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SECOND ORDER EVOLUTION INCLUSIONS GOVERNED BY SWEEPING PROCESS IN BANACH SPACES

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In this paper we prove two existence theorems concerning the existence of solutions for second order evolution inclusions governed by sweeping process with closed convex sets depending on time and state in Banach spaces. This work extends some recent existence theorems cncerning sweeping process from Hilbert spaces to Banach spaces.

1. Introduction

Differential inclusions represent a relevant generalization of differential equations. Moreover, it has several applications in different branches of mathematical sciences such as Control Theory Viscosity, Optimization and Mechanical problems (see [3,11]).

In his leading paper, Moreau [17] proposed and studied the following evolution inclusion (differential inclusion) governed by sweeping process of first order:

$$(\mathrm{Q}) \begin{cases} -u'(t) \in N_{C(t)}(u(t)) \text{ a.e. on } I = [0,T], \\ u(0) = u_0. \end{cases}$$

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where C is a set-valued function from the interval I to the family of non empty closed convex subsets of a Hilbert space H and $N_{C(t)}(u(t))$ is the normal cone of the subset $C(t)$ at the point $u(t)$.

Since then, an important improvements have been developed by weakening assumptions in order to obtain the most general result of existence for sweeping process. For these results, we refer to [4,5,6,7,8,9,10,14,18,19].

To the best of our knowledge, except a recent Ph.D. thesis [2], all previous results concerning the existence of solutions for sweeping process were considered in Hilbert spaces.

Our aim in the present paper is to extend some existence theorems concerning sweeping process from Hilbert spaces to Banach spaces. More precisely, we prove, via discretization technique [9,10], the existence of solutions for the following two second order evolution inclusions problems governed by sweeping process in Banach spaces:

Problem (P_1).

Find two continuous functions $u : [0,T] \to X$ and $v : [0,T] \to X$ such that

$$(P_1)\begin{cases} v(t) = b + \int_0^t u(s)ds, \, \forall t \in I, \\ u(t) \in C(t, v(t)), \, \forall t \in I, \\ u(t) = J^*(u^*(t)), \, \forall t \in [0,T], \\ u^*(t) = J(a) + \int_0^t (u^*)'(s)ds, \, \forall t \in I, \\ -(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + F(t, v(t), u(t)), \text{ a.e. for } t \in I. \end{cases}$$

Where C is a set-valued function defined from $[0,T] \times X$ to the family of non empty closed convex subsets of a separable $p-$uniformly convex and $q-$uniformly smooth Banach space X, (p, $q > 1$), J is the normalized duality mapping, $N_{C(t,x)}(y)$ is the normal cone of the set $C(t,x)$ at the point y and a, b are two given points in X such that $a \in C(0,b)$. The set valued function F is an upper semi continuous defined from $[0,T] \times X \times X$ to the family of non empty convex weakly compact sets of the topological dual space X^* of X such that

$$F(t,x,y) \subseteq (1 + ||x|| + ||y||)B_{X^*}, \forall (t,x,y) \in [0,T[\times X \times X. \tag{1.1}$$

Problem (P_2).

Let z, w be two continuous functions defined from $[-r,0]$ to a separable $p-$uniformly convex and $q-$uniformly smooth Banach space X, (p, $q > 1$).

Find two continuous functions $u : [-r,T] \to X$ and $v : [-r,T] \to X$ such that

$$(\mathrm{P}_2)\begin{cases} v(t) = z(t),\ \forall t \in\ [-r,0];\ v(t) = z(0) + \int_0^t u(s)ds,\ \forall t \in [0,T], \\ u(t) \in C(t,v(t)),\ \forall t \in [0,T], \\ u(t) = J^*(u^*(t)),\ \forall t \in [-r,T], \\ u^*(t) = J(w(t)), \forall t \in [-r,0], \\ u^*(t) = J(w(0)) + \int_0^t (u^*)'(s)ds, \forall t \in [0,T], \\ -(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + G(t,\tau(t)v,\tau(t)u), \text{ for a.e. on } [0,T]. \end{cases}$$

Where G is a set-valued function defined from $[0,T] \times C_X([-r,0]) \times C_X([-r,0])$ to the family of all non empty convex weakly compact subsets of X^*such that

$$\begin{aligned} &G(t,f,g) \subseteq\ (1 + \|f(0)\|\ + \|g(0)\|)B_{X^*}, \\ &\forall (t,f,g) \in [0,T\ [\times C_X([-r,0]) \times C_X([-r,0]) \end{aligned} \tag{1.2}$$

and for each $t \in [0,T]$, $\tau(t) : C_X([-r,0]) \longrightarrow C_X([0,T])$, $(\tau(t)g)(s) = g(s+t), \forall s \in [-r,0]$.

We note that if X is a Hilbert space, then J is equal to the identity operator on X and X is 2-uniformly convex and 2-uniformly smooth Banach space. So, if X is a Hilbert space, then Problem (P_1) takes the form:

$$(\mathrm{Q}_1)\begin{cases} v(t) = b + \int_0^t u(s)ds,\ \forall t \in I = [0,T], \\ u(t) \in C(t,v(t)),\ \forall t \in I, \\ u(t) = a + \int_0^t\ u'(s)ds,\ \forall t \in I, \\ -u'(t) \in N_{C(v(t))}(u(t)) + F(t,u(t),v(t)), \text{ a.e. for } t \in I. \end{cases}$$

Then, Problem (P_1) extends problem (Q_1), which was studied in Prop. 4.2 ([9]), from Hilbert spaces to Banach spaces.

