk'$$
$$\leq |k_1 - k_2| + \Delta t L |k_1 - k_2| + \Delta t ||G||_\infty |k_1 - k_2| + \Delta t L |k_1 - k_2|,$$

where L is the Lipschitz constant of P and Q.

Therefore there exists a constant C, such that

$$AB \leq 1 + C\Delta t.$$

Now, $\frac{1}{AB}\psi \in U$ and

$$\rho(\mu^{n+1}, \nu_N^n) \leq \sup_{\phi\in U} AB \left| \int_{T_1} \frac{1}{AB}\psi(k)(\mu^n(dk) - \mu_N^n(dk)) \right| \tag{15}$$
$$\leq (1 + C\Delta t)\rho(\mu^n, \mu_N^n).$$

For the second term of (13) holds

$$\rho(\nu_N^n, \mu_N^{n+1}) = \sup_{\phi\in U} \left| \int_{T_1} \phi(k) \left(1 - \Delta t \int_{T_1} Q(k,k')dk'\right) \mu_N^n(dk) + \right.$$
$$\left. +\Delta t \int_{T_2} \phi(k)P(k,k')\mu_N^n(dk')dk - \frac{1}{N}\sum_{i=1}^{N} \phi(\tilde{k}_i^{n+1}) \right|$$
$$\leq \sup_{\phi\in U} \left| \int_{T_1} \phi(k) \left(1 - \Delta t \int_{T_1} Q(k,k')dk'\right) \mu_N^n(dk) - \right.$$
$$\left. -\frac{1}{N} \sum_{i=N_c+1}^{N} \phi(\tilde{k}_i^{n+1}) \right|$$
$$+ \sup_{\phi\in U} \left| \Delta t \int_{T_2} \phi(k)P(k,k')\mu_N^n(dk')dk - \right.$$
$$\left. -\frac{1}{N}\sum_{i=1}^{N_c} \phi(\tilde{k}_i^{n+1}) \right| =: S_1 + S_2.$$

The second term is easy to handle:

$$S_2 := \sup_{\phi\in U} \left| \Delta t \int_{T_2} \phi(k)P(k,k')\mu_N^n(dk')dk - \frac{q_n\Delta t + \xi^n}{N_c} \sum_{i=1}^{N_c} \phi(\tilde{k}_i^{n+1}) \right|$$
$$\leq \sup_{\phi\in U} \left| q^n\Delta t \left\{ \int_{T_1} \phi(k) \left(\int_{T_1} \frac{1}{q^n} P(k,k')\mu_N^n(dk') \right) dk - \frac{1}{N_c}\sum_{i=1}^{N_c} \phi(\tilde{k}_i^{n+1}) \right\} \right|$$
$$+ \sup_{\phi\in U} \left| \frac{\xi^n}{N_c} \sum_{i=1}^{N_c} \phi(\tilde{k}_i^{n+1}) \right|.$$

Let $\tilde{\mu}^{n+1} := \frac{1}{N_c}\sum_{i=1}^{N_c}\delta(k-\tilde{k}_i^{n+1})$ and $\int_{T_1}\frac{1}{q^n}P(k,k')\mu_N^n(dk')$ be the density function of the measure $\gamma^n \in M_1$, then it follows from Koksma's theorem and the explicit calculation of the discrepancy

$$
\begin{aligned}
S_2 &\leq \sup_{\phi\in U}|q^n\Delta t V(\phi)D(\tilde{\mu}^{n+1},\gamma^n)| + \frac{1}{2N} \leq q^n\Delta t \sup_{k\in[0,1]}\left|\int_k^0 \tilde{\mu}^{n+1}(dk')\right| + \frac{1}{2N} \\
&\leq q^n\Delta t \sup_{1\leq i\leq N_c}\left|\frac{i}{N_c} - \frac{2i-1}{2Nq^n\Delta t}\right| + \frac{1}{2N} \\
&\leq \frac{q^n\Delta t}{2Nq^n\Delta t} + q^n\Delta t \sup_{1\leq i\leq N_c}\left|\frac{i}{N_cNq^n\Delta t}(Nq^n\Delta t - N_c)\right| + \frac{1}{2N} \\
&\leq \frac{1}{N} + q^n\Delta t\cdot\frac{|\xi^n|N}{Nq^n\Delta t} \leq \frac{3}{2N}.
\end{aligned}
$$

With (12) follows

$$
S_2 \leq \frac{3q^n\Delta t}{2N_c - 1}. \tag{16}
$$

We call again $J := \left\{j \in N : j = \left[\frac{i-N_c}{1-q^n\Delta t}\right], i = N_c + 1(1)N\right\}$. Then we know

