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STANLEY'S CONJECTURE, COVER DEPTH AND EXTREMAL SIMPLICIAL COMPLEXES

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A famous conjecture by R. Stanley relates the depth of a module, an algebraic invariant, with the so-called Stanley depth, a geometric one. We describe two related geometric notions, the cover depth and the greedy depth, and we study their relations with the Stanley depth for Stanley-Reisner rings of simplicial complexes. This leads to a quest for the existence of extremely non-partitionable simplicial complexes. We include several open problems and questions.

This paper is a report about a research project suggested by J. Herzog at the summer school P.R.A.G.MAT.I.C. 2008 at the University of Catania. In particular, the paper describes a direction where we expect that possible counterexamples can be found at least for a weaker version of Stanley's conjecture.

1. Introduction

Let S be a polynomial ring in n variables and M a finitely generated $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a decomposition of M as a direct sum of $\Bbbk$-vector spaces

$$M = \bigoplus_{i=1}^{k} m_i \Bbbk[Z_i]$$

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where $m_i\mathbb{k}[Z_i]$ with homogenous m_i is a Stanley space of dimension $|Z_i|$ for each i. The *Stanley depth* of a Stanley decomposition is given by the minimal dimension of the appearing Stanley spaces and the *Stanley depth* of the module M is defined as the maximal Stanley depth of any Stanley decomposition of M.

While there always exists some Stanley decomposition, Stanley conjectured in [21] that one can even find a Stanley decomposition of Stanley depth as least as large as the depth of M.

Stanley's conjecture: $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$.

The conjecture is true in polynomial rings with at most five variables [4, 18] and has also been checked in many other cases [1, 2, 4, 8–10, 13, 15, 18]. However, it is still open in general and it seems to be widely believed that it is wrong. The main bottleneck in finding counterexamples is the huge increase in complexity and computation time when calculating the Stanley depth for concrete examples [15, 17].

The main two cases of interest are: monomial ideals $I \subseteq S$, and Stanley-Reisner rings $\mathbb{k}[\Delta]$ of simplicial complexes Δ. In the latter case, we note why Stanley's conjecture is difficult: it relates an algebraic/homological notion, like $\operatorname{depth}(\mathbb{k}[\Delta])$, with a combinatorial/geometric one, $\operatorname{sdepth}(\mathbb{k}[\Delta])$. For instance, depth depends on the characteristic of the base field $\mathbb{k}$, while Stanley depth does not. We relax this hard question to a more tractable one, Problem 5.1, which is purely combinatorial. Essentially, we would like to know, how small the Stanley depth of a simplicial complex can be in relation to the minimal dimension of its maximal faces, the so-called cover depth of Δ. This problem can be rephrased as the quest to find simplicial complexes that are as non-partitionable as possible. So, while it has recently been shown that it suffices to solve Stanley's conjecture in the Cohen-Macaulay case [14], our approach aims in the possibly opposite direction. As a first step, we consider simplicial complexes that are extremely non-shellable.

This paper is organized in the following way. First, we recall the notion of Stanley depth and interprete it combinatorially for Stanley-Reisner rings of simplicial complexes. Then, we discuss the notions of cover depth and greedy depth that 'measure' pureness respectively shellability. In the final section, we report about our quest for counterexamples that are extremely non-shellable. We leave the question open if similar extremely non-partitionable complexes can be found.

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2. Stanley decompositions and Stanley depth

In this section, we recall the basic definitions and some properties of Stanley decompositions and Stanley depth for S-modules. Everything is known and we refer the interested reader to [13–15, 20] if not stated otherwise. We will focus on the case of squarefree monomial ideals and Stanley-Reisner rings of simplicial complexes.

Throughout, we assume that I is a squarefree monomial ideal in a polynomial ring S over a field $\mathbb{k}$. The number of variables in S will be assumed to be n or made clear by the context. We note that most of the definitions make sense also in the non-squarefree case where there are a number of results on its own (see e.g. [12]). From now on, let M be a $\mathbb{Z}^n$-graded S-module of the form I or S/I. In the latter case $M = S/I$, we call the set of monomials in $S\setminus I$ the *standard monomials* of S/I.

A *Stanley space (of M)* is a $\mathbb{k}$-subspace of M of the form $m\mathbb{k}[Z]$, where $m \in M$ is a homogeneous element, $Z \subseteq \{x_1, \dots, x_n\}$, and the natural $\mathbb{k}$-linear map $\mathbb{k}[Z] \to m\mathbb{k}[Z]$ is a bijection. In particular, a $\mathbb{k}$-basis of $m\mathbb{k}[Z]$ is given by the elements mm' for all monomials $m' \in k[Z]$. The dimension of a Stanley space $m\mathbb{k}[Z]$ is defined as $|Z|$.

