A NOTE ON A PROBLEM OF CAPOBIANCO AND MOLLUZZO

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The aim of this paper is to solve a problem of Capobianco and Molluzzo [2, pag. 65]. More exactly, we show that for any two integers \( n \) and \( m \), \( 1 < n < m \), there exist a graph \( G \), such that \( k(G) = n \) and \( k[L(G)] = m \), where \( k(G) \) denotes the connectivity of \( G \), and \( L(G) \) the line graph of \( G \).

Introduction.

Graphs, considered here, are finite and simple (without loops or multiple lines (edges)), and Harary [4] is followed for terminology and notation. Chartrand and Stewart [3] have shown that the line graph \( L(G) \), of a point (vertex) \( n \)-connected graph \( G \), is itself \( n \)-connected, if \( n \geq 2 \). In their book [2], Capobianco and Molluzzo, using \( K_{1,n} \) as their example, have noted that the difference between the point (vertex) connectivity of a graph \( k(G) \) and its line graph \( k[L(G)] \) can be arbitrarily large. They then have posed the following open problem: «It is not known whether, for any two integers \( n, m, 1 < n < m \), there exists a graph \( G \), such that \( k(G) = n \) and \( k[L(G)] = m \) [2, pag. 65]. □

In this paper, we shall solve this problem, by constructing the desired graph.

**Proposition.** For any two non negative integers \( r \) and \( s \), we have

\[
k[L(K_{r,s})] = r + s - 2.
\]

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* *Entrato in Redazione il 4 ottobre 1988*
Proof. The result follows immediately, since \( L(K_{r,s}) = K_r \times K_s \) [5], and \( k(K_r \times K_s) = r + s - 2 \) [1, problem 10.7, pag. 125]. \( \square \)

The main result.

The main result of this paper is the following

THEOREM 1. For any two integers \( n, m \), \( 1 < n < m \), there exists a graph \( G_{n,m} \), such that \( k(G_{n,m}) = n \) and \( k[L(G_{n,m})] = m \).

Proof. If \( m \geq 2n - 2 \), then \( m - n + 2 \geq n \). Thus, \( k(K_{n,m-n+2}) = n \) and, by the Proposition, \( k[L(K_{n,m-n+2})] = m \). Therefore, we set \( G_{n,m} = K_{n,m-n+2} \), when \( m \geq 2n - 2 \).

If \( n = 2 \) or \( n = 3 \) and \( m > n \), then \( m \geq 2n - 2 \), and \( G_{n,m} \) is defined as above. Thus, we assume that \( n \geq 4 \) and \( n < m < 2n - 2 \).

Let \( H \) and \( H' \) be complete graphs on the disjoint sets \( V = \{v_1, \ldots, v_{2n}\} \) and \( V' = \{v'_1, \ldots, v'_{2n}\} \), respectively. We form \( G_{n,m} \), by adding to \( H \cup H' \) the lines (edges) \( e_i = (v_i, v'_i) \), \( 1 \leq i \leq n \) and the lines \( e_{n+j} = (v_j, v'_{j+1}) \), \( 1 \leq j \leq m - n \) (note that \( m - n < n - 2 \)). Since \( G_{n,m} - v_1 - \ldots - v_n \) is isomorphic to \( K_n \cup K_{2n} \), we have that \( k(G_{n,m}) \leq n \). One can use Menger's Theorem to show that \( k(G_{n,m}) = n \), since \( H \) and \( H' \) are \((2n-1)\)-connected, and the desired family of paths connecting a point (vertex) of \( H \) to one of \( H' \) is easily constructed.

Now, we must show that \( k[L(G_{n,m})] = m \). Since the lines \( e_1, \ldots, e_m \) of \( G_{n,m} \) form a disconnecting set of points for \( L(G_{n,m}) \), then \( k[L(G_{n,m})] \leq m \).

We shall complete the proof, by showing that any disconnecting set \( \{x_1, \ldots, x_k\} \) of points of \( L(G_{n,m}) \), with \( k \leq m \), must contain the set \( \{e_1, \ldots, e_m\} \). Hence, the sets coincide.

First, note that the degree of any point of \( H - x_1 - \ldots - x_k \) is positive, since \( H = K_{2n} \), and \( k \leq m < 2n - 1 \). The same holds for \( H' \), as well.

Moreover, both \( H - x_1 - \ldots - x_k \) and \( H' - x_1 - \ldots - x_k \) are connected, since \( K_{2n} \) is \((2n-1)\)-connected, and \( k < 2n - 1 \). Thus, if some \( e_i \) is not in \( \{x_1, \ldots, x_k\} \), then \( e_i, H \) and \( H' \) all lie in the same component of \( G_{n,m} - x_1 - \ldots - x_k \). Then, it follows that \( G_{n,m} - x_1 - \ldots - x_k \) is connected, i.e., a contradiction. Therefore, every \( e_i \) is in \( \{x_1, \ldots, x_k\} \), so
that \( k = m \). Thus, \( k[L(G_{n,m})] = m \), completing the proof. \( \square \)

Zamfirescu [6] has proved that \( \lambda[L(G)] \geq 2n - 2 \), if \( \lambda(G) \geq n \) (\( \lambda(G) \) denotes the line (edge) connectivity of \( G \)). But, this suggests a line connectivity version of the question of Capobianco and Molluzzo. The examples constructed above suffice to answer this question, since, for \( m \geq 2n - 2 \), the graph \( G_{n,m} = K_{n,m-n+2} \) satisfies \( \lambda(G_{n,m}) = n \) and \( \lambda[L(G_{n,m})] = m \). Thus, we have established the following

**Theorem 2.** For any integers \( n > 1 \) and \( m \geq 2n - 2 \), there exists a graph \( G_{n,m} \), such that \( \lambda(G_{n,m}) = n \), and \( \lambda[L(G_{n,m})] = m \). \( \square \)

**References**


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