Moreover, Al-yusof [2] proved in Th. 2.2.3 the existence of solutions of the second order sweeping process :

$$(\mathrm{Q}_2)\begin{cases} J(v(t)) = J(b) + \int_0^t J(u(s))ds,\ \forall t \in I, \\ u(t) \in C(v(t)),\ \forall t \in I, u(0) = u_0, \\ -(J(u))'(t) \in N_{C(v(t))}(u(t)) + F(t,J(v(t)),J(u(t))) \\ +G(t,J(v(t)),J(u(t)), \text{ a.e. for } t \in I. \end{cases}$$

Where C is a set-valued function defined from a closed neighborhood of a given point b in a separable $p-$uniformly convex and $q-$uniformly smooth Banach space X and taking a closed convex values in X and F is a set-valued function defined from $[0,\infty[\times X^* \times X^*$, taking convex weakly compact values in the topological dual,X^*, of X and scalarly upper semi continuous on the set $[0,\frac{\nu}{l}] \times \{J(x) \times J(y) : y \in C(x)\}$, for some positive constant ν and l . The set valued G is a uniformly continuous set-valued function defined from $[0,\infty[\times X^* \times X^*$and taking non empty compact values in the topological dual, X^*, of X and such that

$$F(t,x,y) \subseteq (1+||x|| + ||y||)B_{X^*}, \forall (t,x,y) \in [0,\infty[\times X^* \times X^*. \tag{1.3}$$

$$G(t,x,y) \subseteq (1+||x|| + ||y||)B_{X^*}, \forall (t,x,y) \in [0,\infty[\times X^* \times X^*. \tag{1.4}$$

We would like to mention that in (Q_2) the sweeping process depends on state only, while in our problems the sweeping process depends on state and time. Also, in Problem (P_2) there is a delay while in (Q_2) there is not delay. Finally, our work extends many results in the literature concerning the existence of solutions for some evolution inclusions governed by sweeping process from Hilbert spaces to Banach spaces. For more informations about differential inclusions we refer to [15,16,18].

2. Preliminaries and Notations

Let $I = [0,T]$, $(T > 0)$ and X be a Banach space with topological dual space X^*. Let

$$\begin{aligned} B &= \{x \in X : ||x|| \leq 1\}, \\ B_* &= \{z \in X^* : ||z|| \leq 1\} \end{aligned}$$

and

$$S = \{x \in X : ||x|| = 1\}.$$

Definition 2.1. ([1], Def. 2.1.1). The Banach space X is said to be strictly convex if

$$x, y \in S \text{ with } x \neq y \Rightarrow ||(1-\lambda)x + \lambda y|| < 1 \text{ for } \lambda \in]0,1[.$$

This means that the unit sphere S contains no line segments.

Definition 2.2. ([1], Def. 2.2.1). A Banach space X is said to be uniformly convex if for any $\varepsilon, 0 < \varepsilon \leq 2$, the inequalities $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon)$ such that $||\frac{1}{2}(x+y)|| \leq 1-\delta$.

Definition 2.3. ([1], Def. 2.3.1). The function

$$\delta_X(\varepsilon) = \inf\{1 - ||\frac{1}{2}(x+y)|| : ||x|| \leq 1, ||y|| \leq 1, ||x-y|| \geq \varepsilon\}$$

is said to the modulus of convexity of X.

Definition 2.4. ([12], [13]). Let $p > 1$ be a real number. A Banach space X is said to be $p-$ uniformly convex if there exist a constant $\lambda > 0$ such that

$$\delta_X(\varepsilon) \geq \lambda \varepsilon^p, \forall \varepsilon \in (0,2].$$

Remark 2.5. (1) Every $p-$ uniformly convex space ($p > 1$) is uniformly convex.

(2) Every Hilbert space is 2–uniformly convex.

Definition 2.6. ([1], Def. 2.4.1), Let X^*be the topological dual of a Banach X. Then the multivalued mapping

$$\begin{aligned} J &: X \to 2^{X^*}, \\ J(x) &= \{y \in X^* :< x, y >= ||x||^2 = ||y||^2\} \end{aligned}$$

is said to be the normalized duality mapping (or duality mapping) in X. And the multivalued mapping

$$\begin{aligned} J^* &: X^* \to 2^{X}, \\ J^*(y) &= \{x \in X :< y, x >= ||x||^2 = ||y||^2\} \end{aligned}$$

is called the normalized duality mapping (or duality mapping) in X^*.

In the following theorem we recall some properties of the duality mapping.

Theorem 2.7. *([1], Prop. 2.4.5, 2.4.12 and 2.4.15).*

1. *If X is a Hilbert space then $J(x) = \{x\}$, for all $x \in X$.*
2. *For each $x \in X$, $J(x)$ is a non empty closed convex and bounded subset of X^*.*
3. *$J(\lambda x) = \lambda J(x)$, $\forall x \in X$ and $\forall \lambda \in \mathbb{R}$.*
4. *If X^* is strictly convex, J is single valued.*
5. *If X is strictly convex, J is one to one, i.e. $x \neq y \Rightarrow J(x) \cap J(y) = \phi$.*
6. *If X is reflexive, then*

$$\cup\{J(x) : x \in X\} = X^*.$$

7. *If X is reflexive strictly convex space with strictly convex conjugate space X^*, then J and J^* are one-to-one, onto and signle-valued mapping and*

$$J^{-1} = J^*,\ JJ^* = I_{X^*} \text{ and } J^*J = I_X,$$

where I_X is the identity mapping on X and $I_{X^*}$is the identity mapping on X^*.

For more properties of the duality mapping we refer to [1].

Definition 2.8. ([1], Def. 2.6.1). A Banach space X is said to be smooth if for each $x \in S$ there exists a unique functional $f_x \in X^*$ such that $< x, f_x >= ||x||$ and $||f_x|| = 1$.

For example L^P, $p > 1$ is smooth but L^1 and $L^\infty$are not smooth.

Definition 2.9. ([1], Def. 2.7.1). The function

$$\begin{aligned}
\rho_X \quad &: \quad [0,\infty[\to [0,\infty[\\
\rho_X(t) \quad &= \quad \sup\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| = 1, ||y|| = t\} \\
&= \quad \sup\{\frac{1}{2}(||x+ty|| + ||x-ty||) - 1 : ||x|| = ||y|| = 1\}, t > 0,
\end{aligned}$$

is said to be the modulus of smoothness of X. It is easy to check that $\rho_X(0) = 0$ and $\rho_X(t) \geq 0$ for all $t \geq 0$.

Definition 2.10. ([1], Def. 2.8.1). The Banach space X is said to be uniformly smooth if

$$\rho_X'(0) = \lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

For example ℓ_p spaces, $(1 < p \leq 2)$, are uniformly smooth.

Definition 2.11. ([12],[13]). Let $q > 1$ be a real number. A Banach space X is said to be $q-$ uniformly smooth if there exist a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \forall t > 0.$$

Clearly, every $q-$uniformly smooth is uniformly smooth. Indeed

$$0 \leq \lim_{t\to 0}\frac{\rho_X(t)}{t} \leq \lim_{t\to 0} ct^{q-1} = 0.$$

So,

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

Remark 2.12. Every Hilbert space is 2- uniformly smooth.

