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EXTENDED SOLUTIONS OF A SYSTEM OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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This paper deals with extended solutions of a system of nonlinear integro-differential equations. This system is obtained in the process of applying the Galerkin method for some initial-boundary value problems.

Contents

1. Introduction

The Galerkin method has received considerable attention as a powerful numerical solution technique to differential equations. It has been widely used as a main tool in the study of wave equations with different boundary value types

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(see [4–12, 14]) and the references therein. In [5] Lê applies the Galerkin method to show the solvability of the following initial-boundary value problem with the unknown $u(x,t)$, $0<x<1$, $0<t<T$:

$$\frac{\partial^2 u}{\partial t^2}(x,t)-\mu(t)\frac{\partial^2 u}{\partial x^2}(x,t)+F\left(u(x,t),\frac{\partial u}{\partial t}(x,t)\right)=f(x,t), \tag{1.1}$$

$$u(0,t)=0, \tag{1.2}$$

$$-\mu(t)\frac{\partial u}{\partial x}(1,t)=Q(t), \tag{1.3}$$

$$u(x,0)=u_0(x),\ \frac{\partial u}{\partial t}(x,0)=u_1(x), \tag{1.4}$$

where $T>0$ is given, and

$$Q(t)=K_1(t)u(1,t)+\lambda_1(t)\frac{\partial u}{\partial t}(1,t)-g(t)-\int_0^t k(t-s)u(1,s)ds, \tag{1.5}$$

$$F\left(u,\frac{\partial u}{\partial t}\right)=K|u|^{p-2}u+\lambda\left|\frac{\partial u}{\partial t}\right|^{q-2}\frac{\partial u}{\partial t} \qquad \text{for } p,\, q\geq 2, \tag{1.6}$$

with given constants K and λ, and given functions u_0,u_1,f,μ,g,k,K_1, and λ_1. The system (1.1)-(1.6) is known as a mathematical model describing the shock of a rigid body and a viscoelastic bar, see [5, 9] and the references therein. In the integral equation of the unknown boundary value $u(1,t)$, curiosity of the appearance of the convolution $k*u(1,\cdot)$ over $(0,t)$ is usually an interesting topic. For a clarification, we would like to refer to [13], in which the mechanical motivation of the above convolution is specifically detailed by a practical model for the collision between a free-fall hammer of a pile-driver and an elastic pile. Since [13] is now online, let us omit the details of this considerable theme.

By applying the Galerkin method, the author in [5] constructs solutions of certain type of finite-dimensional approximations for this system (see also [1, 15, 16, 18]). Specifically, by considering $\{\omega_j\}$ as a denumerable and orthonormal basis of

$$V=\{v\in H^1(0,1):v(0)=0\},$$

the approximate solutions of the problem (1.1)-(1.6) are presented as follows:

$$u_m(x,t)=\sum_{j=1}^{m}c_{mj}(t)\omega_j(x), \tag{1.7}$$

where the unknown coefficients c_{mj} satisfy the following system of ordinary linear differential equations:

$$\left\langle \frac{\partial^2 u_m}{\partial t^2}(\cdot,t), \omega_j \right\rangle + \left\langle \frac{\partial u_m}{\partial x}(\cdot,t), \omega_j' \right\rangle + Q_m(t)\omega_j(1) +$$
$$+ \left\langle F\left(u_m(\cdot,t), \frac{\partial u_m}{\partial t}(\cdot,t)\right), \omega_j \right\rangle = \left\langle f(\cdot,t), \omega_j \right\rangle, \quad 1 \le j \le m, \tag{1.8}$$

$$Q_m(t) = u_m(1,t) + \frac{\partial u_m}{\partial t}(1,t) + g(t) - \int_0^t k(t-s) u_m(1,s)ds, \tag{1.9}$$

with

$$u_m(\cdot,0) = u_{0m} = \sum_{j=1}^m \alpha_{mj}\omega_j \to u_0 \qquad \text{strongly in} \quad V, \tag{1.10}$$
$$\frac{\partial u_m}{\partial t}(\cdot,0) = u_{1m} = \sum_{j=1}^m \beta_{mj}\omega_j \to u_1 \qquad \text{strongly in} \quad L^2, \tag{1.11}$$

here $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(0,1)$. By a similar argument to that of the case