A *Stanley decomposition* $\mathscr{D}$ of M is a decomposition of M as a $\mathbb{k}$-vector space into a direct sum of Stanley spaces:

$$\mathscr{D}: \quad M = \bigoplus_{i=1}^{k} m_i \mathbb{k}[Z_i]$$

where $m_i\mathbb{k}[Z_i]$ are Stanley spaces of M.

The *Stanley depth* of a Stanley decomposition $\mathscr{D}$ is given by the minimal dimension of the Stanley spaces appearing in the decomposition.

$$\operatorname{sdepth}(\mathscr{D}) = \min_{i=1,\dots,k} |Z_i|$$

The *Stanley depth* of M is defined as the maximal Stanley depth of any Stanley decomposition of M.

$$\operatorname{sdepth}(M) = \max_{\mathscr{D}} \operatorname{sdepth}(\mathscr{D})$$

Remark 2.1 ([13, 15])**.** The Stanley depth of M can be obtained by a Stanley decomposition

$$\mathscr{D}: \quad M = \bigoplus_{i=1}^{k} m_i \mathbb{k}[Z_i]$$

where m_i is squarefree and $\operatorname{supp}(m_i) \subseteq Z_i$ for all i. Such $\mathscr{D}$ is called a *squarefree Stanley decomposition*.

Furthermore, there is a bijective correspondence between squarefree Stanley decompositions and *partitions into intervals* of the poset of squarefree monomials of I, respectively, squarefree standard monomials of S/I.

Example 2.2. Let $2 \leq d < n$ and consider the squarefree monomial ideal

$$I_{n,d} = (x^{\alpha} \; : \; \alpha \in \{0,1\}^n, |\alpha| = d)$$

This is the Stanley-Reisner ideal of the $(d-2)$-skeleton of the $(n-1)$-dimensional simplex. For $d = 1$, we get the maximal ideal $\mathfrak{m} = I_{n,1}$.

It was shown in [15] that $\operatorname{depth}(I_{n,d}) = d$. Moreover, the authors remark that it is a hard combinatorial problem to find the exact value of $\operatorname{sdepth}(I_{n,d})$.

Let us deduce an upper bound on the Stanley depth. We apply the same simple counting idea that was used to bound $\operatorname{sdepth}(\mathfrak{m})$ in [15].

Assume that $\operatorname{sdepth}(I_{n,d}) = k$.

Following Remark 2.1, this means that there is a partition of the poset of all squarefree monomials of degree at least d into intervals such that the upper bound of each interval is a squarefree monomial of degree at least k.

In particular, there is such an interval for every squarefree monomial x^{α} of degree d and every such interval contains at least $k-d$ squarefree monomials of degree $d+1$. Since all $\binom{n}{d}$ such intervals are pairwise disjoint and there are only $\binom{n}{d+1}$ squarefree monomials of degree $d+1$ in total, this implies

$$\binom{n}{d}(k-d) \leq \binom{n}{d+1} = \frac{n-d}{d+1}\binom{n}{d}$$

or equivalently, $k \leq d + \frac{n-d}{d+1} = \frac{n+d^2}{d+1}$. This shows that $\operatorname{sdepth}(I_{n,d}) \leq \frac{n+d^2}{d+1}$.

Let Δ be a simplicial complex on $[n] := \{1,\ldots,n\}$. The *Stanley-Reisner ring* of Δ is defined as $\Bbbk[\Delta] := S/I_{\Delta}$, where

$$I_{\Delta} = (x^{\alpha} \; : \; \alpha \notin \Delta)$$

is the Stanley-Reisner ideal of Δ.

We note that the poset of squarefree standard monomials of $\Bbbk[\Delta]$ is in bijective correspondence to the face poset of Δ.

We say that Δ is *k-partitionable*, if there is a partition of the face poset of Δ into intervals $[G_i, F_i]$, such that k equals the minimum of $|F_i|$. Hence, the Stanley depth $\operatorname{sdepth}(\Bbbk[\Delta])$ equals the maximal k such that Δ is k-partitionable.

It has been recently shown in [14] that Stanley's conjecture holds for arbitrary Δ if it holds for all Cohen-Macaulay Δ. Recall that Δ is called *Cohen-Macaulay*, if $\operatorname{depth}(\Bbbk[\Delta]) = \dim(\Bbbk[\Delta])$. Since $\dim(\Bbbk[\Delta]) = \dim(\Delta) + 1$, this implies that Stanley's conjecture for Stanley-Reisner rings is equivalent to another famous conjecture of Stanley [22]: Any Cohen-Macaulay simplicial complex Δ is $\dim(\Delta) + 1$-partitionable, that is, partitionable in the usual sense.

