A NOTE ON A PROBLEM OF CAPOBIANCO AND MOLLUZZO

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The aim of this paper is to solve a problem of Capobianco and Molluzzo [2, pag. 65]. More exactly, we show that for any two integers n and m, $1 < n < m$, there exist a graph G, such that $k(G) = n$ and $k[L(G)] = m$, where $k(G)$ denotes the connectivity of G, and $L(G)$ the line graph of G.

Introduction.

Graphs, considered here, are finite and simple (without loops or multiple lines (edges)), and Harary [4] is followed for terminology and notation. Chartrand and Stewart [3] have shown that the line graph $L(G)$, of a point (vertex) n-connected graph G, is itself n-connected, if $n \geq 2$. In their book [2], Capobianco and Molluzzo, using $K_{1,n}$ as their example, have noted that the difference between the point (vertex) connectivity of a graph $k(G)$ and its line graph $k[L(G)]$ can be arbitrarily large. They then have posed the following open problem: «It is not known whether, for any two integers $n, m, 1 < n < m$, there exists a graph G, such that $k(G) = n$ and $k[L(G)] = m$»[2, pag. 65]. ❑

In this paper, we shall solve this problem, by constructing the desired graph.

PROPOSITION. *For any two non negative integers r and s, we have*

$$k[L(K_{r,s})] = r + s - 2.$$

* Entrato in Redazione il 4 ottobre 1988

Proof. The result follows immediately, since $L(K_{r,s}) = K_r \times K_s$ [5], and $k(K_r \times K_s) = r + s - 2$ [1, problem 10.7, pag. 125]. ❑

The main result.

The main result of this paper is the following

THEOREM 1. *For any two integers $n, m, 1 < n < m$, there exists a graph $G_{n,m}$, such that $k(G_{n,m}) = n$ and $k[L(G_{n,m})] = m$.*

Proof. If $m \geq 2n-2$, then $m-n+2 \geq n$. Thus, $k(K_{n,m-n+2}) = n$ and, by the Proposition, $k[L(K_{n,m-n+2})] = m$. Therefore, we set $G_{n,m} = K_{n,m-n+2}$, when $m \geq 2n-2$.

If $n = 2$ or $n = 3$ and $m > n$, then $m \geq 2n-2$, and $G_{n,m}$ is defined as above. Thus, we assume that $n \geq 4$ and $n < m < 2n-2$. Let H and H' be complete graphs on the disjoint sets $V = \{v_1, \ldots, v_{2n}\}$ and $V' = \{v'_1, \ldots, v'_{2n}\}$, respectively. We form $G_{n,m}$, by adding to $H \cup H'$ the lines (edges) $e_i = (v_i, v'_i)$, $1 \leq i \leq n$ and the lines $e_{n+j} = (v_j, v'_{j+1})$, $1 \leq j \leq m-n$ (note that $m - n < n - 2$). Since $G_{n,m} - v_1 - \ldots - v_n$ is isomorphic to $K_n \cup K_{2n}$, we have that $k(G_{n,m}) \leq n$. One can use Menger's Theorem to show that $k(G_{n,m}) = n$, since H and H' are $(2n-1)$-connected, and the desired family of paths connecting a point (vertex) of H to one of H' is easily constructed.

Now, we must show that $k[L(G_{n,m})] = m$. Since the lines $e_1, \ldots, e_m$ of $G_{n,m}$ form a disconnecting set of points for $L(G_{n,m})$, then $k[L(G_{n,m})] \leq m$.

We shall complete the proof, by showing that any disconnecting set $\{x_1, \ldots, x_k\}$ of points of $L(G_{n,m})$, with $k \leq m$, must contain the set $\{e_1, \ldots, e_m\}$. Hence, the sets coincide.

First, note that the degree of any point of $H - x_1 - \ldots - x_k$ is positive, since $H = K_{2n}$, and $k \leq m < 2n-1$. The same holds for H', as well.

Moreover, both $H - x_1 - \ldots - x_k$ and $H' - x_1 - \ldots - x_k$ are connected, since K_{2n} is $(2n-1)$-connected, and $k < 2n-1$. Thus, if some e_i is not in $\{x_1, \ldots, x_k\}$, then e_i, H and H' all lie in the same component of $G_{n,m} - x_1 - \ldots - x_k$. Then, it follows that $G_{n,m} - x_1 - \ldots - x_k$ is connected, i.e., a contradiction. Therefore, every e_i is in $\{x_1, \ldots, x_k\}$, so that $k = m$. Thus, $k[L(G_{n,m})] = m$, completing the proof. ❑

Zamfirescu [6] has proved that $\lambda[L(G)] \geq 2n - 2$, if $\lambda(G) \geq n$ ($\lambda(G)$ denotes the line (edge) connectivity of G). But, this suggests a line connectivity version of the question of Capobianco and Molluzzo. The examples constructed above suffice to answer this question, since, for $m \geq 2n - 2$, the graph $G_{n,m} = K_{n,m-n+2}$ satisfies $\lambda(G_{n,m}) = n$ and $\lambda[L(G_{n,m}] = m$. Thus, we have established the following

THEOREM 2. *For any integers $n > 1$ and $m \geq 2n - 2$, there exists a graph $G_{n,m}$, such that $\lambda(G_{n,m}) = n$, and $\lambda[L(G_{n,m})] = m$.* ❑

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