$$F\left(u, \frac{\partial u}{\partial t}\right) = Ku + \lambda \frac{\partial u}{\partial t}$$

in [12, Section 3.1], one can see that also in the general case of (1.6) the unknown coefficients c_{mj} are the solution of the following system:

$$c_i''(t) + \sum_{j=1}^m \left[\langle w_j', w_i' \rangle + w_j(1)w_i(1)\right] c_j(t) + \sum_{j=1}^m w_j(1)w_i(1)c_j'(t) -$$
$$- \sum_{j=1}^m w_j(1)w_i(1) \int_0^t k(t-s)c_j(s)ds + \langle F\left(u(\cdot,t), \frac{\partial u}{\partial t}(\cdot,t)\right), w_i \rangle =$$
$$= -g(t)w_i(1) + \langle f(\cdot,t), w_i \rangle, \tag{1.12}$$
$$c_i(0) = \alpha_i, c_i'(0) = \beta_i, \quad 1 \le i \le m \tag{1.13}$$

with the unknown functions c_j. Let us denote

$$c(t) = (c_1(t), c_2(t), \cdots, c_m(t)),$$

and for each $1 \le i \le m$,

$$F_{1i}\big(t, c(t), c'(t)\big) = -\sum_{j=1}^{m} \Big[\langle \omega_j', \omega_i' \rangle + \omega_j(1)\omega_i(1)\Big] c_j(t) -$$

$$-\sum_{j=1}^{m} \omega_j(1)\omega_i(1)c_j'(t) - g(t)\omega_i(1) + \langle f(\cdot,t), \omega_i \rangle -$$

$$-\Big\langle F\Big(\sum_{j=1}^{m} \omega_j c_j(t), \sum_{j=1}^{m} \omega_j c_j'(t)\Big), \omega_i \Big\rangle =$$

$$= -\sum_{j=1}^{m} \Big[\langle \omega_j', \omega_i' \rangle + \omega_j(1)\omega_i(1)\Big] c_j(t) - \sum_{j=1}^{m} \omega_j(1)\omega_i(1)c_j'(t) - g(t)\omega_i(1) +$$

$$+\langle f(t), \omega_i \rangle - \sum_{j=1}^{m} K \Big\langle \Big| \sum_{j=1}^{m} \omega_j c_j(t) \Big|^{p-2} \omega_j c_j(t), \omega_i \Big\rangle +$$

$$+\sum_{j=1}^{m} \lambda \Big\langle \Big| \sum_{j=1}^{m} \omega_j c_j'(t) \Big|^{q-2} \omega_j c_j'(t), \omega_i \Big\rangle, \tag{1.14}$$

and

$$F_{2i}\big(c(t)\big) = \sum_{j=1}^{m} \omega_j(1)\omega_i(1)c_j(t). \tag{1.15}$$

Then by setting $\alpha_i = c_i(0)$ and $\beta_i = c_i'(0)$, $1 \le i \le m$, and considering the multi-variable functions $F_{1i} : [0,T] \times \mathbb{R}^{2m} \to \mathbb{R}$ and $F_{2i} : \mathbb{R}^m \to \mathbb{R}$ described in (1.14) and (1.15), we can rewrite the system (1.12)-(1.13) as the following equivalent system of integro-differential equations:

$$c_i(t) = \alpha_i + \beta_i t + \int_0^t \Big(\int_0^\tau (Gc)_i(s) ds\Big) d\tau, \tag{1.16}$$

$$(Gc)_i(t) = F_{1i}\big(t, c(t), c'(t)\big) + \int_0^t k(t-s) F_{2i}\big(c(s)\big) ds, \tag{1.17}$$

where $0 \le t \le T$, $1 \le i \le m$; α_i and β_i are given constants, and k, $F_{1i} : [0,T] \times \mathbb{R}^{2m} \to \mathbb{R}, F_{2i} : \mathbb{R}^m \to \mathbb{R}$ are given functions.