Example 2.3. Consider the following simplicial complex Δ.

It is easy to check that there is no partition of the face poset of Δ into intervals $[G_1,\{1,2\}],[G_2,\{3,4\}]$, and thus $\operatorname{sdepth}(\Bbbk[\Delta]) < 2$. But there is a partition of the face set of Δ into three intervals which corresponds to a squarefree Stanley decomposition of $\Bbbk[\Delta]$ of Stanley depth 1, see Figure 1.

In this example, we actually have

$$\operatorname{depth}(\Bbbk(\Delta)) = \operatorname{sdepth}(\Bbbk[\Delta]) = 1 < 2 = \dim(\Bbbk[\Delta])$$

In particular, Δ is not Cohen-Maccaulay.

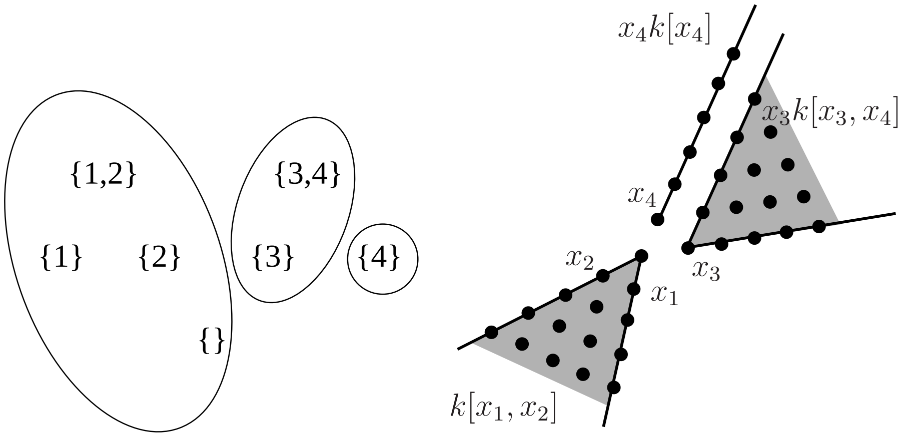

Figure 1: Face poset partition and corresponding Stanley decomposition

3. Cover depth

There is a natural weaker version of Stanley decompositions, called Stanley covers, that was presented by J. Herzog in [16]. The corresponding notion of cover depth has a number of interesting properties.

A *Stanley cover* $\mathscr{C}$ of M is a decomposition of M as a (not necessarily direct) sum of Stanley spaces:

$$\mathscr{C}: \quad M = \sum_{i=1}^{k} m_i \Bbbk[Z_i]$$

where $m_i\Bbbk[Z_i]$ are Stanley spaces of M.

The *cover depth* of such a decomposition is defined as the minimal dimension of the Stanley spaces appearing in the sum.

Then *cover depth* of the module M is the maximum of all cover depths of Stanley covers $\mathscr{C}$ of M.

$$\operatorname{cdepth}(M) = \max_{\mathscr{C}} \operatorname{cdepth}(\mathscr{C})$$

It is clear that every Stanley decomposition also is a Stanley cover and thus

$$\operatorname{sdepth}(M) \leq \operatorname{cdepth}(M)$$

Furthermore, every monomial ideal $M = I$ has cover depth $\operatorname{cdepth}(I) = n = \dim(I)$, since $\sum_{i=1}^{r} m_i S$ is Stanley cover of I, if the m_i are chosen as a minimal set of generators of I.

There is a general relation between cover depth, depth and dimension of M, if the module has for instance the form $M = I/J$. We prove an auxiliary result first. Recall that $\operatorname{Ass}(M)$ is the set of all prime ideals associated to M, that is it consists of all prime ideals of S which are the annihilator of some element $m \in M$.

Lemma 3.1. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then there exists a Stanley cover*

$$M = \sum_{i=1}^{k} m_i \Bbbk[Z_i]$$

such that $(x_j \,:\, j \notin Z_i) \in \operatorname{Ass}(M)$ for all $i = 1, \ldots, k$.

Proof. Since M is a finitely generated $\mathbb{Z}^n$-graded S-module, we have

$$M = Su_1 + \ldots + Su_l$$

for some homogenous elements $u_1, \ldots, u_l \in M$. It also holds that $Su_i \cong S/I$ for some monomial ideal I and $\operatorname{Ass}(Su_i) \subseteq \operatorname{Ass}(M)$ for all u_i. Hence, we may assume $M = S/I$ for a monomial ideal I. Let $I = \cap_{i=1}^{r} Q_i$ be a minimal decomposition of I into irreducible monomial ideals.