$$
\tilde{k}_i^{n+1} = k_j^n \qquad j \in J, i = N_c + 1(1)N.
$$

Therefore

$$
\begin{aligned}
(17)\quad S_1 :&= \sup_{\phi\in N}\left|\int_{T_1}\phi(k)\left(1 - \Delta t\int_{T_1}Q(k,k')dk'\right)\mu_N^n(dk') - \frac{1}{N}\sum_{j\in J}\phi(k_j^n)\right| \\
&= \sup_{\phi\in U}\left|\frac{1}{N}\sum_{j\notin J}\phi(k_j^n)\left(1 - \Delta t\int_{T_1}Q(k_j^n,k')dk'\right) - \right. \\
&\qquad \left. -\frac{1}{N}\sum_{j\in J}\phi(k_j^n)\Delta t\int_{T_1}Q(k_j^n,k')dk'\right| \\
&\leq \sup_{\phi\in U}\left|\frac{q^n\Delta t + \xi^n}{N_c}\sum_{j\notin J}\phi(k_j^n)\left(1 - \Delta t\int_{T_1}Q(k_j^n,k')dk'\right) - \right.
\end{aligned}
$$

$$-\frac{1-q^n\Delta t-\xi^n}{N-N_c}\sum_{j\in J}\phi(k_j^n)\Delta t\int_{T_1}Q(k_j^n,k')dk'\Bigg|$$

$$\leq \sup_{\phi\in U}\Bigg|\frac{q^n\Delta t}{N_c}\sum_{j\notin J}\phi(k_j^n)\left(1-\Delta t\int_{T_1}Q(k_j^n,k')dk'\right)-$$

$$-\frac{1-q^n\Delta t}{N-N_c}\Delta t\sum_{j\in U}\phi(k_j^n)\int_{T_1}Q(k_j^n,k')dk'\Bigg|$$

$$+\sup_{\phi\in U}|\xi^n|\Bigg|\frac{1}{N_c}\sum_{j\notin J}\phi(k_j^n)\left(1-\Delta t\int_{T_1}Q(k_j^n,k')dk'\right)+$$

$$+\frac{\Delta t}{N-N_c}\sum_{j\in J}\phi(k_j^n)\int_{T_1}Q(k_j^n,k')dk'\Bigg|$$

$$\leq R_1+R_2.$$

For the first term we get

$$R_1=\sup_{\phi\in U}\Delta t\Bigg|\int_{T_2}Q(k,k')\mu_N^n(dk')dk\cdot \tag{18}$$

$$\cdot\frac{1}{N_c}\sum_{j\notin J}\phi(k_j^n)\left(1-\Delta t\int_{T_1}Q(k_j^n,k')dk'\right)-$$

$$-\left(1-\Delta t\int_{T_2}Q(k,k')\mu_N^n(dk')dk\right)\cdot$$

$$\cdot\frac{1}{N-N_c}\sum_{j\in J}\phi(k_j^n)\int_{T_1}Q(k_j^n,k')dk'\Bigg|.$$

Now it holds

$$N_c\geq Nq^n\Delta t-\frac{1}{2}=-\frac{1}{2}+N\Delta t\int_{T_2}Q(k,k')\mu_N^n(dk')dk$$

$$\geq -\frac{1}{2}+N\Delta t\min\{Q(k,k'):(k,k')\in T_2\}.$$

Since $\min Q(k,k') > 0$ we can choose $N \uparrow \infty$ and $\Delta t \downarrow 0$, such that $N_c \uparrow \infty$. Then

$$\frac{1}{N}\sum_{j\notin J}\delta(k-k_j^n) \qquad \text{and} \qquad \frac{1}{N-N_c}\sum_{j\in J}\delta(k-k_j^n)$$

have the same limit σ^n by construction and from (18) follows in the limit

$$\begin{aligned}
\Delta t \Bigg| &\int_{T_2} Q(k,k')\sigma^n(dk')dk \int_{T_1}\phi(k)\left(1-\Delta t\int_{T_1}Q(k,k')dk'\right)\sigma^n(dk)-\\
&-\left(1-\Delta t\int_{T_2}Q(k,k')\sigma^n(dk')dk\right)\cdot\int_{T_2}\phi(k)Q(k,k')\sigma^n(dk)dk'\Bigg|\\
=\Delta t\Bigg|&\int_{T_2}Q(k,k')\sigma^n(dk')dk\int_{T_1}\phi(k)\sigma^n(dk)-\\
&-\int_{T_2}\phi(k)Q(k,k')\sigma^n(dk)dk'\Bigg|\\
=\tilde{C}\Delta t&
\end{aligned}$$

and, in general, $\tilde{C} > 0$. This is not enough to show convergence. However, under the additional assumption

$$\int_{T_1} Q(k,k')dk' = \int_{T_1} P(k',k)dk' = \text{constant} =: q \tag{19}$$

we can proceed. Then (18) takes the form

$$R_1 = \sup_{\phi\in U}\Delta t q(1-q\Delta t)\left|\frac{1}{N_c}\sum_{j\notin J}\phi(k_j^n) - \frac{1}{N-N_c}\sum_{j\in J}\phi(k_j^n)\right|$$

and we estimate the discrepany D of these two point sets.

$$D \le \max_i\left|\frac{j-i}{N_c} - \frac{i}{N-N_c}\right| + \frac{1}{N_c},$$

$$\text{where} \quad j = \left[\frac{i}{1-q\Delta t}\right], \quad i = 1(1)N-N_c$$