It is worth mentioning that the appearance of the system (1.16)-(1.17) from (1.16)-(1.17) gives an important connection between systems of partial differential equations and systems of integro-differential equations. Besides giving this relationship between different aspects of differential equations, [12, Remark 3] gives another interest for the study of system (1.16)-(1.17) by describing this system as a generalization of the Lotka-Volterra system, see [2, 3, 17]. Indeed it is shown that in the case of $n = 2$, by suitable choices of k, F_{1i} and F_{2i}, $i = 1, 2$, the system (1.16)-(1.17) admits the following differential form of Lotka-Volterra equations:

$$c_1'(t) = \beta_1 + c_1(t)\big(a_{11} + a_{12}c_2(t)\big), \tag{1.18}$$
$$c_2'(t) = \beta_2 + c_2(t)\big(a_{21} + a_{22}c_1(t)\big), \tag{1.19}$$

where $\beta_i, a_{ij} \in \mathbb{R}$ for $i, j = 1, 2$.

Lê and Pascali in [12] prove that the system (1.16)-(1.17) is solvable in $C^1([0,T^*];\mathbb{R}^m)$ for some $T^* \in (0,T]$, when $k \in L^1(0,T)$, $F_{1i} \in C\left(\mathbb{R}^{2m+1};\mathbb{R}\right)$ and $F_{2i} \in C(\mathbb{R}^m;\mathbb{R})$, $1 \le i \le m$. Later, the system (1.16)-(1.17) is generalized to the following system of integro-differential equations:

$$c_i(t) = G_i(t) + \int_0^t N_i(t-\tau)(Vc)_i(\tau)\,d\tau, \quad 0 \le t \le T, 1 \le i \le m, \tag{1.20}$$

$$(Vc)_i(t) = H_{1i}\left(t, c(t), c'(t), \cdots, c^{(k)}(t)\right) +$$
$$+ \int_0^t k_i(t-\tau) H_{2i}\left(t, \tau, c(\tau), c'(\tau), \cdots, c^{(k)}(\tau)\right) d\tau \tag{1.21}$$

for $k, m \in \mathbb{N}$, and given functions G_i, N_i, k_i, H_{1i}, H_{2i}, $1 \le i \le m$. Noticeably, [12, Section 4] gives a sketch of the proof for the solvability of the system (1.20)-(1.21) in $C^k([0,T^*];\mathbb{R}^m)$ for some $T^* \in (0,T]$. Here $C^k([0,T];\mathbb{R}^m)$ denotes the Banach space of all of $k-$differential functions

$$u : [0,T] \to \mathbb{R}^m,$$

$$u(t) = (u_1(t), u_2(t), \cdots, u_m(t))$$

endowed with the norm

$$\|u\|_{C^k([0,T];\mathbb{R}^m)} = \sum_{j=0}^{k} \|u^{(j)}\|_0,$$

$$\|u\|_0 = \sup_{0 \le t \le T} |u(t)|_1, \quad |u(t)|_1 = \sum_{i=1}^{m} |u_i(t)|.$$

To guarantee the existence of local solutions Lê and Pascali make the following assumptions on G_j, N_j, k_j, H_{1j} and H_{2j}, $1 \le j \le m$:

(B_G) $G_j \in C^k([0,T];\mathbb{R})$,

(B_k) $k_j \in L^1(0,T)$,

(B_N) $N_j \in C^k([0,T];\mathbb{R})$, and for each $0 \le p \le k$ there exists a constant $C_{jp} > 0$ such that $N_j^{(p)}$ satisfy the Lipschitz condition:

$$\left| N_j^{(p)}(t) - N_j^{(p)}\left(\widetilde{t}\right) \right| \le C_{jp}|t - \widetilde{t}| \text{ for all } \left(t, \widetilde{t}\right) \in [0,T]^2,$$

(B_H) $H_{1j} \in C\left(\mathbb{R}^{(k+1)m+1};\mathbb{R}\right)$, $H_{2j} \in C\left(\mathbb{R}^{(k+1)m+2};\mathbb{R}\right)$.

For the assumption (B_H) more restricted, in this paper we obtain extended solutions of the system (1.20)-(1.21) by extending the local solutions given in [12, Theorem 2]. More precisely, as mentioned, while the solvability of this system only exists on $[0,T^*] \subseteq [0,T]$ for $T > 0$ given, in this project this solvability is extended in to the full interval $[0,T]$.