Consider the module S/Q for some irreducible monomial ideal Q, say, $Q = (x_{i_1}^{t_1}, \ldots, x_{i_s}^{t_s})$. We write $Z'_Q = \{x_{i_1}, \ldots, x_{i_s}\}$ and $Z_Q = \{x_1, \ldots, x_n\} \setminus Z'_Q$. It is a basic fact that $\operatorname{Ass}(S/Q) = \{(x_{i_1}, \ldots, x_{i_s})\}$. It is also easy to see that a Stanley cover of S/Q is given by

$$S/Q = \sum_{m \in \Bbbk[Z'_Q], m \notin Q} m\Bbbk[Z_Q]$$

where $(x_i : x_i \notin Z_Q) = \operatorname{Ass}(S/Q)$.

Now, the statement follows directly from

$$M = S/I = \sum_{i=1}^{r} S/Q_i = \sum_{i=1}^{r} \sum_{m \in \mathbb{k}[Z'_{Q_i}], m \notin Q_i} m\mathbb{k}[Z_{Q_i}]$$

and the fact that $\mathrm{Ass}(M) = \cup_{i=1}^{r} \mathrm{Ass}(S/Q_i)$. □

Remark 3.2. Let us point out the resemblance of this result with another notion, called the fdepth, [15]. Here, let $\mathscr{F}$ be a *prime filtration* of M, i.e.,

$$\mathscr{F} \; : \; 0 = M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

where $M_i/M_{i-1} \cong (S/\mathscr{P}_i)(-a_i)$ with $a_i \in \mathbb{Z}^n$ and $\mathscr{P}_i$ a monomial prime ideal. Then it is easy to see that this yields a Stanley decomposition of M. However, in contrast to Lemma 3.1 the occuring prime ideals $\mathrm{supp}(\mathscr{F})$ are in general not contained in $\mathrm{Ass}(M)$. Actually, the maximal Stanley depth that can be achieved by prime filtrations, $\mathrm{fdepth}(M)$, is always equal or smaller than $\mathrm{depth}(M)$. We also remark that $\mathrm{fdepth}(\mathbb{k}[\Delta])$ can be read off the poset of Δ, see Corollary 2.8 in [15]. In particular, it is independent of the characteristic of the base field.

Let us write $\mathrm{mon}(M)$ for the set of homogeneous elements of our $\mathbb{Z}^n$-graded module M. For any $m \in \mathrm{mon}(M)$, let $\mathrm{cdepth}(m)$ denote the maximal dimension of a Stanley space generated by m, that is

$$\mathrm{cdepth}(m) := \max\{|Z_i| : m\mathbb{k}[Z_i] \text{ is a Stanley space of } M\}.$$

It is easy to see that $\mathrm{cdepth}(m) = \dim(Sm)$. The reader perhaps suspects a relation to the cover depth of M and will not be disappointed.

Theorem 3.3 ([16])**.** *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then the following statements hold:*

(i) $\min\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\} \leq \min\{\dim S/\mathscr{P} \; : \; \mathscr{P} \in \mathrm{Ass}(M)\}$

(ii) $\min\{\dim S/\mathscr{P} \; : \; \mathscr{P} \in \mathrm{Ass}(M)\} \leq \mathrm{cdepth}(M)$

(iii) $\mathrm{cdepth}(M) \leq \max\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$

(iv) $\max\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\} = \max\{\dim S/\mathscr{P} \; : \; \mathscr{P} \in \mathrm{Ass}(M)\} = \dim(M)$

Furthermore, if $\dim_{\mathbb{k}} M_a \leq 1$ *holds for all* $a \in \mathbb{Z}^n$ *then*

$$\min\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\} = \mathrm{cdepth}(M)$$

Proof. (i). For every $\mathscr{P} \in \mathrm{Ass}(M)$, it holds that $S/\mathscr{P} \cong Sm$ for some homogeneous m. Thus, $\dim(S/\mathscr{P}) = \dim(Sm) = \mathrm{cdepth}(m)$.

(ii) is a direct consequence of Lemma 3.1.

(iii). Choose a Stanley cover $\mathscr{C}$ of optimal cover depth $\mathrm{cdepth}(M)$. Then $\mathscr{C}$ contains a Stanley space $m\Bbbk[Z]$ such that $\mathrm{cdepth}(M) = |Z|$. Hence, $\mathrm{cdepth}(M) \leq \mathrm{cdepth}(m) \leq \max\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$.