Theorem 2.13. *([12],[13]). Let X be a Banach space and $p > 1$.*

1. *If X is $p-$uniformly convex, then X^* is $p'-$uniformly smooth where $p' = \frac{p}{p-1}$.*
2. *If X is $p-$uniformly smooth, then X^* is $p'-$uniformly convex where $p' = \frac{p}{p-1}$.*

Now, let X be a Banach space and X^* be its topological dual. Let

$$V : X^* \times X \to \mathbb{R}, V_* : X \times X^* \to \mathbb{R}$$

be two functions defined by:

$$\begin{aligned} V(\varphi,x) &= ||\varphi||^2 - 2 < \varphi,x > +||x||^2, \\ V_*(x,\varphi) &= V(\varphi,x). \end{aligned}$$

Definition 2.14. ([2], Def. 0.0.6). Let X be a Banach space, E be a non empty subset of X and $\varphi \in X^*$. If there exists a point $z \in E$ satisfying

$$V(\varphi,z) = d_E^V(\varphi),$$

where $d_E^V(\varphi) = \inf_{x\in E} V(\varphi,x)$, then z is called a generalized projection of φ onto E.

The set of all such points is denoted by $\pi_E(\varphi)$, i.e.

$$\pi_E(\varphi) = \{z \in E : V(\varphi,z) = d_E^V(\varphi)\} \subseteq X.$$

Now, we list in the following theorem some properties of V and $\pi_E(\varphi)$ (see [2], Ch. 0).

Theorem 2.15. *Let X be a Banach space and X^*be its topological dual.*

1. *$V(J(x),x) = 0$.*
2. *If X is uniformly convex or uniformly smooth, then*

$$V(\varphi,x) = 0 \Leftrightarrow \varphi = J(x), \forall x \in X \text{ and } \varphi \in X^*.$$

3. *If X is a Hilbert space, then*

$$V(\varphi,x) = ||\varphi - x||^2.$$

4. *If X is reflexive and E is a non empty closed and convex subset of X, then*

   (a) $\pi_E(\varphi) \neq \phi$, $\forall \varphi \in X^*$.

   (b) X is strictly convex if and only if $\pi_E(\varphi)$ is singleton for all $\varphi \in X^*$.

   (c) If X is also smooth, then for any given $\varphi \in X^*$,

$$z \in \pi_E(\varphi) \Leftrightarrow < \varphi - J(z), x - z > \ \leq 0, \forall x \in E.$$

Definition 2.16. ([2]). Let X be a Banach space with topological dual X^*, E be a non empty closed convex subset of X and $z \in X$. The convex normal cone of E at z is defined by

$$N_E(z) = \{\varphi \in X^* : < \varphi, x - z > \leq 0, \forall x \in E\}.$$

For more details about the convex normal cone see [2,3].

Theorem 2.17. *([2], prop. 0.0.1 and prop. 0.0.8). Let E be a non empty, closed and convex subset of Banach space X and $z \in X$. Then*

1. *$N_E(z) \cap B_* = \partial d_E(z)$, where $\partial d_E(z)$ is the subdifferential of the function $z \to d_E(z)$ ($d_E(z)$ is the distance from z to E).*

2. *If X is reflexive and smooth, then*

$$z \in \pi_E(\varphi) \Leftrightarrow \varphi - J(z) \in N_E(z), \ \forall \varphi \in X^* \text{ and } \forall z \in X.$$

Theorem 2.18. *([2], Lemma 1.4.1 and prop. 1.4.2), Let $p, q > 1$, X be a $p-$uniformly convex and $q-$uniformly smooth Banach space and let E be a non empty bounded subset of X, then there exist two constant $\alpha > 0$, $\beta > 0$ such that*

$$\alpha||x - y||^p \leq V(J(x), y) \leq \beta||x - y||^q, \ \forall x, y \in E.$$

Moreover, if E, in addition, is closed and $\varphi \in X^*$, then

$$d_E^V(\varphi) = 0 \Leftrightarrow J^*(\varphi) \in E.$$

By (Th. 2.5) and the relation between V and V^*, we can conclude easily the following lemma.

Lemma 2.19. *Let $p, q > 1$, X be a $p-$uniformly convex and $q-$uniformly smooth Banach space and let S be a bounded set of X^*, then there exist two nonnegative constant α, β such that*

$$\alpha||\varphi - \psi||^{q'} \leq V_*(J^*(\varphi), \psi) \leq \beta||\varphi - \psi||^{p'}, \ \forall \varphi, \psi \in S,$$

where p', q' are the conjugate numbers of p, q respectively.

3. Main Results

Existence of a solution for problem (P_1)

Theorem 3.1. *Let p, $q > 1$, X be a separable $p-$ uniformly convex and $q-$ uniformly smooth Banach space and $I = [0,T]$ where $T > 0$. Let C be a set-valued function defined from $I \times X$ to the family of non empty closed convex subsets of X satisfying the following two conditions:*

(C_1) for all φ, $\psi \in X^*$, all x, $y \in X$ and all t, $t' \in I$, there are positive real numbers λ, γ, a', b' and c' such that a', b', $c' \in [q', \infty[$ with $q' = \frac{q}{q-1}$ and that for every φ, $\psi \in X^*$, every x, $y \in X$ and every t, $t' \in I$

$$|d^V_{C(t,x)}(\psi) - d^V_{C(t',y)}(\varphi)| \leq \lambda(|t-t'|^{a'} + ||\psi - \varphi||^{b'}) + \gamma||y-x||^{c'}.$$

(C_2) there is a convex compact subset of X^*, K, such that

$$J(C(t,x)) \subset K, \forall (t,x) \in I \times X.$$

Let F be a scalarly upper semi continuous set-valued function defined from $I \times X \times X$ with convex weakly compact values in X^* such that

$$F(t,x,y) \subset (1 + ||x|| + ||y||)B_* \quad \text{for every } (t,x,y) \in graph(C).$$

Then for every $b \in X$ and every $a \in C(0,b)$, there exist two absolutely continuous functions $v : I \to X$, $u^* : I \to X^*$ and a continuous function $u : I \to X$ such that:

$$(P_1) \begin{cases} v(t) = b + \int_0^t u(s)ds, \forall t \in I, \\ u(t) \in C(t, v(t)), \forall t \in I, \\ u(t) = J^*(u^*(t)), \forall t \in I, \\ u^*(t) = J(a) + \int_0^t (u^*)'(s)ds, \forall t \in I, \\ -(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + F(t, v(t), u(t)), \text{ a.e. for } t \in I. \end{cases}$$