Remark: i counts the points in J, $j-i$ the points in $\{1,\dots,N\}\backslash J$ until i and $\max_i \left|\frac{j-i}{N_c} - \frac{i}{N-N_c}\right|$ calculates the "discrepancy-function" in $i \in J$. $\frac{1}{N_c}$ take into account the "discrepancy-function" in the other points.

$$\frac{i}{N-N_c} - \frac{j-i}{N_c} = i\left(\frac{1}{N-N_c} - \frac{q\Delta t}{1-q\Delta t}\cdot\frac{1}{N_c}\right) + \frac{\eta}{N_c} \qquad \eta \in [0,1]$$

$$= i\left(N^{-1}(1-q\Delta t-\xi)^{-1} - N^{-1}(1-q\Delta t)^{-1}\left(1+\frac{\xi}{q\Delta t}\right)^{-1} + \frac{\eta}{N_c}\right)$$

$$= \frac{i}{N}\cdot\frac{\xi}{q\Delta t}(1-q\Delta t-\xi)^{-1}(1-q\Delta t)^{-1}\left(1+\frac{\xi}{q\Delta t}\right)^{-1} + \frac{\eta}{N_c}.$$

If N_c is large enough, then $(1-q\Delta t)\left(1+\frac{\xi}{q\Delta t}\right) < 1$ since $|\xi| \leq \frac{1}{2N}$

$$\frac{i}{N-N_c} - \frac{j-i}{N_c} < \frac{2i|\xi|}{Nq\Delta t} + \frac{\eta}{N_c} < \frac{2i|\xi|}{q\Delta t}\cdot\frac{q\Delta t}{N_c-\frac{1}{2}} + \frac{\eta}{N_c}$$

$$< \frac{1}{N}\frac{N-N_c}{N_c-\frac{1}{2}} + \frac{\eta}{N_c} < \frac{1-\frac{N}{N_c}}{N_c-\frac{1}{2}} + \frac{1}{N_c-\frac{1}{2}} < \frac{4}{2N_c-1}.$$

And therefore

$$D < \frac{6}{2N_c-1}$$

which gives

$$R_1 \leq \Delta tq(1-\Delta tq)\cdot\frac{6}{2N_c-1}.$$

For R_2 we have

$$R_2 \leq \sup_{\phi\in U}\frac{1}{2N}\left|\frac{1}{N_c}\sum_{j\notin U}(1-q\Delta t)\phi(k_j^n) + \frac{q\Delta t}{N-N_c}\sum_{j\in J}\phi(k_j^n)\right| \leq \frac{1}{2N}$$

which leads to

$$S_1 \leq q\Delta t(1-q\Delta t)\frac{6}{2N_c-1} + \frac{q\Delta t}{4N_c-2} < q\Delta t\frac{13}{4N_c-2}$$

and finally

$$\rho(\nu_N^n, \mu_N^{n+1}) \leq S_1 + S_2 < \frac{10q\Delta t}{2N_c - 1}. \tag{20}$$

Together with (15) we have now

$$\rho(\mu^{n+1}, \mu_N^{n+1}) \leq (1 + C\Delta t)\rho(\mu^n, \mu_N^n) + \frac{10q\Delta t}{2N_c - 1}$$

and by iteration

$$\rho(\mu^{n+1}, \mu_N^{n+1}) \leq (1 + C\Delta t)^{n+1}\rho(\mu^0, \mu_N^0) + \frac{10q\Delta t}{2N_c - 1}\sum_{\nu=0}^{n}(1 + C\Delta t)^\nu.$$

Let be $t = n\Delta t$, $t \in [0, t_0]$, then

$$\rho(\mu^n, \mu_N^n) \leq (1 + C\Delta t)^n\rho(\mu^0, \mu_N^0) + \frac{10q\Delta t}{2N_c - 1}\,\frac{(1 + C\Delta t)^{n+1} - 1}{C\Delta t}.$$

Since $(1 + C\Delta t)^n \leq \exp(Ct) \leq \exp(Ct_0)$, we showed convergence, if $N_c \to \infty$. We summarize this in the

THEOREM: *Let be $T_j \simeq [0,1]^j$ the j-dimensional torus, $P, Q : T_2 \to \mathbb{R}^+$ uniform Lipschitz-continuous functions with*

$$\int_{T_1} P(k, k')dk = \int_{T_1} Q(k, k')dk' = q.$$

Then the approximation μ_N^n defined by (P1), (P2) converges to μ^n, where

$$\mu^n(dk) = f(t, k)dk$$

and f is the solution of

$$\frac{\partial f}{\partial t} = \int_{T_1} P(k, k')f(t, k')dk' - f(t, k)\int_{T_1} Q(k, k')dk'$$

$$f(0, k) = f_0(k) \geq 0, \qquad \int_{T_1} f_0(k)dk = 1.$$

for every finite time-interval, if $N\Delta t \to \infty$ *and* $\Delta t \to 0$, *provided*

$$\lim_{N\to 0} \mu_N^0 = \mu^0 .$$

4. Conclusion.

We presented a number-thoretical convergence proof of a point approximation of a space homogeneous transport equation. Computer implementation seems promising but still needs some refinements. The extension of the method to the space inhomogeneous case is still under investigation and results will be published in due course.

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