As compared with [12], the result in this paper is more advanced in terms of

- ◇ the mathematical reason: the extension of the solution of the system (1.20)-(1.21) is obtained, as stated, and
- ◇ technical difficulties:
  - ▷ Since more restrictions on the given functions H_{1j} and H_{2j} for $1 \le j \le m$ are added, a proof of the mathematical claim must be clarified. Then one may see how possibly the new result is obtained and the restricted conditions on the given data are adopted,
  - ▷ the key constant M in [12] only depends on (α_j, β_j) for $1 \le j \le m$, correspondent to G_j in this paper. However, here the key constant M depends not only on G_j but also on all given functions of the system (1.20)-(1.21). On the other hand, if the proof were not specified, or were only claimed, "the similarity" between the solvability in this paper and in [12] would be "misunderstood".

2. Extension of the solution for the system (1.20)-(1.21)

For given natural numbers $m, k \ge 1$, we can rewrite the system (1.20)-(1.21) as follows:

$$c_j(t) = (Uc)_j(t), \quad 0 \le t \le T, 1 \le j \le m, \tag{2.1}$$

where

$$(Uc)_j(t) = G_j(t) + \int_0^t N_j(t-\tau)(Vc)_j(\tau)\,d\tau, \qquad 0 \le t \le T, 1 \le j \le m, \tag{2.2}$$

$$(Vc)_j(t) = H_{1j}\left(t, c(t), c'(t), \cdots, c^k(t)\right) + \\ + \int_0^t k_j(t-\tau) H_{2j}\left(t, \tau, c(\tau), c'(\tau), \cdots, c^k(\tau)\right) d\tau, \tag{2.3}$$

with $0 \le t \le T$, and G_j, k_j, N_j, H_{1j} and H_{2j}, for $1 \le j \le m$, given functions. To obtain the extended solution of the system (2.1)-(2.3), we make the following assumptions: for each $1 \le j \le m$,

(E_G) $G_j \in C^k([0,T];\mathbb{R})$,

(E_k) $k_j \in L^1(0,T)$,

(E_N) $N_j \in C^k([0,T];\mathbb{R})$, and for each $0 \le p \le k$ there exists a constant $C_{jp} > 0$ such that $N_j^{(p)}$ satisfy the Lipschitz condition:

$$\left|N_j^{(p)}(t) - N_j^{(p)}\left(\widetilde{t}\right)\right| \le C_{jp}|t - \widetilde{t}| \text{ for all } \left(t, \widetilde{t}\right) \in [0,T]^2,$$

(E_H) $\left(H_{1j}, H_{2j}\right) \in C\left([0,T] \times \mathbb{R}^{(k+1)m};\mathbb{R}\right) \times C\left([0,T]^2 \times \mathbb{R}^{(k+1)m};\mathbb{R}\right)$ and there exists

$$(C_1, C_2) \in L^\infty([0,T];\mathbb{R}_*) \times L^\infty\left([0,T]^2;\mathbb{R}_*\right),$$

$\mathbb{R}_* = \mathbb{R}_+ \cup \{0\}$, such that

$$\left|H_{1j}(t,x)\right| \le C_1(t), \left|H_{2j}(t,\tau,x)\right| \le C_2(t,\tau)$$

for all $x \in \mathbb{R}^{(k+1)m}$ and $(t,\tau) \in [0,T]^2$.

Now we are proving that the assumptions (E_G), (E_k), (E_N) and (E_H) provide an extended solution in $C^k([0,T];\mathbb{R}^m)$ of the system (2.1)-(2.3). Indeed, we find $M > 0$ such that the mapping $U : S \to S$ given by (2.1)-(2.3) has a fixed point in the set

$$S = \left\{c \in C^k([0,T];\mathbb{R}^m) : \|c\|_k \le M\right\}.$$

To do this, we prove the following properties of the operator U, or steps:

(1) U is a selfmap of S,

(2) $U : Y \to Y$ is continuous,

(3) $\overline{US}$ is a compact subset of Y.

Step 1. U is a selfmap of S.