(iv). Recall that $\dim(M) = \max\{\dim S/\mathscr{P} \ : \ \mathscr{P} \in \mathrm{Ass}(M)\}$. Now, let $m \in M$ be homogeneous such that $\mathrm{cdepth}(m) = \max\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$. It holds that $\mathrm{cdepth}(m) = \dim(Sm) \leq \dim(M)$. For the converse inequality, let $\mathscr{P} \in \mathrm{Ass}(M)$ be such that $\dim(S/\mathscr{P}) = \dim(M)$. For $m \in M$ homogeneous with $Sm \cong S/\mathscr{P}$ (up to a shift), it follows that $\dim(M) = \dim(Sm) = \mathrm{cdepth}(m) \leq \max\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$.

Finally, let $\dim_\Bbbk M_a \leq 1$ for all $a \in \mathbb{Z}^n$. Choose again a Stanley cover $\mathscr{C}$ such that $\mathrm{cdepth}(\mathscr{C}) = \mathrm{cdepth}(M)$ and let $m \in M$ be a homogeneous element with $\mathrm{cdepth}(m) = \min\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$. By our assumption, $m \in m_i\Bbbk[Z_i]$ for some Stanley space in $\mathscr{C}$. Hence, $\mathrm{cdepth}(m) \geq |Z_i| \geq \mathrm{cdepth}(\mathscr{C}) = \mathrm{cdepth}(M)$. □

Corollary 3.4. $\mathrm{depth}(M) \leq \min\{\dim S/\mathscr{P} \ : \ \mathscr{P} \in \mathrm{Ass}(M)\} \leq \mathrm{cdepth}(M) \leq \dim(M)$, *with equality in the second place, if* $\dim_\Bbbk M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$.

Proof. The first inequality can be found in [7, Proposition 1.2.13]. The other statements follow directly from Theorem 3.3. □

Example 3.5. Let $S = \Bbbk[x]$, and let $F := Se_1 \oplus Se_2$ $\mathbb{Z}$–graded via $\deg(e_1) = \deg(e_2) = 1$. We define M as the quotient of F modulo the $\mathbb{Z}$-graded submodule generated by $xe_1 + xe_2$. Let us denote by w_1, w_2 the images of e_1, e_2 in M. Then $Sw_1 + Sw_2$ is a Stanley cover, hence $\mathrm{cdepth}(M) = 1 = \dim(M)$. Since $\mathrm{Ann}(w_1) = \mathrm{Ann}(w_2) = (0)$ and $\mathrm{Ann}(w_1 + w_2) = (x)$, we get $\mathrm{Ass}(M) = \{(0), (x)\}$. Therefore, $\mathrm{cdepth}(M) = 1$, but $\min\{\dim S/\mathscr{P} \ : \ \mathscr{P} \in \mathrm{Ass}(M)\} = 0$.

Question 3.6. Does $\mathrm{depth}(M) \leq \min\{\mathrm{cdepth}(m) : m \in \mathrm{mon}(M)\}$ also hold without the assumption $\dim_\Bbbk M_a \leq 1$ for all $a \in \mathbb{Z}^n$?

For Stanley-Reisner rings of simplicial complexes, the cover depth has a very concrete geometric meaning. We see that the difference between the cover depth and the dimension is a measure of the non-pureness of simplicial complexes.

Corollary 3.7. *Let* $M = S/I = k[\Delta]$ *be the Stanley-Reisner ring of a simplicial complex* Δ*. Then*

$$\mathrm{cdepth}(M) = \min\{|F| \ : \ F \textit{ is a facet of } \Delta\}$$

and

$$\dim(M) = \max\{|F| \,:\, F \text{ is a facet of } \Delta\}.$$

4. Greedy depth

In the case of Stanley-Reisner rings, the geometric pictures in Section 2 lead immediately to a naive way of producing Stanley decompositions. As we already noted, there exist explicit calculations of the Stanley depth for certain monomial ideals. We would like to use those to bound the Stanley depth of Stanley-Reisner rings.

We begin with an illustrating example.

Example 4.1. Let $S = \mathbb{k}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be the polynomial ring in $2n$ variables and consider the S-module $M = S/I$ where the monomial ideal I is given by

$$I = (x_1,\ldots,x_n)(y_1,\ldots,y_n)$$

that is, I is generated by all monomials x_iy_j for $1 \le i,j \le n$. We have described the special case $n = 2$ in Example 2.3.