Proof. First, we note that since the Banach space X is $p-$uniformly convex, it is reflexive and strictly convex (Th. 2.2.4, [1] and Remark 2.1). Since X is $q-$uniformly smooth, its conjugate X^* is $q'-$uniformly convex, $q' = \frac{q}{q-1}$ (Th.2.2) and hence, $X^*$is strictly convex. Then by (Th.2.1) the duality mapping J is single valued, continuous, one to one, onto and $JJ^* = I_{X^*}$ and $J^*J = I_X$. For notational convenience, we take $T = 1$. Let $h : I \times X \times X \to X^*$ be a scalarly measurable selection of F (see [11]). Let $n \geq 1$ be a fixed integer. We consider a partition of $I = [0,1]$ by the points $t_i^n = ie_n$, $e_n = \frac{1}{n}$, $0 \leq i \leq n$. Let $I_0^n = \{t_0^n\} = \{0\}$, $I_{i+1}^n = \left]t_i^n, t_{i+1}^n\right]$, $0 \leq i \leq n-1$. We construct three sequences of continuous functions $v_n : I \to X$, $u_n^* : I \to X^*$ and $u_n : I \to X$ as follows:

$$v_n(0) = v_n(t_0^n) = b, u_n^*(0) = u_n^*(t_0^n) = J(a); \; u_n(0) = J^*(u_n^*(0)) = J^*(J(a)) = a.$$

For $t \in I_1^n$ we define

$$v_n(t) = b + (t - t_0^n)a, \tag{1}$$

$$u_n^*(t) = \frac{t_1^n - t}{e_n} J(x_0^n) + \frac{t - t_0^n}{e_n} J(x_1^n), \tag{2}$$

$$u_n(t) = J^*(u_n^*(t)), \tag{3}$$

where $x_0^n = a \in C(0,b)$ and

$$x_1^n = \pi_{C(t_1^n, v_n(t_1^n))}(J(x_0^n) - e_n h(t_0^n, v_n(t_0^n), u_n(t_0^n))). \tag{4}$$

Since X is strictly convex and since the set $C(t,x)$ is closed and convex subset of X, the point x_1^n exists (Th. 2.3). By the properties of the generalized projection, Def. 2.11 and from (3.4) we get

$$J(x_0^n) - e_n h(t_0^n, v_n(t_0^n), u_n(t_0^n)) - J(x_1^n) \in N_{C(t_1^n, v_n(t_1^n))}(x_1^n), \tag{5}$$

and

$$x_1^n \in C(t_1^n, v_n(t_1^n)). \tag{6}$$

From (3.2) and (3.5), for each $t \in]t_0^n, t_1^n[$ we get

$$(u_n^*)'(t) = \frac{J(x_1^n) - J(x_0^n)}{e_n} \in -N_{C(t_1^n, v_n(t_1^n))}(x_1^n) - h(t_0^n, v_n(t_0^n), u_n(t_0^n)). \tag{7}$$

Moreover, by (3.3)

$$u_n(t_0^n) = J^*(u_n^*(t_0^n)) = J^*(J(a)) = a = x_0^n$$

and

$$u_n(t_1^n) = J^*(u_n^*(t_1^n)) = J^*(J(x_1^n)) = x_1^n.$$

Then the relation (3.7) can be written as the form:

$$(u_n^*)'(t) \in -N_{C(t_1^n, v_n(t_1^n))}(u_n(t_1^n)) - h(t_0^n, v_n(t_0^n), u_n(t_0^n)). \tag{8}$$

By induction for $0 \leq i \leq n-1$ and for $t \in I_{i+1}^n$ we define

$$v_n(t) = v_n(t_i^n) + (t - t_i^n)u_n(t_i^n), \tag{9}$$

$$u_n^*(t) = \frac{t_{i+1}^n - t}{e_n} J(x_i^n) + \frac{t - t_i^n}{e_n} J(x_{i+1}^n), \tag{10}$$

and

$$u_n(t) = J^*(u_n^*(t)) \tag{11}$$

where

$$x_{i+1}^n = \pi_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n))). \tag{12}$$

According to the relations (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned} u_n(t_{i+1}^n) &= J^*(u_n^*(t_{i+1}^n)) \\ &= J^*(J(x_{i+1}^n)) \\ &= x_{i+1}^n. \end{aligned} \tag{13}$$

So,

$$u_n(t_{i+1}^n) \in C(t_{i+1}^n, v_n(t_{i+1}^n)). \tag{14}$$

Also,

$$(u_n^*)'(t) = \frac{J(x_{i+1}^n) - J(x_i^n)}{e_n} \in -N_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(x_{i+1}^n) - h(t_i^n, v_n(t_i^n), u_n(t_i^n)).$$

By (3.13), this relation implies that

$$(u_n^*)'(t) \in -N_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(u_n(t_{i+1}^n)) - h(t_i^n, v_n(t_i^n), u_n(t_i^n)). \tag{15}$$

Thus the functions v_n, u_n and u_n^* are defined on I, for each positive integer n.

Now, for each positive integer n we define also, two real valued functions θ_n, $\delta_n : I \to I$ as follows:

$$\theta_n(0) = \delta_n(0) = 0,\ \theta_n(t) = t_{i+1}^n,\ \delta_n(t) = t_i^n \text{ for } t \in I_{i+1}^n.$$

Then by (3.9) we get

$$v_n'(t) = u_n(\delta_n(t)),\ \forall n \geq 1 \text{ and } \forall t \in I,$$

this gets us

$$v_n(t) = b + \int_0^t u_n(\delta_n(s))ds \ , \forall n \geq 1 \text{ and } \forall t \in I. \tag{16}$$

Furthermore, by (3.14), (3.15) and condition (C_2) we have for all $n \geq 1$

$$u_n(\theta_n(t)) \in C(\theta_n(t), v_n(\theta_n(t))), \forall t \in I, \tag{17}$$

$$u_n^*(\theta_n(t)) \in K, \forall t \in I, \tag{18}$$

$$-(u_n^*)'(t) \in N_{C(\theta_n(t),v_n(\theta_n(t)))}(u_n(\theta_n(t))) + h(\delta_n(t), v_n(\delta_n(t)), u_n(\delta_n(t))) \tag{19}$$

for a.e. on I, and

$$h(\delta_n(t), v_n(\delta_n(t)), u_n(\delta_n(t))) \in F(\delta_n(t), v_n(\delta_n(t)), u_n(\delta_n(t))). \tag{20}$$