First note that $(Vc)_j \in C([0,T];\mathbb{R})$ for each $c \in Y = C^k([0,T];\mathbb{R}^m)$ and $1 \leq j \leq m$. On the other hand, for each $1 \leq j \leq m$ and $0 \leq p \leq k$, from (2.2) we get

$$(Uc)_j^{(p)}(t) = G_j^{(p)}(t) + \int_0^t N_j^{(p)}(t-\tau)(Vc)_j(\tau)\,d\tau. \tag{2.4}$$

This along with the above assumptions imply that $Uc \in Y$, meaning that $UY \subseteq Y$. Now for $c \in S$, we deduce from (2.4) that

$$\begin{aligned}\left|(Uc)_j^{(p)}(t)\right| &\leq \left|G_j^{(p)}(t)\right| + \int_0^t \left|N_j^{(p)}(t-\tau)(Vc)_j(\tau)\right| d\tau \\ &\leq \left\|G_j^{(p)}\right\|_0 + T\left\|N_j^{(p)}\right\|_0 \left\|(Vc)_j\right\|_0\end{aligned} \tag{2.5}$$

for $p = 0, 1, \ldots, k$. Thus

$$\|Uc\|_k \leq \sum_{p=0}^{k}\sum_{j=1}^{m}\left\|G_j^{(p)}\right\|_0 + T\left(\sum_{p=0}^{k}\sum_{j=1}^{m}\left\|N_j^{(p)}\right\|_0\left\|(Vc)_j\right\|_0\right). \tag{2.6}$$

Now we are estimating $\left\|(Vc)_j\right\|_0$. For $1 \leq j \leq m$ let

$$N(H_{1j}) = \sup\left\{\left|H_{1j}\left(\tau, t_1, \ldots, t_{(k+1)m}\right)\right| : \left(\tau, t_1, \ldots, t_{(k+1)m}\right) \in [0,T] \times [-M,M]^{(k+1)m}\right\},$$

$$N(H_{2j}) = \sup\left\{\left|H_{2j}\left(\tau_1, \tau_2, t_1, \ldots, t_{(k+1)m}\right)\right| : \left(\tau_1, \tau_2, t_1, \ldots, t_{(k+1)m}\right) \in [0,T]^2 \times [-M,M]^{(k+1)m}\right\},$$

and

$$N(H_{1j}, H_{2j}, k) = N(H_{1j}) + \left\|k_j\right\|_{L^1(0,T)} N(H_{2j}). \tag{2.7}$$

Note that by the assumptions of the theorem the values $N(H_{1j})$, $N(H_{2j})$, and hence $N(H_{1j}, H_{2j}, k)$ are bounded and well-defined (independent of M). Now from (2.3) we get

$$\begin{aligned}\left|(Vc)_j(\tau)\right| &\leq \left|H_1\left(\tau, c(\tau), c'(\tau)\cdots c^k(\tau)\right)\right| + \\ &+ \int_0^\tau |k(\tau-s)|\left|H_2\left(\tau, s, c(s), c'(s)\cdots c^k(s)\right)\right| ds,\end{aligned} \tag{2.8}$$

implying

$$\left\|(Vc)_j\right\|_0 \leq N(H_{1j}, H_{2j}, k) \qquad \text{for all } c \in S. \tag{2.9}$$

Applying (2.6) and (2.9) yields

$$\|Uc\|_k \leq \sum_{p=0}^{k}\sum_{j=1}^{m}\left\|G_j^{(p)}\right\|_0 + T\left(\sum_{p=0}^{k}\sum_{j=1}^{m}\left\|N_j^{(p)}\right\|_0 N(H_{1j},H_{2j},k)\right). \tag{2.10}$$

By choosing the constant M large enough, such that

$$\sum_{p=0}^{k}\sum_{j=1}^{m}\left\|G_j^{(p)}\right\|_0 + T\left(\sum_{p=0}^{k}\sum_{j=1}^{m}\left\|N_j^{(p)}\right\|_0 N(H_{1j},H_{2j},k)\right) \leq M$$

and applying (2.10), we have $\|Uc\|_k \leq M$ for all $c \in S$, meaning that $US \subseteq S$.
Step 2. $U : Y \to Y$ is continuous.