Let $\mathscr{D}$ be a Stanley decomposition of M. Then $1 \in M$ is contained in exactly one Stanley space $1\mathbb{k}[Z]$ of $\mathscr{D}$ for $Z \subseteq \{1,\ldots,n\}$. Therefore, either $Z = \emptyset$ or Z is contained in only one of the sets $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$. Let's w.l.o.g. assume that $Z \subseteq \{y_1,\ldots,y_n\}$. Since $\mathscr{D}$ is a Stanley decomposition of M, it induces a Stanley decomposition of the maximal ideal $\mathfrak{m} = (x_1,\ldots,x_n) \subset \mathbb{k}[x_1,\ldots,x_n]$. Thus $\operatorname{sdepth}(M) \le \operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$ (see Example 5.2). By choosing first a Stanley decomposition $\mathscr{D}'$ of $\mathfrak{m}$ with $\operatorname{sdepth}(\mathscr{D}') = \operatorname{sdepth}(\mathfrak{m})$, and then adding the Stanley space $\mathbb{k}[y_1,\ldots,y_n]$ to obtain a Stanley decomposition of M, we get:

$$\operatorname{sdepth}(M) = \left\lceil \frac{n}{2} \right\rceil.$$

Question 4.2. The ideal I in the previous example can be seen as the edge ideal of a complete bipartite graph on $2n$ vertices. Here is a generalization: Let I be the edge ideal of a complete k-partite hypergraph $\mathbf{H}_n^k$. Here, $\mathbf{H}_n^k$ has kn vertices divided into k independent sets $V^{(i)}$ (for $i = 1,\ldots,k$) each with n vertices $v_1^{(i)},\ldots,v_n^{(i)}$, and $\mathbf{H}_n^k$ has n^k (hyper-)edges consisting of exactly k vertices. Then I is a squarefree monomial ideal in the polynomial ring $\mathbb{k}[v_j^{(i)} \,:\, i \in \{1,\ldots,k\},\, j \in \{1,\ldots,n\}]$:

$$I = (v_1^{(1)},\ldots,v_n^{(1)}) \cdots (v_1^{(k)},\ldots,v_n^{(k)})$$

What is $\operatorname{sdepth}(S/I)$ in this case?

Essentially, Example 4.1 uses a naive construction that assembles two 'local' Stanley decompositions into a 'global' one. Let us make this approach rigorous in general.

Let $M = S/I = k[\Delta]$ be a Stanley-Reisner ring of a simplicial complex Δ and let $\mathscr{F} = \{F_1, \dots, F_l\}$ be an ordering of the facets of Δ. For $i = 1, \dots, l$, define the squarefree monomial ideal I_i in $\Bbbk[F_i]$ by

$$I_i = (x^\alpha : \alpha \in \{0,1\}^n, \operatorname{supp}(\alpha) \subseteq F_i, \operatorname{supp}(\alpha) \not\subseteq F_j\ \forall j < i)$$

Note that I_i is generated by the squarefree monomials corresponding to the minimal faces in F_i that are not contained in any previously considered facet F_j, $j < i$. The squarefree monomials in I_i correspond to an order filter in the face lattice of Δ with maximal element F_i. Now, construct a Stanley decomposition $\mathscr{D}_i$ of optimal Stanley depth for every I_i

$$\mathscr{D}_i \ : \quad I_i = \bigoplus_j m_j^{(i)} \Bbbk[Z_j^{(i)}]$$

and assemble those decompositions into a Stanley decomposition $\mathscr{D}$ of M:

$$\mathscr{D} \ : \quad M = \Bbbk[\Delta] = \bigoplus_i \bigoplus_j m_j^{(i)} \Bbbk[Z_j^{(i)}]$$

Definition 4.3. A Stanley decomposition $\mathscr{D}$ of $\Bbbk[\Delta]$ is called *greedy* if it can be constructed as described above for some ordering of the facets of Δ. The *greedy depth*, $\operatorname{gdepth}(\Bbbk[\Delta])$, is defined as the maximal depth of all greedy Stanley decompositions of Δ.

By definition, the following chain of inequalities holds:

$$\operatorname{gdepth}(\Bbbk[\Delta]) \le \operatorname{sdepth}(\Bbbk[\Delta]) \le \operatorname{cdepth}(\Bbbk[\Delta])$$

Note that in Example 4.1 the greedy depth coincides with the Stanley depth.

We will see that there is a straightforward relation between greedy depth and shellability. Recall the definition of (non-pure) shellability as in [6, Proposition 2.5].

Definition 4.4. Let Δ be a (not necessarily pure) simplicial complex with set of facets $\mathscr{F}(\Delta)$. Then, Δ is *shellable*, if there is a linear ordering $F_1, \dots, F_l$ of the facets in $\mathscr{F}(\Delta)$ such that the set

$$\{F \subset F_i : F \notin \langle F_1, \dots, F_{i-1} \rangle\}$$

has a unique minimal element for all $i = 2, \dots, l$.

It is easy to see that a linear ordering $F_1,\dots,F_l$ of the facets of Δ is a shelling order if and only if all ideals I_i as defined in the greedy construction for that ordering are principal.

Proposition 4.5. *If Δ is a shellable simplicial complex, then*

$$\operatorname{gdepth}(\Bbbk[\Delta]) = \operatorname{cdepth}(\Bbbk[\Delta])$$

If Δ is pure, then the converse holds as well.