By condition (C_2) and (3.18) we get

$$||J(x_{i+1}^n)|| = ||u_n^*(\theta_n(t))|| \leq k. \tag{21}$$

and

$$||u_n(\delta_n(t))|| \leq k. \tag{22}$$

Then, by (3.22) and definition of v_n, for every $i = 0, 1, ..., n-1$ we obtain

$$\begin{aligned} v_n(t_{i+1}^n) &= v_n(t_i^n) + (t_{i+1}^n - t_i^n)u_n(t_i^n) \\ &= v_n(0) + e_n u_n(t_0^n) + e_n u_n(t_1^n) + ... + e_n u_n(t_i^n). \end{aligned}$$

Then, by (3.22), for every $i = 0, 1, 2, ..., n-1$ we have

$$\begin{aligned} ||v_n(t_{i+1}^n)|| &\leq b + e_n a + e_n k + ... + e_n k \\ &\leq b + a + i e_n k \\ &\leq b + a + k. \end{aligned} \tag{23}$$

So, by (3.20), (3.22) and (3.23)

$$\begin{aligned} ||h(t_i^n, v_n(t_i^n), u_n(t_i^n))|| &\leq 1 + ||v_n(t_i^n)|| + ||u_n(t_i^n)|| \\ &\leq 1 + b + a + k + k \\ &= 1 + b + a + 2k = \mu \end{aligned} \tag{24}$$

Now let us show that the sequence (u_n^*) converges uniformly to a Lipschitz continuos function $u^* : I \to X^*$. Let us first prove the following claim.

Claim. there is a constant $\rho > 0$ such that

$$||u_n^*(t_1) - u_n^*(t_2)|| \leq \rho|t_1 - t_2|, \forall t_1, t_2 \in I.$$

Since $X^*$is $q'-$ uniformly convex ($q' = \frac{q}{q-1}$), (Th. 2.2) then by (3.21), (3.24) and lemma 2.1 there exists a positive real number α such that for all $n \geq 1$ and all $i = 0, 1, 2, ..., n-1$,

$$\begin{aligned}
& \alpha||J(x_{i+1}^n) - (J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)))||^{q'} \\
\leq \; & V_*(J^*(J(x_{i+1}^n)), J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)))) \\
= \; & ||x_{i+1}^n||^2 - 2 < x_{i+1}^n, J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)) \\
> \; & +||J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n))||^2 \\
= \; & V(J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)), x_{i+1}^n) \\
= \; & d^V_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)) \\
& \text{(by (3.12))} \\
= \; & d^V_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(J(x_i^n) - e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n)) - d^V_{C(t_i^n, v_n(t_i^n))}(J(x_i^n)) \\
& \text{(by (Th. 2.5))} \\
\leq \; & \lambda\left[|t_{i+1}^n - t_i^n|^{a'} + ||e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n))||^{b'}\right] + \gamma||v_n(t_{i+1}^n) - v_n(t_i^n)||^{c'} \\
& \text{(by condition (C}_1\text{))} \\
\leq \; & \lambda\left[(e_n)^{a'} + (e_n)^{b'}(\mu)^{b'}\right] + \gamma(e_n)^{c'}||u_n(t_i^n)||^{c'}, \text{ (by (3.9))} \\
= \; & \lambda\left[(e_n)^{a'} + (e_n)^{b'}(\mu)^{b'}\right] + \gamma(e_n)^{c'}(k)^{c'} \\
\leq \; & \lambda\left[(e_n)^{q'} + (e_n)^{q'}(\mu)^{b'}\right] + \gamma(e_n)^{q'}(k)^{c'} \\
= \; & (e_n)^{q'}\left[\lambda + \lambda(\mu)^{b'} + \gamma(k)^{c'}\right].
\end{aligned}$$

Then

$$||J(x_{i+1}^n) - J(x_i^n) + e_n h(t_i^n, v_n(t_i^n), u_n(t_i^n))|| \leq \left[\frac{\lambda + \lambda(\mu)^{b'} + \gamma(k)^{c'}}{\alpha}\right]^{\frac{1}{q'}} e_n,$$

which gives us

$$||\frac{J(x_{i+1}^n) - J(x_i^n)}{e_n} + h(t_i^n, v_n(t_i^n), u_n(t_i^n))|| \leq \left[\frac{\lambda + \lambda(\mu)^{b'} + \gamma(k)^{c'}}{\alpha}\right]^{\frac{1}{q'}} = \delta.$$

Coming back to the definition of u_n^*, we get for all $n \geq 1$ and for almost all $t \in I$,

$$||(u_n^*)'(t)|| \leq \delta + \mu = \rho. \tag{25}$$

Then for all $n \geq 1$ and all $t_1, t_2 \in I$ $(t_1 < t_2)$

$$||u_n^*(t_2) - u_n^*(t_1))|| \leq \int_{t_1}^{t_2} ||(u_n^*)'(s)||ds \leq |t_2 - t_1|\rho, \tag{26}$$

this proved the claim.

Now by (3.25), for all $n \geq 1$ and all $t \in I$

$$\begin{aligned} & \lim_{n\to\infty} ||u_n^*(\theta_n(t)) - u_n^*(t)|| \\ \leq & \lim_{n\to\infty} \int_t^{\theta_n(t)} ||(u_n^*)'(s)||ds \\ \leq & \lim_{n\to\infty} \rho |\theta_n(t) - t| \\ = & \ 0. \end{aligned} \tag{27}$$

From (3.18) and (3.27) we conclude that for every $t \in I$, the set $\{u_n^*(t) : n \geq 1\}$ is a relatively compact subset of X^*. By (3.26) and theorem 0.0.4 [3] there exists a Lipschitz continuos function $u^* : I \to X^*$ such that u_n^* converges uniformly in $C_{X^*}(I)$ to u^* and $u^*(t) = J(x_0^n) + \int_0^t (u^*)'(s)ds$, for all $t \in I$. We define $u : I \to X$ by $u(t) = J^*(u^*(t))$, since J^* is continuous on the compact set K, then for all $t \in I$

$$\begin{aligned} \lim_{n\to\infty} u_n(t) &= \lim_{n\to\infty} J^*(u_n^*(t)) \\ &= J^*(u^*(t)) \\ &= u(t). \end{aligned} \tag{28}$$