First note that from (2.2) for each $(y_1,y_2) \in Y^2$, $0 \leq p \leq k$ and $1 \leq j \leq m$ we have

$$\begin{aligned}\left|(Uy_1)_j^{(p)}(t) - (Uy_2)_j^{(p)}(t)\right| &= \int_0^t \left|N_j^{(p)}(t-\tau)\right|\left|(Vy_1)_j(\tau) - (Vy_2)_j(\tau)\right| d\tau,\\ &\leq T\left\|N_j^{(p)}\right\|_0 \|Vy_1 - Vy_2\|_0,\end{aligned}$$

implying

$$\left\|(Uy_1)^{(p)} - (Uy_2)^{(p)}\right\|_0 \leq T\left(\sum_{j=1}^{m}\left\|N_j^{(p)}\right\|_0\right)\|Vy_1 - Vy_2\|_0,$$

and hence

$$\|Uy_1 - Uy_2\|_k \leq T\left(\sum_{p=0}^{k}\sum_{j=1}^{m}\left\|N_j^{(p)}\right\|_0\right)\|Vy_1 - Vy_2\|_0.$$

Therefore, to prove the continuity of $U : Y \to Y$, it is enough to prove the continuity of $V : Y \to C([0,T];\mathbb{R}^m)$.
Let $x_n \to x_0$ in Y, which is equivalent to

$$x_n^{(p)} \to x_0^{(p)} \text{ in } C([0,T];\mathbb{R}^m)$$

for $p = 0,1,...,k$. Then one can find a constant $M_0 > 0$ such that

$$x_n^{(p)}(s), x_0^{(p)}(s) \in [-M_0, M_0]^m$$

for all $s \in [0,T]$, and $p = 0,1,...,k$. Since H_{1j} is uniformly continuous on

$$[0,T] \times [-M_0,M_0]^{(k+1)m},$$

we get

$$\sup_{0\le t\le T}\Big|H_{1j}\left(t,x_n(t),x_n'(t),\cdots,x_0^{(k)}(t)\right)-$$
$$-H_{1j}\left(t,x_0(t),x_0'(t),\cdots,x_0^{(k)}(t)\right)\Big|\to 0 \tag{2.11}$$

when $n\to\infty$. Similarly, by the uniform continuity of H_{2j} on

$$[0,T]^2\times[-M_0,M_0]^{(k+1)m},$$

we have

$$\sup_{0\le t,s\le T}\Big|H_{2j}\left(t,s,x_n(s),x_n'(s),\cdots,x_n^{(k)}(s)\right)-$$
$$-H_{2j}\left(t,s,x_0(s),x_0'(s),\cdots,x_0^{(k)}(s)\right)\Big|\to 0 \tag{2.12}$$

when $n\to\infty$. Now considering (2.3) we have

$$\|Vx_n-Vx_0\|_0\le$$
$$\le\sup_{0\le t\le T}\sum_{j=1}^{m}\Big|H_{1j}\left(t,x_n(t),x_n'(t),\cdots,x_n^{(k)}(t)\right)-$$
$$-H_{1j}\left(t,x_0(t),x_0'(t),\cdots,x_0^{(k)}(t)\right)\Big|+$$
$$+\sup_{0\le t\le T}\sum_{j=1}^{m}\int_0^t|k(t,s)|\Big|\Big[H_{2j}\left(t,s,x_n(s),x_n'(s),\cdots,x_n^{(k)}(s)\right)-$$
$$-H_{2j}\left(t,s,x_0(s),x_0'(s),\cdots,x_0^{(k)}(s)\right)\Big]\Big|ds\le$$
$$\le\sup_{0\le t\le T}\sum_{j=1}^{m}\Big|H_{1j}\left(t,x_n(t),x_n'(t),\cdots,x_n^{(k)}(t)\right)-$$
$$-H_{1j}\left(t,x_0(t),x_0'(t),\cdots,x_0^{(k)}(t)\right)\Big|+$$
$$+\|k\|_{L^1(0,T)}\sup_{0\le t,s\le T}\sum_{j=1}^{m}\Big|\Big[H_{2j}\left(t,s,x_n(s),x_n'(s),\cdots,x_n^{(k)}(s)\right)-$$
$$-H_{2j}\left(t,s,x_0(s),x_0'(s),\cdots,x_0^{(k)}(s)\right)\Big]\Big|.$$

Therefore, by applying (2.11) and (2.12) we get

$$\|Vx_n-Vx_0\|_0\to 0,\quad n\to\infty,$$

which implies the continuity of $V : Y \to C([0,T];\mathbb{R}^m)$. This completes the proof of this step.

Step 3. $\overline{US}$ is a compact subset of Y.