Proof. Let $F_1,\dots,F_l$ be a shelling ordering of the facets of Δ. Then, for $i = 1,\dots,l$, the ideal I_i is principal and generated by the squarefree monomial corresponding to the unique minimal element of

$$\{F \subset F_i : F \notin \langle F_1,\dots,F_{i-1}\rangle\}$$

Thus, $I_i \subseteq \Bbbk[F_i]$ has a Stanley decomposition of depth $|F_i|$ and the Stanley depth of the corresponding greedy decomposition is

$$\min\{|F_i| : i = 1,\dots,l\} = \operatorname{cdepth}(\Bbbk[\Delta]).$$

If Δ is pure, then $\operatorname{gdepth}(\Bbbk[\Delta]) = \operatorname{cdepth}(\Bbbk[\Delta])$ if and only if the Stanley depth of all ideals I_i is equal to $|F_i|$ for some ordering $F_1,\dots,F_l$ of the facets. But this is equivalent to I_i being principal because every Stanley decomposition of I_i has exactly one full-dimensional Stanley space. Finally, all I_i are principal if and only if $F_1,\dots,F_l$ is a shelling order. □

For Δ being a Cohen-Macaulay and thus pure simplicial complex, it has been shown in [13] that

$$\operatorname{sdepth}(\Bbbk[\Delta]) \geq \operatorname{depth}(\Bbbk[\Delta]) \quad \Leftrightarrow \quad \Delta \text{ is partitionable.}$$

Proposition 4.5 can be seen as a weaker analog of this.

Remark 4.6. For a pure simplicial complex Δ there is another well-known criterion for shellability [11, 15]: Δ is shellable if and only if $\Bbbk[\Delta]$ is clean, that is there exists a prime filtration such that $\operatorname{supp}(\mathscr{F}) = \min(M)$. In other words, $\operatorname{fdepth}(\Bbbk[\Delta]) = \dim(\Bbbk[\Delta])$, see Remark 3.2.

5. Construction of extreme examples

The main problem in this area are missing examples. Even in the situation of Stanley-Reisner rings, it is not known precisely how much the various geometric notions (sdepth, cdepth, gdepth) can differ.

We found it natural to ask the following question.

Problem 5.1. Is there a constant $C < n$ such that for every Stanley-Reisner ring M the inequality

$$\operatorname{cdepth}(M) \leq C \operatorname{sdepth}(M)$$

holds?

We note that the inequality is always true for $C \geq n$.

Our motivation is clear: The existence of such a $C < n$ would imply a non-trivial weaker version of Stanley's conjecture:

$$\operatorname{depth}(M) \leq \operatorname{cdepth}(M) \leq C \operatorname{sdepth}(M).$$

Let's look at an example first.

Example 5.2. Consider the maximal monomial ideal $\mathfrak{m} \subset S$ and its powers $\mathfrak{m}^k$. It is clear that $\operatorname{cdepth}(\mathfrak{m}^k) = n$. Note that $\mathfrak{m}^k$ is not squarefree for $k > 1$.

Using some simple counting argument, it has been shown in [10] that

$$\operatorname{sdepth}(\mathfrak{m}^k) \leq \left\lceil \frac{n}{k+1} \right\rceil$$

and the author conjectures that actually equality holds for $k > 1$.

It is known that $\operatorname{sdepth}(\mathfrak{m}) = \left\lfloor \frac{n}{2} \right\rfloor$ but the proof is already quite involved ([5]).

Example 5.3 (Example 2.2 continued)**.** Recall the squarefree monomial ideal

$$I_{n,d} = (x^\alpha \ : \ \alpha \in \{0,1\}^n, |\alpha| = d)$$

We showed in Example 2.2 that $\operatorname{cdepth}(I_{n,d}) = n$ and $\operatorname{sdepth}(I_{n,d}) \leq \frac{n+d^2}{d+1}$.

We invite the reader to check that for any $C \geq 1$ the choice $n = 2C(2C-1)$ and $d = 2C-1$ implies that $\operatorname{sdepth}(I_{n,d}) < n/C$, so

$$\operatorname{cdepth}(I_{n,d}) > C \operatorname{sdepth}(I_{n,d}).$$

Example 5.2 and Example 5.3 show that a constant C as in Problem 5.1 cannot exist for general S-modules M. However, we think that it should be possible to find counterexamples even for Stanley-Reisner rings. This would again support the common belief that one cannot solve Stanley's conjecture by purely combinatorial or geometric methods. In the following, we present some partial results.

Let us introduce a class of simplicial complexes that behave nicely with respect to their depth and greedy depth.