Moreover, for all $t \in I$,

$$\begin{aligned} \lim_{n\to\infty} u_n(\theta_n(t)) &= \lim_{n\to\infty} J^*(u_n^*(\theta_n(t))) \\ &= J^*(u^*(t)) \\ &= u(t), \end{aligned} \tag{29}$$

and

$$\begin{aligned} \lim_{n\to\infty} u_n(\delta_n(t)) &= \lim_{n\to\infty} J^*(u_n^*(\delta_n(t))) \\ &= J^*(u_n^*(t)) \\ &= u(t). \end{aligned} \tag{30}$$

We note that for all $n \geq 1$ and all $t \in I$,

$$u_n(\delta_n(t)) \in J^*(K),$$

which ensures that the sequence $\{u_n(\delta_n)\}$ is uniformly bounded. So, by (3.16) and (3.30) for all $t \in I$

$$\lim_{n\to\infty} v_n(t) = v(t),$$

where

$$v(t) = b + \int_0^t u(s)ds. \tag{31}$$

Let us show that

$$u(t) \in C(t, v(t)) \text{ , } \forall t \in I. \tag{32}$$

Let $t \in I$ and n be a fixed positive integer, by (3.17) and condition (C_1), we have

$$\begin{aligned}
& d^V_{C(t,v(t))}(J(u_n(\theta_n(t)))) \\
= \; & d^V_{C(t,v(t))}(J(u_n(\theta_n(t)))) - d^V_{C(\theta_n(t),v_n(\theta_n(t)))}(J(u_n(\theta_n(t)))) \\
\leq \; & \lambda \; (|\theta_n(t) - t|^{a'}) + \gamma \, ||v_n(\theta_n(t)) - v(t)||^{c'}.
\end{aligned}$$

Then

$$\begin{aligned}
& d^V_{C(t,v(t))}(J(u(t))) \\
\leq \; & |d^V_{C(t,v(t))}(J(u(t))) - d^V_{C(t,v(t))}(J(u_n(\theta_n(t)))| + d^V_{C(t,v(t))}(J(u_n(\theta_n(t))) \\
\leq \; & \gamma \, ||J(u(t)) - J(u_n(\theta_n(t)))||^{c'} + \lambda \; (|\theta_n(t) - t|^{a'}) + \gamma \, ||v_n(\theta_n(t)) - v(t)||^{c'}.
\end{aligned}$$

By passing to the limit when $n \to \infty$ in the preceding inequality, we get $u(t) \in C(t, v(t))$.

Now, by (3.24), the sequence (g_n) which is defined by

$$g_n(t) = h(\delta_n(t), v_n(\delta_n(t)), u_n(\delta_n(t))),$$

is uniformly bounded by μ in X^*. Since X is separable and reflexive ,then, we can assume that the sequence (g_n) converges to a function $g \in L^\infty(I, X^*)$ with respect to the topology $\sigma(L^\infty(I, X^*), L^1(I, X^*))$. By invoking the scalarly upper semi continuous of F we get (see Th. V-14 [11])

$$g(t) \in F(t, v(t), u(t)), \text{ a.e. on } I. \tag{33}$$

Finally, we proceed to show that

$$(u^*)'(t) \in -N_{C(t,v(t))}(u(t)) - g(t), \text{ a.e.}$$

From (3.19), (3.20) and (3.25) we get for every $n \geq 1$ and for almost $t \in I$,

$$-(u_n^*)'(t) - g_n(t) \in N_{C(\theta_n(t),v_n(\theta_n(t)))}(u_n(\theta_n(t))) \cap \delta B_*,$$

so, by (Th. 2.4) we get for every $n \geq 1$ and for almost all $t \in I$

$$-(u_n^*)'(t) - g_n(t) \in \delta \partial d_{C(\theta_n(t),v_n(\theta_n(t)))}(u_n(\theta_n(t))). \tag{34}$$

By Mazurs lemma we get for almost all $t \in I$,

$$(u^*)'(t) + g(t) \in \bigcap_n \overline{co} \left\{ (u_m^*)'(t) + g_m(t) : m \geq n \right\},$$

Hence $\overline{co}$ denotes the closed convex hull. Fix any $t \in I$ such that the preceding relation is satisfied. let $z \in X$, the last above relation yields:

$$< (u^*)'(t) + g(t), z > \;\leq\; \inf_n \sup_{m \geq n} \; < (u_m^*)'(t) + g_m(t), z >,$$

hence according to (3.34) we get

$$\begin{aligned} < \;& (u^*)'(t) + g(t), z > \\ \leq \;& \limsup_n \sigma < -\delta \partial d_{C(\theta_n(t),v_n(\theta_n(t)))} \, (u_n(\theta_n(t))), z >, \end{aligned} \tag{35}$$

where $\sigma(D,z)$ is the support function of the convex closed set D at the point z.

According to (prop. 0.0.2, [2]) the relation (3.35) gets us

$$\begin{aligned} < \;& (u^*)'(t) + g(t), z > \\ \leq \;& \sigma < -\delta \partial d_{C(t,v(t))} \, (u(t)), z >, \end{aligned}$$

as the set $\partial d_{C(t,v(t))} \, (u(t))$ is closed and convex and $u(t) \in C(t,v(t))$, $\forall t \in I$, we obtain

$$(u^*)'(t) + g(t) \in -(\rho + \mu) \; \partial \; d_{C(t,v(t))} \, (u(t)).$$

Thus

$$-(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + g(t) \text{ a.e.},$$

this means

$$-(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + F(t, v(t), u(t)),$$

which completes the proof.