Let $c \in S$ and $t_1, t_2 \in [0,T]$. Then from (2.4) we get

$$\begin{aligned}
\left|(Uc)_j^{(p)}(t_1) - (Uc)_j^{(p)}(t_2)\right| &\le \left|G_j^{(p)}(t_1) - G_j^{(p)}(t_2)\right| + \\
&+ \left|\int_0^{t_1} N_j^{(p)}(t_1-\tau)(Vc)_j(\tau)\,d\tau - \int_0^{t_2} N_j^{(p)}(t_2-\tau)(Vc)_j(\tau)\,d\tau\right| \\
&\le \left|G_j^{(p)}(t_1) - G_j^{(p)}(t_2)\right| + \int_0^{t_2}\left|N_j^{(p)}(t_1-\tau) - N_j^{(p)}(t_2-\tau)\right|\left|(Vc)_j(\tau)\right| d\tau + \\
&+ \left|\int_{t_1}^{t_2} N_j^{(p)}(t_1-\tau)(Vc)_j(\tau)\,d\tau\right| \\
&\le \left|G_j^{(p)}(t_1) - G_j^{(p)}(t_2)\right| + TC_{jp}\left\|(Vc)\right\|_0 |t_1-t_2| + \left\|N^{(p)}\right\|_0 \left\|(Vc)\right\|_0 |t_1-t_2|.
\end{aligned} \tag{2.13}$$

On the other hand, $G_j \in C^k([0,T];\mathbb{R})$. Therefore by (2.13) and using Arzela-Ascoli theorem we conclude that $\overline{US}$ is compact in Y. In summary, by the Schauder fixed-point theorem $U : S \to S$ has fixed point $c \in S$, which implies the existence of the solution c for the system (2.1)-(2.3). Therefore we have the following result:

Theorem 2.1. *The assumptions (E_G), (E_k), (E_N) and (E_H) provide an extended solution $c \in C^k([0,T];\mathbb{R}^m)$ of the system (1.20)-(1.21).*

Remark 2.2. It is worth noting out that the nonlinear damping-source term of the equation (1.1) $F\left(u, \frac{\partial u}{\partial t}\right)$ given by (1.6) is correspondent to that in [12] with respect to $p = q = 2$, a linear case. This difference does not actually give any changes of the systems of integro-differential equations studied in both this paper and in [12]. By starting from the nonlinear initial-boundary value problem (1.1)-(1.6), we would like to show that these systems of integro-differential equations obviously cover the system deduced from the process of applying the Galerkin approximation for not only the problem (1.1)-(1.6), but also many other nonlinear initial-boundary problems.

Remark 2.3. Observing the assumption (E_H) and the solution of the system (1.20)-(1.21), one can see that the functions H_{1i} and H_{2i}, $1 \le i \le m$, are adapted to the problem (1.1)-(1.6) in terms of the continuity and the boundedness in time of these functions.

3. Some further problems

In this paper, the first open problem in [12, Section 5] has been solved. Here let us recall some further interesting questions regarding system the (1.20)-(1.21) as follows:

1. The solvability by a suitable iterative procedure, even with less restricted data.

2. Numerical solutions with respect to some special given data.

3. One can consider the solution existence when H_{ij} are not continuous for each $i \in \{1,2\}$ and $j = \overline{1,m}$, we can suppose that H_{ij} satisfy the following conditions:

   - $H_{1j}(t,\cdot)$ and $H_{2j}(t,\tau,\cdot)$ are bounded on bounded sets of $\mathbb{R}^{(k+1)m}$ for $(t,\tau) \in \mathbb{R}^2_+$,
   - $H_{1j}(\cdot,\xi,\eta)$ and $H_{2j}(\cdot,\cdot,\xi,\eta)$ are measurable on $\mathbb{R}_+$ and on $\mathbb{R}^2_+$, respectively, for every fixed $(\xi,\eta) \in \mathbb{R}^{2(k+1)m}$,
   - $H_{1j}(t,\cdot,\cdot)$ and $H_{2j}(t,\tau,\cdot,\cdot)$ are continuous $\mathbb{R}^{2(k+1)m}$ for all $(t,\tau) \in \mathbb{R}^2_+$.

4. Approximating the solutions by sequences of polynomials.

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