Definition 5.4. A simplicial complex Δ is called (n,k)-*extremal* if Δ is pure of dimension $n-1$ and has the following two properties:

(i) Any two facets intersect in a face having at most k vertices.

(ii) For any facet F and any face $G \subseteq F$ with $|G| = k$ there exists a facet F' such that $F \cap F' = G$.

Lemma 5.5. *Let Δ be a (n,k)-extremal simplicial complex. Then*

$$\operatorname{gdepth}(\Bbbk[\Delta]) \leq \operatorname{sdepth}(I_{n,k+1})$$

Proof. Choose an ordering $F_1,\dots,F_l$ of the maximal faces of Δ. Then in the greedy construction $I_l = I_{n,k+1}$. □

Proposition 5.6. *Let Δ be a (n,k)-extremal simplicial complex. Then*

$$\operatorname{depth}(\Bbbk[\Delta]) \leq k+1$$

If Δ is $(n,1)$-extremal (for $n \geq 1$) and connected, then $\operatorname{depth}(\Bbbk[\Delta]) = 2$.

Proof. It is well known that the depth of a Stanley-Reisner ring can be calculated as follows:

$$\operatorname{depth}(\Bbbk[\Delta]) = \min\{i : \exists F \in \Delta : \tilde{H}_{i-|F|-1}(\operatorname{lk}F;\Bbbk) \neq 0\}.$$

Let F be any face of Δ with k vertices. Then, $\operatorname{lk}F$ is a disjoint union of at least two simplices with $n-k \geq 1$ vertices each. In particular, $\operatorname{lk}F$ is disconnected and thus $\tilde{H}_1(\operatorname{lk}F;\Bbbk) \neq 0$. This show that $\operatorname{depth}(\Bbbk[\Delta]) \leq k+1$.

Now, assume that Δ is $(n,1)$-extremal and connected with $\operatorname{depth}(\Bbbk[\Delta]) = i < 2$.

If $i = 0$, then there is a face F with $\tilde{H}_{-|F|-1}(\operatorname{lk}F;\Bbbk) \neq 0$. This implies $|F| = 0$ and thus $\operatorname{lk}F = \Delta$. Therefore, Δ has non-trivial homology in dimension -1, hence, $\Delta = \{\emptyset\}$, a contradiction to $\dim(\Delta) = n-1 \geq 0$.

If $i = 1$, then there is a face F with $\tilde{H}_{-|F|}(\operatorname{lk}F;\Bbbk) \neq 0$. Again, we must have $|F| = 0$ and $\operatorname{lk}F = \Delta$. However, Δ was assumed to be connected and thus $\tilde{H}_0(\Delta;\Bbbk) = 0$, a contradiction. □

We think that (n, k)–extremal simplicial complexes are good candidates when trying to prove that there is no constant $C < n$ as in Problem 5.1. This leads to the following question.

Question 5.7. For which $1 \leq n$ and $k \leq n-1$ does some (n,k)-extremal simplicial complex Δ exist?

We can give constructions in some special cases.

Example 5.8. The boundary of the n-simplex is $(n,n-1)$-extremal.

Example 5.9. Cut off each vertex of a regular n-simplex by an affine hyperplane intersecting the adjacent edges at their midpoints. The $n+1$ intersection polytopes are $(n-1)$-simplices that form an $(n,1)$-extremal simplicial complex. For $n=3$, this yields the subcomplex of the boundary of the 3-dimensional crosspolytope that one gets by leaving out 4 triangles that pairwise do not share an edge.

Remark 5.10. For $n \geq 9$, Lemma 5.5 implies

$$\mathrm{gdepth}(\Bbbk[\Delta]) \leq \mathrm{sdepth}(I_{n,2}) \leq \frac{n+4}{3} < \frac{n}{2} = \frac{\mathrm{cdepth}(\Bbbk[\Delta])}{2}$$

for any $(n,1)$-extremal simplicial complex Δ. In particular, in view of Proposition 4.5 this means that such complexes are extremely non-shellable.

Of course, showing something about the greedy depth is not what we are really after. The real question is if we can prove a similar inequality for the Stanley depth. In particular, we are interested in the behaviour of the Stanley depth of (n,k)-extremal simplicial complexes. An answer to the next question could show the way to a general method.

Question 5.11. Let C be fixed and let Δ be a $(n,1)$-extremal simplicial complex (e.g. the one in Example 5.9).

(i) Does $\mathrm{gdepth}(\Bbbk[\Delta]) = \mathrm{sdepth}(\Bbbk[\Delta])$ hold, at least for sufficiently large n?

(ii) Does $\mathrm{sdepth}(\Bbbk[\Delta]) < \frac{n}{C}$ hold for sufficiently large n?

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