□

Existence of a solution for problem (P_2)

Theorem 3.2. *Let p, $q > 1$, X be a separable $p-$ uniformly convex and $q-$ uniformly smooth Banach space and $I = [0,T]$ where $T > 0$. Let C be a set-valued function defined from $I \times X$ to the family of non empty closed convex subsets of X and satisfying the following two conditions:*

(C_1) for all φ, $\psi \in X^*$, all x, $y \in X$ and all t, $t' \in I$, there are positive real numbers λ, γ, a', b' and c' such that a', b', $c' \in [q', \infty[$ with $q' = \frac{q}{q-1}$ and that for every φ, $\psi \in X^*$, every x, $y \in X$ and every t, $t' \in I$

$$|d^V_{C(t,x)}(\psi) - d^V_{C(t',y)}(\varphi)| \leq \lambda(|t-t'|^{a'} + ||\psi - \varphi||^{b'}) + \gamma||y-x||^{c'}.$$

(C_2) there is a convex compact subset of X^*, K, such that

$$J(C(t,x)) \subset K, \forall (t,x) \in I \times X.$$

Let G be a scalarly upper semi continuous set-valued function defined from $[0,T] \times C_X([-r,0]) \times C_X([-r,0])$ to the family of all non empty convex weakly compact subsets of X^*, such that $G(t,f,g) \subset (1 + ||f(0)|| + ||g(0)||)B_*$, for all $(t,f,g) \in [0,T] \times C_X([-r,0]) \times C_X([-r,0])$, where $C_X([-r,0])$ is the set of all continuous functions from $[-r,0]$ to X.

Then, for every w, $z \in C_X([-r,0])$ with $w(0) \in C(0,z(0))$, there exists three continuous functions $v : [-r,T] \to X$, $u : [-r,T] \to X$ and $u^* : [-r,T] \to X$ such that

$$(P_2)\begin{cases} v(t) = z(t),\ \forall t \in\ [-r,0];\ v(t) = z(0) + \int_0^t u(s)ds,\ \forall t \in [0,T], \\ u^*(t) = J(w(t)),\ \forall t \in [-r,0], \\ u^*(t) = J(w(0)) + \int_0^t (u^*)'(s)ds,\ \forall t \in [0,T], \\ u(t) = J^*(u^*(t)),\ \forall t \in [-r,T], \\ u(t) \in C(t,v(t)),\ \forall t \in [0,T], \\ -(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + G(t,\tau(t)v,\tau(t)u),\ \text{for a.e. on } [0,T]. \end{cases}$$

Proof. The proof will be a careful adaptation of notations of the proof of Theorem 3.1. We will focus the differences. Let

$$h : [0,T] \times C_X([-r,0]) \times C_X([-r,0]) \to X^*$$

be a measurable selection of G. For notations convenience, we take $T = 1$. Let $n \geq 1$ be a fixed integer, we put $e_n = \frac{1}{n} < 1$ and we will consider the following partition of the interval $[0,1]$:

$$t_i^n = ie_n \text{ where } 0 \leq i \leq n.$$

Let $I_{i+1}^n = \left]t_i^n, t_{i+1}^n\right]$ when $0 \leq i \leq n-1$ and $I_0^n = \{t_0^n\} = \{0\}$. We set $a = w(0)$, $b = z(0)$. We construct three sequences of continuous functions v_n, u_n^* and u_n as follows. Put $v_n(t) = z(t)$ and $u_n^*(t) = J(w(t))$ for all $t \in [-r, 0]$. Thus $v_n(0) = v_n(t_0^n) = z(0) = b$, $u_n^*(0) = u_n^*(t_0^n) = J(w(0)) = J(a)$. Also, we set

$$u_n(t) = J^*(u_n^*(t)), \forall t \in [-r, 0].$$

For each $t \in I_1^n$, we define

$$v_n(t) = z(0) + (t - t_0^n)w(0), \tag{36}$$

$$u_n^*(t) = \frac{t_1^n - t}{e_n} J(x_0^n) + \frac{t - t_0^n}{e_n} J(x_1^n), \tag{37}$$

and

$$u_n(t) = J^*(u_n^*(t)), \tag{38}$$

where $x_0^n = a$ and

$$x_1^n = \pi_{C(t_1^n, v_n(t_1^n))}(J(x_0^n) - e_n h(t_0^n, \tau(t_0^n)v_n, \tau(t_0^n)u_n). \tag{39}$$

According to the definition of the generalized projection we have

$$u_n(t_1^n) = J^*(u_n^*(t_1^n)) = J^*(J(x_1^n)) = x_1^n \in C(t_1^n, v_n(t_1^n)) \tag{40}$$

and

$$\frac{J(x_1^n) - J(x_0^n)}{e_n} \in -N_{C(t_1^n, v_n(t_1^n))}(x_1^n) - h(t_0^n, \tau(t_0^n)v_n, \tau(t_0^n)u_n).$$

Then for almost every $t \in]t_0^n, t_1^n[$,

$$-(u_n^*)'(t) \in N_{C(t_1^n, v_n(t_1^n))}(u_n(t_1^n)) + h(t_0^n, \tau(t_0^n)v_n, \tau(t_0^n)u_n). \tag{41}$$

By induction for $0 \leq i \leq n$ we set for $t \in I_{i+1}^n$

$$v_n(t) = v_n(t_i^n) + (t - t_i^n)u_n(t_i^n), \tag{42}$$

$$u_n^*(t) = \frac{t_{i+1}^n - t}{e_n} J(x_i^n) + \frac{t - t_i^n}{e_n} J(x_{i+1}^n), \tag{43}$$

$$u_n(t) = J^*(u_n^*(t)). \tag{44}$$

where

$$x_{i+1}^n = \pi_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(J(x_i^n) - e_n h(t_i^n, \tau(t_i^n)v_n, \tau(t_i^n)u_n)). \tag{45}$$

Again, according to the definition of the generalized projection we get

$$u_n(t_{i+1}^n) = J^*(u_n^*(t_{i+1}^n)) = J^*(J(x_{i+1}^n)) = x_{i+1}^n \in C(t_{i+1}^n, v_n(t_{i+1}^n)) \tag{46}$$

and

$$\frac{J(x_{i+1}^n) - J(x_i^n)}{e_n} \in -N_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(x_{i+1}^n) - h(t_i^n, \tau(t_i^n)v_n, \tau(t_i^n)u_n).$$

Then, for almost $t \in \left]t_i^n, t_{i+1}^n\right[$,

$$(u_n^*)'(t) \in -N_{C(t_{i+1}^n, v_n(t_{i+1}^n))}(u_n(t_{i+1}^n)) - h(t_i^n, \tau(t_i^n)v_n, \tau(t_i^n)u_n). \tag{47}$$

Now, for each integer $n \geq 1$ we define two functions $\theta_n,\ \delta_n : [0,1] \to [0,1]$ as follows: $\theta_n(0) = \delta_n(0) = 0$, $\theta_n(t) = t_{i+1}^n$, $\delta_n(t) = t_i^n$ for $t \in I_{i+1}^n$, $0 \leq i \leq n-1$. Clearly

$$\lim_{n\to\infty} (\theta_n(t)) = \lim_{n\to\infty} (\delta_n(t)) = t,\ \forall t \in [0,1], \tag{48}$$

then from (3.42), (3.44), (3.46) and (3.47) for every $n \geq 1$ we have

$$v_n(t) = v_n(\delta_n(t)) + \int_0^t u_n(\delta_n(s))ds,\ \forall t \in [0,1], \tag{49}$$

$$u_n^*(t) = J(w(0)) + \int_0^t (u_n^*)'(s)ds,\ \forall t \in [0,1], \tag{50}$$

$$u_n(t) = J^*(u_n^*(t)),\ \forall t \in [0,1], \tag{51}$$

$$u_n(\theta_n(t)) \in C(\theta_n(t), v_n(\theta_n(t)),\ \forall t \in [0,1],$$

$$-(u_n^*)'(t) \in N_{C(\theta_n(t), v_n(\theta_n(t)))}(u_n(\theta_n(t))) + h(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n) \tag{52}$$

a.e. for $t \in [0,1]$,

$$h(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n) \in F(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n). \tag{53}$$

a.e. for $t \in [0,1]$. As in the proof of theorem 3.1 we can show that

1. The sequence (u_n^*) converges uniformly on $[0,1]$ to an absolutely continuous function u^* on $[0,1]$ and

$$u^*(t) = J(w(0)) + \int_0^t (u^*)'(s)ds,\ \forall t \in [0,1]. \tag{54}$$

2. The sequence (u_n) converges point by point to a continuous function u such that

$$u(t) = J^*(u^*(t)),\, \forall t \in [0,1]. \tag{55}$$

3. The sequence (v_n) converges uniformly on $[0,1]$ to an absolutely continuous function v with

$$v(t) = J(z(0)) + \int_0^t u(s)ds,\, \forall t \in [0,1]. \tag{56}$$

We extend the definition of u^*, u and v on $[-r,0]$ as: $u^*(t) = J(w(t))$, $u(t) = J^*(u^*(t)) = w(t)$ and $v(t) = z(t)$. So, the sequences u_n^*, u_n and v_n converge point by point to u^*, u and v on $[-r,1]$ respectively.

Now for every $n \geq 1$ and $t \in [0,1]$ we define

$$\begin{aligned} g_n &: [0,1] \to X^* \\ g_n(t) &= h(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n), \end{aligned}$$

then $g_n(t) \in G(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n)$, a.e. for every $t \in I$. So, by (3.22) and (3.23) we obtain for all $n \geq 1$ and almost for all $t \in I$,

$$\begin{aligned} ||g_n(t)|| &= ||h(\delta_n(t), \tau(\delta_n(t))v_n, \tau(\delta_n(t))u_n)|| \\ &\leq 1 + ||v_n(\delta_n(t))|| + ||u_n(\delta_n(t))|| \\ &\leq 1 + a + b + k + k \\ &= 1 + a + b + 2k = \mu. \end{aligned}$$

Then, we can assume that, with respect to the topology $\sigma(L^\infty(I,X^*), L^1(I,X^*))$, the sequence (g_n) converges to a function $g \in L^\infty(I,X^*)$.

Claim.

$$g(t) \in G(t, \tau(t)v, \tau(t)u), \text{a.e.} \tag{57}$$

Let $t \in I$. We have

$$
\begin{aligned}
& ||\tau(\delta_n(t))u_n - \tau(t)u|| \\
\leq\ & ||\tau(\delta_n(t))u_n - \tau(t)u_n|| + ||\tau(t)u_n - \tau(t)u|| \\
\leq\ & \sup_{-r\leq s\leq 0} ||u_n(\delta_n(t)+s) - u_n(t+s)|| + ||\tau(t)u_n - \tau(t)u|| \\
\leq\ & \sup_{\substack{-r\leq s_1\leq s_2\leq T \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| + ||\tau(t)u_n - \tau(t)u|| \\
\leq\ & \sup_{\substack{-r\leq s_1,s_2\leq 0 \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| + \sup_{\substack{-r\leq s_1\leq 0\leq s_2\leq T \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| \\
& + \sup_{\substack{0\leq s_1\leq s_2\leq T \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| + ||\tau(t)u_n - \tau(t)u|| \\
\leq\ & \sup_{\substack{-r\leq s_1,s_2\leq 0 \\ |s_1-s_2|\leq \frac{1}{n}}} ||w(s_1) - w(s_2)|| + \sup_{\substack{-r\leq s_1\leq 0 \\ |s_1|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(0)|| \\
& + \sup_{\substack{0\leq s_2\leq T \\ |s_2|\leq \frac{1}{n}}} ||u_n(0) - u_n(s_2)|| + \sup_{\substack{0\leq s_1\leq s_2\leq T \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| + ||\tau(t)u_n - \tau(t)u|| \\
\leq\ & 2\sup_{\substack{-r\leq s_1,s_2\leq 0 \\ |s_1-s_2|\leq \frac{1}{n}}} ||w(s_1) - w(s_2)|| + 2\sup_{\substack{0\leq s_1,s_2\leq T \\ |s_1-s_2|\leq \frac{1}{n}}} ||u_n(s_1) - u_n(s_2)|| + ||\tau(t)u_n - \tau(t)u||.
\end{aligned}
$$

By the continuity of w, the uniform convergence of u_n towards u and the preceding estimate, we get

$$\lim_{n\to\infty} ||\tau(\delta_n(t))u_n - \tau(t)u|| = 0. \tag{58}$$

Similarly we prove that

$$\lim_{n\to\infty} ||\tau(\delta_n(t))v_n - \tau(t)v|| = 0. \tag{59}$$

using (3.58) and (3.59) and by invoking the scalarly upper semi contiuity of G and a closure type result in ([11], theorem VI-14) we get the desired claim.

Finally, as in the proof of (Th. 3.1) we can prove that

$$-(u^*)'(t) \in N_{C(t,v(t))}(u(t)) + G(t,\tau(t)v,\tau(t)u), \text{ a.e. on } [0,1],$$

which completes the proof. $\square$

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